Florentin Smarandache

My High School Math Notebook

Vol. 2

(Algebra (9th to 12th grades), Trigonometry)

Educational Publisher
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(1973 – 1974)
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Preface

Since childhood I got accustomed to study with a pen in my hand.

I extracted theorems and formulas, together with the definitions, from my text books.

It was easier, later, for me, to prepare for the tests, especially for the final exams at the end of the semester.

I kept (and still do today) small notebooks where I collected not only mathematical but any idea I read in various domains.

These two volumes reflect my 1973-1974 high school studies in mathematics.

Besides the textbooks I added information I collected from various mathematic books of solved problems I was studying at that time.

In Romania in the 1970s and 1980s the university admission exams were very challenging. Only the best students were admitted to superior studies. For science and technical universities, in average, one out of three candidates could succeed, since the number of places was limited. For medicine it was the worst: only one out of ten!

The first volume contains: Arithmetic, Plane Geometry, and Space Geometry.

The second volume contains: Algebra (9^{th} to 12^{th} grades), and Trigonometry.

Florentin Smarandache

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ALGEBRA GRADE 9^{TH}

$\pi = 3.141592635...$ is an irrational number

The module of a real number x is the positive value of that number.

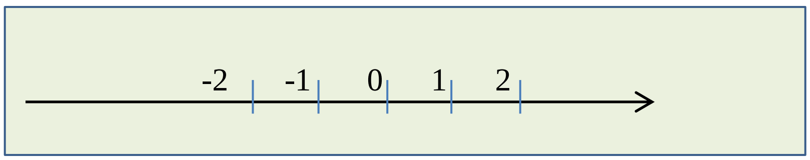
$$|x| = \begin{cases} -x \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ +x \text{ if } x > 0 \end{cases}$$

$|+3| = 3;$

$|-3| = 3;$

$|0| = 0$

To any real number we can associate a point on the line and only one; If on a line we select a point O called origin and a point which will represent the number 1 (the unity segment), the line is called the real axis or the real line.



Operations with real numbers

Addition

The addition is the operation in which to any pair of real numbers x, y corresponds a real number and only one, denoted $x + y$, and which has:

a) The common sign of x, y, if they have the same sign, and the module is the sum of their modules.
b) The sign of the number whose module is larger and the module is the difference of their modules, in the case that the numbers have different signs.

Multiplication

The multiplication is the operation through which to an pair of real numbers x, y corresponds a real number an if d only one, denoted $x \cdot y$ (xy or $x \times y$), and which has:

a) The sign of $"+"$ if x, y have the same sign, and as module the product of their modules.
b) The sign of $"-"$ if x, y have different signs, and as module the product of their modules.

Algebraic computation

An algebraic expression is a succession of real numbers written with their signs.
Variables are the letters that intervene in an algebraic expression.
Constants are the real numbers (coefficients of the variable).

Example

$E(a,b,c) = 2a^2 + 4b^3 - c$

a,b,c are variables

+2, +4, -1 are constants

$E(a,b,c)$ is an algebraic expression

Algebraic expression:

- Monomial
- Polynomial
  - Binomial
  - Trinomial
  - Etc.

The monomial expression is a succession of signs between which the first is a constant, and the following are different variables separated by the sign of the operation of multiplication.

Examples:

$7x^2 + yz^3; 2x; 4; 0$

Monomial

- The literal part: x^2yz^3; x
- The coefficient of the monomial: $7; 4; 0$

The degree of a monomial

a) The degree of a monomial with only one variable is the exponent of the power of that variable.
b) The degree of a monomial which contains multiple variable is the sum of the power of the exponents of the variables.
c) The degree of a monomial in relation to one of its variable is exactly the exponent of the respective variable.

Similar monomials

Similar monomials are those monomials which have the same variables and each variable has the same exponential power.

The sum of several monomials is a similar monomial with the given monomials.

Observation:

If a letter is at the denominator and has a higher exponent than at that from the numerator (example: $4x^2 : 2x^5$) or it is only at the denominator, then we don't obtain a monomial.

Polynomial

A polynomial is a sum of monomials

The numeric value of a polynomial results from the substitution of its variables wit real numbers.

$$P(x, y) = 2xy + 3x - 1$$

$$P(1,0) = 2 \cdot 1 \cdot 0 + 3 \cdot 1 - 1 = 2$$

Polynomial of a reduced form

The polynomial of a reduced form is the polynomial which can be represented as a sum of similar monomials.

The degree of a polynomial

The degree of a polynomial in relation to a variable is the highest exponent of the variable.

The degree of a polynomial relative to more variables is equal to the sum of the highest exponents of the variables.

Homogeneous polynomials

The homogeneous polynomials are the polynomials give in a reduced form whose terms are all of the sane degree.

Examples:

$2x;5x^7;4x^2+3xy-y^2;x+y+z$

To ordinate the terms of a polynomial means to write its terms in a certain order, following certain criteria.

A polynomial with one variable is ordinated by the ascending or descending powers of the variables.

The canonical form of a polynomial $P(x)$ is the ordinate polynomial by the descending powers of x.

The incomplete polynomial is the polynomial which ordinated by the powers of x has some terms of different degrees (smaller than the highest power of x).

Example:

$-2x^6+3x^5+2$ is an incomplete polynomial (are missing ax^4,bx^3,cx^3,dx)

A polynomial with several variables can be ordinated by the powers of a given variable.

Example:

$P(x,y,z)=-3y^3+xy^2+(3x^3-zx)y+(2x^3-2x^2z-5x^2)$ is ordinated by the powers of y.

The ordination by the homogeneous polynomials

Any given polynomial of a reduced form can be ordinated as a sum of homogeneous polynomials.

Example:

$P(x,y,z)=2x^3y+3x++4y-xy+z^3-5=(2x^3y)+(z^3)+(-xy)+(3x+4y)+(-5)$

Operations with polynomials

Opposed polynomials are two polynomials whose sum is the null polynomial.

The sum of polynomials

If $P(x)+Q(x)=S(x)\Rightarrow grS(x)\le max\left[P(x),Q(x)\right]$

The difference of polynomials

If $P(x)-Q(x)=D(x)\Rightarrow grD(x)\leq max\left[P(x),Q(x)\right]$

The product of polynomials

If $P(x)\cdot Q(x)=D(x)\Rightarrow gr.D(x)=gr.P(x)+gr.Q(x)$

The division of polynomials

- The division without remainder

  If $\dfrac{P(x)}{Q(x)}=I(x)\Rightarrow gr.I(x)=gr.P(x)-gr.Q(x)$

- The division with reminder

If $\dfrac{P(x)}{Q(x)}=I(x)$ and $R(x)\Rightarrow gr.I(x)=gr.P(x)-gr.Q(x)$ and $gr.R(x)<I(x)$.

To divide two polynomials of the same variable, we'll ordinate the polynomials by their descending powers of the variable.

A polynomial monomial is a polynomial in which the coefficients of the unknown have a maximum degree of 1.

To divide two polynomials that have several variables, we'll ordinate the polynomials by the powers of a variable and proceed with the division as usual.

Theorem

The remainder of a division of a polynomial $P(x)$ by $x-a$ is equal with the numeric value of the polynomial for which $x-a$, that is $R=P(a)$

Consequence

If $P(x)$ is divisible by $x-a$, then $P(a)=0$.

Formulae used in computations

1) **The product of a sum and difference**

   $(x+y)(x-y)=x^2-y^2$

2) **The square of a binomial**

   $(x+y)^2=x^2+2xy+y^2$

1. The method of expressing relative to a common factor
2. The usage of the formulae in computations

   a) $x^2-y^2=(x+y)(x-y)$

   b) $x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$

   $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$

c) $\left(x^2+2xy+y^2\right)=\left(x+y\right)^2$

$\left(x^2-2xy+y^2\right)=\left(x-y\right)^2$

d) $x^3+3x^2y+3xy^2+y^3=\left(x+y\right)^3$

$x^3-3x^2y+3xy^2-y^3=\left(x-y\right)^3$

e) $x^2+y^2+z^2+2xy+2xz+2yz=\left(x+y+z\right)^2$

f) $x^2+\left(a+b\right)x+ab=\left(x+a\right)\left(x+b\right)$

3. The method of grouping the terms
4. Combined methods

The greatest common divisor (GCD) of several given polynomials is the polynomial with the greatest degree which divides all given polynomials. To determine it we'll take the common irreducible factors at the smallest exponent.

The smallest common multiple of several given polynomials is the polynomial whose degree is the smallest and which is divided by each of the given polynomials. To determine it, we will form the product of the common and non-common irreducible factors at the highest exponent.

Algebraic fractions (algebraic expressions)

An algebraic fraction is an expression of the format $\frac{P}{Q}$, where P,Q are polynomials

To simplify an algebraic fraction is equivalent with writing it into an irreducible format.

To find the common denominator we have to decompose firstly the polynomials denominators in irreducible factors.

Co-prime polynomials are the polynomials which are prime between them.

Algebraic fraction generalized

An algebraic fraction generalized is a fraction of the format:

$$F\left(x,y\right)=\frac{\frac{x+y}{x-1}}{\frac{2x+5}{y+7}}$$

Equalities, equations, system of equations

Definition

An equality is a proposition that refers to the elements of a set M in which the sign "=" appears only one time.

Equality is formed of

- The left side (side I)
- The right side (the side II)

Equality can be:

- True
- False
-

Definition

An equation is the equality which is true only for certain values of the unknown.

To solve an equation means to find the solutions of the give equation.

The equation E_1 implies the equation E_2 only when all the solutions of the equation E_1 are also solutions of the equation E_2 (but not necessarily any solution of E_2 is a solution of E_1).

$P \Leftrightarrow Q \ if \ P \Rightarrow Q \ and \ Q \Rightarrow P$

Two equations are equivalent only when have the same solutions.

Properties of an equation in the set of real numbers

Theorem

1) If we add to both sides of an equation (equality) the same element, we'll obtain an equation equivalent to the given equation.
2) If we move a term from a side of equality to the other side and change its sign, we'll obtain an equality equivalent to the given one.
3) If we multiply both sides of equality with a number different of zero, we'll obtain an equality equivalent to the given equality.

The equation of first degree with one unknown

Method of solving it

- If there are parenthesis, we'll open them computing the multiplications or divisions
- If there are denominators we'll eliminate them
- The terms that contain the unknown are separated from the rest, trying to obtain:

  $ax = b \Rightarrow x = \frac{b}{a}$

  This equation has only one solution.

The discussion of the equation of first degree with one unknown

Real parameters are determined real numbers but not effectively obvious

The equation can be:

- Determined (compatible) has a solution
- Not determined: has an infinity od solutions
- Impossible (incompatible) it doesn't have any solution

In a discussion are treated the following cases:

a) The case when the coefficient of x is zero
b) The case when the coefficient of x is different of zero. If the equation contains fractions, a condition that is imposed is that the denominators should be different of zero, otherwise the operations are not defined.

System of equations

Theorems

1) If in a system of equations one equation or several are substituted by one equation or several equations equivalent to those substituted, then we obtain a system of equations equivalent to the initial system of equations.
2) If in a system of equations one equation or several are multiplied by a number different of zero, we'll obtain a system of equations equivalent to the given system of equations.
3) If in a system of equations we substitute one equation by an equation in which the left side contains the sum of the left side members of the equations of the system and in the right side the sum of the right members of the equations, then we will obtain a system of equations equivalent to the given system of equations.

Methods of computation

1) The method of substitution
2) The method of reduction

The systems of equations ca be:

- Determined (compatible) has solutions
- Non-determined – it has an infinity of solutions
- Impossible (incompatible) it does not have solutions

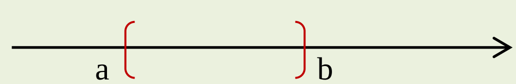
Inequations of first degree

A relation of order is any relation which has the following properties:

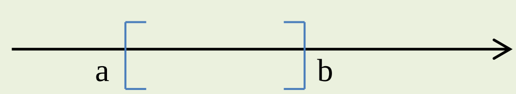
1) $\forall x, y \in \mathbb{R}$, is true at least one of the following propositions: $x > y;\ x = 0;\ x < y$
2) Transitivity
3) Anti-symmetry

Intervals

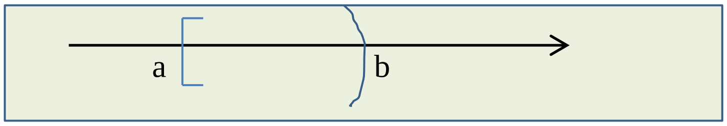
1) Open interval



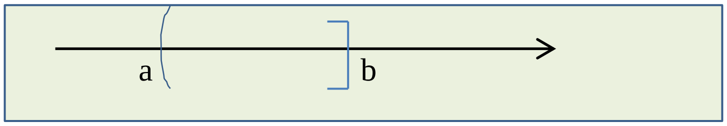
2) Closed interval



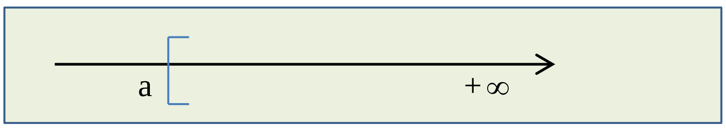
3) Interval closed to left and open to the right



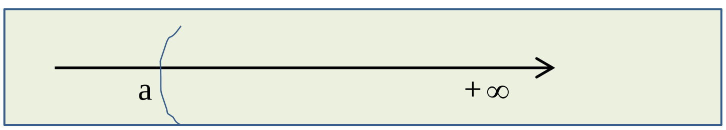
4) Interval open to the left and closed to the right



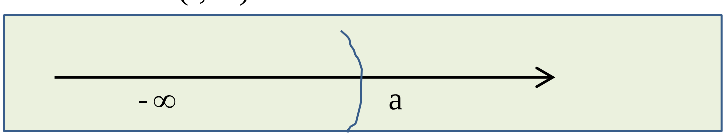
5) Interval closed to the left and unlimited to the right
$[a,+\infty)$



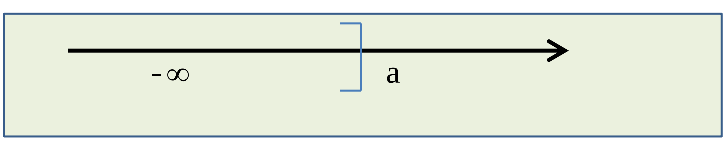
6) Interval open to the left and unlimited to the right
$(a,+\infty)$



7) Interval open to the right and unlimited to the left
$(-,\infty)$



8) Interval closed to the right and unlimited to the left
$(-\infty,a]$



Properties of the Inequality

1) If the same real number is added to both sides of an inequality, it is obtained an inequality equivalent to the given inequality.
2) If a given inequality is multiplied on both sides by the same real positive number, it will be obtained an inequality equivalent to the given inequality.
   If the inequality is multiplied on both sides with a real negative number then the sense of inequality will change.

Powers with irrational exponent

An irrational algebraic expression is he expression which contains the sign of radical:

$\sqrt{a^2-3}; \sqrt[3]{x+\sqrt{y}}; 3+\sqrt{\frac{a}{b}}$

The power of a number is that number multiplied by itself as many times as the power indicates.

$$a^n = \underbrace{a \cdot a \cdot a \cdot \,......\, \cdot a}_{n\ times}$$

a is called the base of the power
n is called the power exponent

- If $x < y \Rightarrow x^n < y^n \ x,y > 0; n \in \mathbb{N}$
- If $x = y \Rightarrow x^n = y^n$
- If $x > y \Rightarrow x^n > y^n$
- But, $x^n = y^n \not\Rightarrow x = y$

Radical's definition

Given a positive number A and a natural number $n \geq 2$, a radical of order n of the number A is another positive number, denoted $\sqrt[n]{A}$, which raised at the power of n reproduces the number A.

$$\sqrt[2k]{A^{2k}} = |A| = \begin{cases} A & if\ A \geq 0 \\ -A\ if & A < 0 \end{cases};$$

$$\sqrt{A^2} = \pm A$$

$$\sqrt{(-5)^2} = -(-5) = 5; \ \sqrt{(-5)^2} = |-5| = 5$$

$$\left(\sqrt[4]{a}\right)^4 = a; \ (\ a \geq 0)$$

Properties of the radicals

1) The amplification of the radical: The value of a radical doesn't change if we multiply by the same natural number the indices of the radical and the exponent of the quantity from inside the radical. $\sqrt[n]{A} = \sqrt[np]{A^p}$
2) The simplification of a radical: The value of a radical doesn't change if we divide by the same number k the indices of the radical and the exponent of the quantity from inside the radical. $\sqrt[np]{A^p} = \sqrt[n]{A}$
   Observation: The simplification of a radical must be done carefully when the indices of the radical is even and there is no precise information about its sign. That's why under the radical we use the module of a quantity: $\sqrt[4]{x^2} = \sqrt[4]{|x|^2} = \sqrt{|x|}$; $\sqrt[6]{x^2} = \sqrt[6]{|x|^2} = \sqrt[3]{|x|}$
3) The root of a certain order from a product is equal to the product of the roots of the same order of the factors, with the condition that each factor has a sense. $\sqrt[n]{A \cdot B \cdot ... \cdot V} = \sqrt[n]{A} \cdot \sqrt[n]{B} \cdot ... \sqrt[n]{V}; \ (\ A,B,...,V > 0)$
   To make the radicals to have the same index, we'll amplify the radicals getting them to the smallest common multiple.

4) The root of a ratio is equal to the ratio of the roots of the same order of the denominator and numerator, with the condition that both factors are positive. $\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$, $\frac{\sqrt[n]{A}}{\sqrt[n]{B}} = \sqrt[n]{\frac{A}{B}}$
5) To raise a radical to a power it is sufficient to raise to that power the expression from inside the radical
6) To extract the root from a radical, the roots indexes are multiplied and the expression under the radical remains unchanged. $\sqrt[n]{\sqrt[k]{A}} = \sqrt[nk]{A}$

The extraction of rational factor from a radical

$$\sqrt[n]{a^n b^k} = a\sqrt[n]{b^k}$$

Introduction of a factor under a radical

$$a\sqrt[n]{b} = \sqrt[n]{a^n}\sqrt[n]{b} = \sqrt[n]{a^n b}$$

Elimination of a radical from the denominator

$$\sqrt[m]{\frac{A}{B}} = \frac{\sqrt[m]{A}}{\sqrt[m]{B}} = \frac{\sqrt[m]{A}\cdot\sqrt[m]{B}}{B} = \frac{\sqrt[m]{AB}}{B};\ (A>0, B>0)$$

The reduction of similar radicals

The rationalization of the denominator of a fraction (we amplify with the conjugate)

Conjugate expression of an irrational factor is another expression, which is not identical null, such that their product does not contain radicals

If at the denominator there is an algebraic expression of radicals of the format $\sqrt[3]{a} \pm \sqrt[3]{b}$, we amplify with $\sqrt[3]{a^2} \pm \sqrt[3]{ab} + \sqrt[3]{b}$.

1) $a^m \cdot a^n = a^{m+n};\ (a \neq 0,\ m > n)$
2) $a^m \div a^n = a^{m-n};\ (a \neq 0)$
3) $(a^n)^m = a^{nm};\ (a \neq 0)$
4) $(ab)^n = a^n b^n$
5) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n};\ (b \neq 0)$
6) $a^0 = 1, (a \neq 0)$
7) $a^{-m} = \frac{1}{a^m}, (a \neq 0)$

   Theorem

   If $\frac{p}{m} = \frac{p'}{m'}$, then $a^{\frac{p}{m}} = a^{\frac{p'}{m'}}$
8) $a^{\frac{m}{n}} = a^{\frac{mk}{nk}} = a^{\frac{m\div k}{n\div k}}$

The equation of second degree with one unknown

The general form is $ax^2 + bx + c = 0;\ (a \neq 0)$

How to find its roots:

$$ax^2+bx+c=0\,|\cdot 4a;\ (a\neq 0)\Rightarrow 4a^2x^2+4abx+4ac=0\Leftrightarrow$$

$$\Leftrightarrow 4a^2x^2+4abx+b^2-b^2+4ac=0\Leftrightarrow (2ax+b)^2-\left(\sqrt{b^2-4ac}\right)^2=0$$

$$\Rightarrow\left(2ax+b+\sqrt{b^2-4ac}\right)\left(2ax+b-\sqrt{b^2-4ac}\right)=0$$

Then the solution will be $x_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

a) In the case in which the coefficient of x is even $b=2b'$ then $x_{1,2}=\dfrac{-b'\pm\sqrt{b'^2-ac}}{a}$

b) In the case in which the coefficient of x is even $b=2b'$ and the coefficient of x^2 is $a=1$, and then $x_{1,2}=-b'\pm\sqrt{b'^2-c}$

Reduced forms of the equation of the second degree

1) In the case $c=0$; $ax^2+bx=0\ \Rightarrow x(ax+b)=0\Rightarrow x_1=0;\ x_2=-\dfrac{b}{a}$

2) In the case $b=0$; $ax^2+c=0\Rightarrow ax^2=-c\Rightarrow x^2=-\dfrac{c}{a}\Rightarrow x_{1,2}=\pm\sqrt{-\dfrac{c}{a}}$

Relations between roots and coefficients

Viète's relation

$$\begin{cases} S=x_1+x_2=-\dfrac{b}{a} \\ P=x_1\cdot x_2=+\dfrac{c}{a} \end{cases}$$

$\Delta=b^2-4ac$:

- If $\Delta<0\Rightarrow$ we have imaginary and conjugated solutions
- If $\Delta=0\Rightarrow$ the solutions are real numbers and equal.
- If $\Delta>0\Rightarrow$ the solutions are real numbers and different.

Irrational equations

Any equation in which the unknown is under the radical constitutes an irrational equation.

To resolve such an equation we'll use the following implication:

$x=y\Rightarrow x^n=y^n$, but $x^n=y^n\not\Rightarrow x=y$.

Precautionary, after determining the solutions of the equation $x^n=y^n$, we need to verify if $x=y$ and eliminate the extra foreign solutions.

We call foreign solutions those solutions which have been introduced through the process of raising to a power.

We must also put the condition that the radical has sense; the solution found must be amongst the values for which the radical makes sense.

If we make successive raising to a power, then we have to put the conditions that both sides are positive.

$$\sqrt[2k]{E(x)} = F(x) \Leftrightarrow \begin{cases} E(x) = [F(x)]^{2k} \\ E(x) \geq 0 \\ F(x) \geq 0 \end{cases}$$

The set of complex numbers (C)

$z = x + iy;\ i = \sqrt{-1}$

The equality of complex numbers

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 = z_2$ if $x_1 = x_2$ and $y_1 = y_2$

The addition of complex numbers

$z_1 + z_2 = (x_1 + x_2) + i\ (y_1 + y_2)$

The subtraction of complex numbers

$z_1 - z_2 = (x_1 - x_2) + i\ (y_1 - y_2)$

The multiplication of complex numbers

$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 - x_2y_1)$

The division of complex numbers

$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2}$, the amplification with the conjugate

Conjugate complex numbers

$z = x + iy$ and $\overline{z} = x - iy$ with the conditions:

a) $z\overline{z} \in \mathbb{R}$

b) $(z + \overline{z}) \in \mathbb{R}$

Quartic equation

The general form: $ax^4 + bx^2 + c = 0$

Solution: we denote $x^2 = y \Rightarrow x^4 = y^2$

$ay^2 + by + c = 0$

$$y_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x_{1,2,3,4} = \pm\sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The following are the conditions that a quartic equation will have all solutions real:

$$\begin{cases} \Delta' = b^2 - 4ac \geq 0 \\ S = -\dfrac{b}{a} \geq 0 \\ P = +\dfrac{c}{a} \geq 0 \end{cases}$$

The reciprocal equation of third degree

A reciprocal equation is the equation whose coefficients at an equal distance from the middle term are equal.

Example:

$$Ax^5 + Bx^4 + Cx^3 + Cx^2 + Bx + A = 0$$

I) $Ax^3 + Bx^2 + Bx + A = 0$

$$A\left(x^3+1\right) + Bx\left(x+1\right) = A\left(x+1\right)\left(x^2 - x + 1\right) + Bx\left(x+1\right) =$$

$$= \left(x+1\right)\left[Ax^2 - \left(A-B\right)x + A\right] = 0$$

$$\Rightarrow \begin{cases} x_1 = -1 \\ x_{2,3} = \dfrac{\left(A-B\right) \pm \sqrt{-3A^2 - 2AB + B^2}}{2A} \end{cases}$$

II) The reciprocal equation of fourth degree

$$Ax^4 + Bx^3 + Cx^2 + Bx + A = 0 \,|\, \div x^2,\ \left(x \neq 0\right)$$

$$Ax^2 + Bx + C + \frac{B}{x} + \frac{A}{x^2} = 0$$

We'll denote $\left(x + \dfrac{1}{x}\right) = y$, then

$$A\left(y^2 - 2\right) + By + C = 0 \ \Rightarrow Ay^2 + By + \left(-2A + C\right) = 0$$

Systems of equations

The equations of second degree with two unknown

$$\underbrace{ax^2 + bxy + cy^2}_{\substack{terms\ of\ second \\ degree}} + \underbrace{dx + ey}_{\substack{terms\ of\ first \\ degree}} + \underbrace{f}_{\substack{free \\ term}} = 0$$

have an infinity of solutions.

I) This

1) Systems formed of an equation of first degree and an equation of second degree

$$\begin{cases} 2x - 3y = 7 \\ 2x^2 - 3xy + y^2 - 5x + 2y + 4 = 0 \end{cases}$$

Is resolved through the substitution method

2) Systems of equations which can be resolved through the reduction method

$$\begin{cases} x^2 - 5xy + y^2 - 3y + 8 = 0 \,|\, \left(-2\right) \\ 2x^2 - 10xy + 2y + 10 = 0 \end{cases}$$

One equation is multiplied by a factor, the equations are added side by side such that one of the unknowns is reduced.

3) Systems of equations in which each equation has on the left side a homogeneous polynomial of second degree and on the right side a constant.

$$\begin{cases} ax^2+bxy+cy^2=d \\ a'x^2+b'xy+c'y^2=d' \end{cases}$$

In particular cases it is applied the reduction method

In general cases it is applied the substitution method

Example

$$\begin{cases} -4x^2-3xy+y^2=-6 \,|\cdot 2 \\ 2x^2-xy-y^2=-4 \,|\cdot(-3) \end{cases}$$

$$\Rightarrow \begin{cases} -8x^2-6xy+2y^2=-12 \\ -6x^2+3xy+3y^2=12 \end{cases} \Rightarrow \begin{cases} -14x^2-3xy+5y^2=0 \,|\div x^2 \\ 2x^2-xy-y^2=-4 \end{cases} \quad (x\neq 0)$$

Homogeneous equations are the equations in which the terms have the same degree.

$5z^2-3y-14=0$;

$$z_{1,2}=\frac{3\pm\sqrt{9+280}}{10}=\frac{3\pm 17}{10} \Rightarrow \begin{cases} 2 \\ -\frac{7}{5} \end{cases}$$

The system is reduced to

$$\begin{cases} \frac{y}{x}=2 \\ 2x^2-xy-y^2=-4 \end{cases} \quad \text{and} \quad \begin{cases} \frac{y}{x}=-\frac{7}{5} \\ 2x^2-xy-y^2=-4 \end{cases}$$

II) Symmetric systems

A symmetric system is a system formed of symmetric equations

An equation is symmetric in x,y if by substituting x by y and y by x, the equation's form does not change.

Examples

$x+y=a$

$4x^2+10xy+4y^2=b$

$2x+2y+xy=c$

Solution

We use the substitution: $x+y=S$, $xy=P$ and we'll find a system which is easier to solve.

If we obtain the solution (a,b), it is obtained also the solution (b,a)

Example

$$\begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases}$$

We denote $x + y = S$ and $xy = P$ then

$$\begin{cases} S^2 - 2P = 5 \\ P = 2 \end{cases} \Rightarrow S^2 - 4 = 5 \Rightarrow S^2 = 9 \Rightarrow S_{1,2} = \pm 3$$

The solutions are: $\begin{cases} S = 3 \\ P = 2 \end{cases}$ and $\begin{cases} S = -3 \\ P = 2 \end{cases}$

Return now to the original unknowns x, y

$$\begin{cases} x + y = 3 \\ xy = 2 \end{cases} \text{ and } \begin{cases} x + y = -3 \\ xy = 2 \end{cases}$$

$z^2 + 3z + 2 = 0 \Rightarrow z_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$ with solutions 2 and 1

$$z^2 + 3z + 2 = 0 \Rightarrow z_{1,2} = \begin{cases} -2 \\ -1 \end{cases}$$

The solutions are

$$\begin{cases} x = 2 \\ y = 1 \end{cases} \qquad \begin{cases} x = 1 \\ y = 2 \end{cases}$$

$$\begin{cases} x = -2 \\ y = -1 \end{cases} \qquad \begin{cases} x = -1 \\ y = -2 \end{cases}$$

III) Other systems are resolved using various operations

- Various substitutions
- Are added or subtract the equations to facilitate certain reductions
- From a system we derive other systems equivalent to the given one

IV) Irrational systems are the systems formed with irrational equations

Methods:

- First we put the conditions that the radicals are positive (if the radical index is even)
- Then the equation is put at the respective power (it is taken into consideration that $A = B \Rightarrow A^n = B^n$, but not vice versa.
- Verify that foreign solutions are not introduced. All the solutions of the initial system are between the solutions of the implied system through raising to the power, but this is not true for the vice versa situation.

Finding the solution for the equation $z^2 = A$ in the set of complex numbers

Let $z = x + iy$, $A = a + bi$

$$z^2 = A \Rightarrow (x + iy)^2 = a + bi$$

$$(x^2 - y^2) + i(2xy) = a + bi$$

We'll use the identification method

$$\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$$

which is an equivalent system of equations; then we resolve the system

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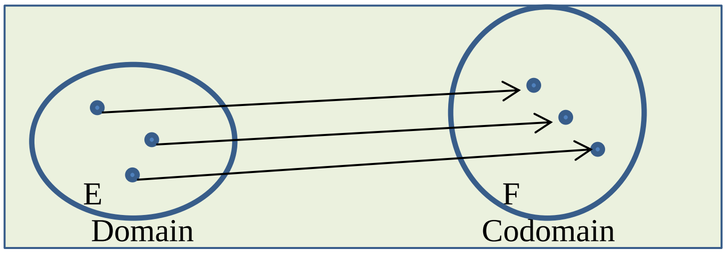
Functions

Given two sets E,F and a relation between the elements of the two sets, such that for any $x\in E$ there exist only one element $y\in F$ in the given relation with x, then we say that it has been defined a function on E with values in F, or an application of the set E in the set F.

- The domain of definition (E) is the set of all the values of x. The arbitrary element x is called the argument of the function
- The set in which the function takes values (F)

$x\to f(x)=y$ (y is the image of x through the function f

$E\xrightarrow{f}F$ or $f:E\to F$



If E or F are finite sets, the function is defined by indicating the correspondence for each element.

$$\begin{array}{l} E=\{a,\ b,\ \ c,\ d\} \\ \quad\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ F=\{3,\ 1,\ \ 2,\ \ 4\ \} \end{array}$$

The functions can be:

- Explicit
- Implicit

The function can be classified as:

- Surjective
- Injective (one-to-one function – bijection)
  Bijective

1) Surjective application (or simple surjection) is a function in which the domain coincides with the codomain $f(E)=F$ for $\forall y\in F, \exists x\in E: y=f(x)$. Any element from F is the image of an element in E.
2) Injective application or bi-univocal is the function which makes that to pairs of different elements to correspond different values. $f(x_1)=f(x_2)\Rightarrow x_1=x_2$
3) Bijective application (bijection) is the function which is injective and surjective in the same time.

Equal functions

The functions f,g, which have the same argument or different arguments are equal: $f=g$ only the following conditions are satisfied:

a) The functions have the same domain (E)
b) Have values on the same codomain (F)

c) $f(a)=g(a)$

Inverse functions

If $f : E \to F$, the function $f^{-1} : F \to E$ is called the invers function of f.

f^{-1} exists when $f(x)$ is a bijection. Its domain is the set which is a codomain for $f(x)$, and its codomain is the domain for $f(x)$.

Computation of a function inverse

$$f(x)=y=\frac{3x+7}{2};\ f^{-1}(x)=?$$

$$y=\frac{3x+7}{2}\Rightarrow x=\frac{2x-7}{3}$$

$$f^{-1}(x)=\frac{2y-7}{3}$$

The method of complete induction

Induction is defined as the process of going from the particular to general.

If a proposition $P(n)$, where $n\in\mathbb{N}$, satisfies the following conditions:

1) $P(a)$ is true, $(a\in\mathbb{N})$
2) $P(n)=P(n+1)$ for any $n\geq a$, in other words: if we suppose that $P(n)$ is true, it results that $P(n+1)$ is true for any $n\geq a$, then $P(n)$ is true for any natural number $n\geq a$.

The number of subsets of a set with n elements is 2^n

Sums

$$S_1=\sum_{1}^{n}k=\frac{n(n+1)}{2}$$

$$S_2=\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_3=\sum_{k=1}^{n}k^3=\left[\frac{n(n+1)}{2}\right]^2=(S_1)^2$$

$$S_4=\sum_{k=1}^{n}k^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

To compute $\sum_{k=1}^{n}k^4$ we start from $(n+1)^5$

$$2^5=(1+1)^5=1^5+5\cdot1^4+10\cdot1^3+10\cdot1^2+5\cdot1^1+1$$

$$3^5=(2+1)^5=2^5+5\cdot2^4+10\cdot2^3+10\cdot2^2+5\cdot2^1+1$$

$$4^5=(3+1)^5=3^5+5\cdot 3^4+10\cdot 3^3+10\cdot 3^2+5\cdot 3^1+1$$

....

$$n^5=\left[(n-1)+1\right]^5=(n-1)^5+5\cdot(n-1)^4+10\cdot(n-1)^3+10\cdot(n-1)^2+5(n-1)^1+1$$

$$(n+1)^5=\left[(n)+1\right]^5=n^5+5\cdot n^4+10\cdot n^3+10\cdot n^2+5\cdot n^1+1$$

$$\sum_{k=1}^{n}k^5+(n+1)^5=\sum_{k=1}^{n}k^5+5\sum_{k=1}^{n}k^4+10S_3+10S_2+5S_1+(n+1)$$

$$(n+1)^5=5S_4+10\frac{n^2(n+1)^2}{4}+10\frac{n(n+1)(2n+1)}{6}+5\frac{n(n+1)}{2}+(n+1)$$

$$(n+1)^5=5S_4+\frac{30n^2(n+1)^2+20n(n+1)(2n+1)+30n(n+1)+12(n+1)}{12}$$

Then

$$S_4=\frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}$$

Properties of the sums

1) $\sum_{1}^{n}ak=a\sum_{1}^{n}k$; $a=const$

2) $\sum_{1}^{n}a=na$; $a=const$

3) $\sum_{1}^{n}(k)(k+1)=\sum_{1}^{n}\left(k^2+k\right)=\sum_{1}^{n}k^2+\sum_{1}^{n}k$

4) $\sum_{k=1}^{n}\frac{A}{k(k+1)}=\sum_{k=1}^{n}\frac{A_1}{k}+\sum_{k=1}^{n}\frac{A_2}{k+1}$

   It has been decomposed in simple fractions.

Combinatory analysis

1) Permutation of n elements is the number of bi-univocal applications of a set of n elements on itself.

   $P(n)=1\cdot 2\cdot 3\cdots n=n!$

   $n!$ is read “factorial of n.

   $(n+1)!=n!(n+1)$

   $(n-1)!=\frac{n!}{n}$

   $0!=1$, (by definition)

2) Arrangements of n elements taken in groups of m elements ($n\geq m$) is the number of applications bi-univocal of a set of m elements in a set of n elements is

$\underbrace{n(n-1)(n-2)\cdots(n-m+1)}_{m\ factors}$; the elements differ by their nature and their position

$$A_n^m = A_n^{m-1}(n-m+1)$$

3) Combinations of n elements grouped by m elements $n \geq m$ are subsets of m elements formed with elements of a set of n elements

$C_n^m = \dfrac{A_n^m}{P_m}$; n is called inferior index and m is called superior index.

$$C_n^m = \frac{n!}{m!(n-m)!}$$

Complementary combinations

$$C_n^m = C_n^{n-m}$$
$$C_n^0 = C_n^n = 1$$
$$C_n^1 = C_n^{n-1} = n$$
$$C_n^{m+1} = C_n^m \cdot \frac{n-m}{m+1}$$
$$C_n^{m+1} + C_n^m = C_{n+1}^{m+1}$$

Sums

1) $C_n^1 + C_n^2 + ... + C_n^{n-1} = 2^n - 2$
2) $C_n^0 + C_n^2 + C_n^4 + ... = C_n^1 + C_n^3 + ... = 2^{n-1}$
3) $2^n - C_n^1 2^{n-1} + C_n^2 2^{n-2} - ... + (-1)^n C_n^n = 1$
4) $C_n^0 - C_n^1 + C_n^2 - ... + (-1)^n C_n^n = 0$
5) $C_n^{m+p} + C_p^1 C_n^{m+p-1} + C_p^2 C_n^{m+p-2} + ... + C_n^m = C_{n+p}^{m+p}$

The binomial theorem (Newton)

$$(x+a)^n = C_n^0 x^n a^0 + C_n^1 x^{n-1} a^1 + C_n^2 x^{n-2} a^2 + ... + C_n^k x^{n-k} a^k + ... + C_n^{n-1} x a^{n-1} + C_n^n x^0 a^n$$

The binomial coefficients: $C_n^0,\ C_n^1,\ C_n^2, \ldots, C_n^n$

The general term is: $T_{k+1} = C_n^k x^{n-k} a^k$; $(k = 0,1,...,n)$

The left side of the binomial formula (Newton) is an homogeneous polynomial in relation to x and a (and of n degree).

The binomial coefficients at the extremities or at an equal distance of the two extremities are equal (reciprocal polynomial).

Real functions of real argument

A function of real argument is a function whose domain is the set of real numbers

A real function is that for which the codomain is a set of real numbers.

The graph of a function of real argument $f : E \to F$ is the set of points $M\left(x, f\left(x\right)\right)$

The intersection of the graph with the axes

a) Intersection with axis $Ox : y = 0 \Rightarrow x = a$
b) Intersection with axis $Oy : x = 0 \Rightarrow y = b$

Monotone functions

A real function of real argument $f : E \to F$ is strictly increasing on an interval $I \subset E$, if for any $x_1 \in I$ and $x_2 \in I$ such that $x_1 < x_2$, we have $f\left(x_1\right) < f\left(x_2\right)$

Function of first degree

$$f\left(x\right) = ax + b$$

Any function of first degree is an bi-univocal application of the set of real numbers on the set of real numbers (is a bijective application).

Therefore any function of first degree $f\left(x\right) = ax + b$ has an inverse $f^{-1}\left(x\right) = \frac{1}{a}x - \frac{1}{b}$, and the inverse is a function of first degree.

The function of first degree is strictly monotone on $\mathbb{R}$.

The graphic of a function of the first degree is a line; if $f\left(x\right) = ax; b = 0$ the line passes through the origin, if $f\left(x\right) = ax + b$, the line passes parallel to the line whose equation is $f\left(x\right) = ax$ at the distance b on the axis Oy.

$tag\alpha = a$, α is the angle of the line with the axis Ox.

To construct a line in a Cartesian system we need two points (usually we take the intersections of the line with the axes.

The function of second degree

$$ax^2 + bx + c = a\left(x - x_1\right)\left(x - x_2\right)$$

where x_1, x_2 are the solutions of the equation.

How to write a polynomial of second degree as sum or difference of two squares

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

a) $\Delta > 0$

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{\Delta}}{2a}\right)^2\right]$$

b) $\Delta = 0$

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$$

c) $\Delta < 0$

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{-\Delta}}{2a}\right)^2\right]$$

Extremes

1) If $a>0$ $y=ax^2+bx+c$ admits a minimum

$$x_{min}=-\frac{b}{2a}$$

$$y_{min}=\frac{-\Delta}{4a}=\frac{4ac-b^2}{4a}$$

2) If $a<0$, $y=ax^2+bx+c$ admits a maximum

$$x_{max}=-\frac{b}{2a}$$

$$y_{max}=\frac{-\Delta}{4a}=\frac{4ac-b^2}{4a}$$

Intervals of strict monotony

1) If $a>0$, a function of second degree is strictly decreasing on an interval $\left(-\infty,-\frac{b}{2a}\right)$ and is strictly increasing on an interval $\left(-\frac{b}{2a},+\infty\right)$.
2) If $a<0$, the function of second degree is strictly increasing on the interval $\left(-\infty,-\frac{b}{2a}\right)$ and strictly decreasing on the interval $\left(-\frac{b}{2a},+\infty\right)$.

A function $f:R\to R$ is symmetric relative to a line (D) which passes through a point x_0 on the axis Ox and it is parallel to Oy if $f(x_0+h)=f(x_0-h)$ for any $h\neq 0$.

The function of second degree admits as symmetrical axis a line parallel to Oy and passing through the function's point of extreme.

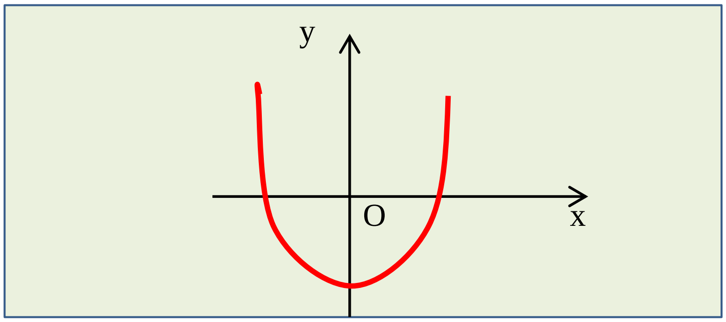
Any value of a function is taken twice: ones in the interval $\left(-\infty,-\frac{b}{2a}\right)$ and the second time on the interval $\left(-\frac{b}{2a},+\infty\right)$, with the exception of the minimum and maximum values which are taken only ones; therefore the function of second degree is not bi-univocal.

The intersection with the axes

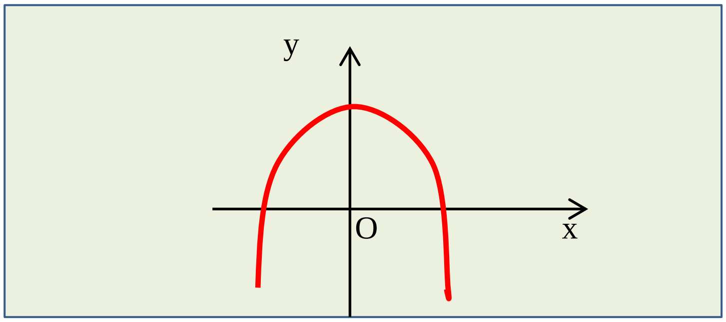
- When $\Delta<0$ then the function does not interest the axis Ox
- When $\Delta=0$ the function intersects the axis $\Delta=0$ in only one point.
- When $\Delta>0$ the function intersects Ox in two distinct points

Graphic – parabola

1) $a > 0$



2) $a < 0$



Parabola is the set of points which are at an equal distance from a fixed point F called focus and from a fixed line (d) called directrix.

$V\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$ the vertex

$F\left(-\frac{b}{2a}, \frac{1-\Delta}{4a}\right)$ the focus

$(d)\ y = \frac{-1-\Delta}{4a}$

The sign of the function of the second degree

a) If $\Delta < 0$, then the function of second degree has the sense of a

b) If $\Delta = 0$, then the function has the sign of a with the exception of the values $x = x_1 = x_2$, where $f(x) = 0$

c) If $\Delta > 0$, then the function has a contrary sense of a between the solutions, and the same sense as a in the exterior of the solutions.

To resolve a system of inequations we intersect the solutions of each inequation.

The nature of the solutions of an equation of second degree with real coefficients which depend of a real parameter

1) $\Delta > 0 \Rightarrow x_1$ and x_2 are real and $x_1 \neq x_2$

2) $\Delta = 0 \Rightarrow x_1 = x_2 \in \mathbb{R}$

3) $\Delta < 0 \Rightarrow x_1 \neq x_2$ are solution imaginary conjugated

The discussion of an equation of second degree with real coefficients dependent of a real parameter

$$S = \frac{b}{a}$$

$$P = \frac{c}{a}$$

$$\Delta = \frac{b^2 - 4ac}{4a}$$

It is studied their signs and are taken all the intervals, as well as the points which delimit the intervals.

Example

Provide the analysis of the following equation

$$x^2 - 2(\lambda - 1)x + 4\lambda - 4 = 0$$

$$S = 2(\lambda - 1)$$

λ	-∞	1	+∞
S	- - - - - - - - - - - - - - - - - - -	0	+ +

$$P = 4(\lambda - 1)$$

λ	-∞	1	+∞
P	- - - - - - - - - - - - - - - - - - -	0	+ +

$$\Delta = \lambda^2 - 2\lambda + 1 - 4\lambda + 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$$

λ	-∞	1		5	+∞
Δ	+ + + + + + + + + + ++ + + + +	0	- - - - - - - - -	0	+ + + + + + + + + + + +

Interval	Δ	S	P	Conclusions
$\lambda \in (-\infty, 1)$	+	-	-	The real solutions $x_1 < 0, x_2 > 0$; $\lvert x_1 \rvert > x_2$
$\lambda = 1$	0	0	0	The real equal solutions $x_1 = x_2 = 0$
$\lambda \in (1, 5)$	-	+	+	The imaginary solutions conjugated
$\lambda = 5$	0	+	+	The real equal solutions $x_1 = x_2 = 4$
$\lambda \in (5, +\infty)$	+	+	+	The real different solutions $x_1 > 0, x_2 > 0$

Exponential functions

Properties

1) $f(x_1) \cdot f(x_2) = f(x_1 + x_2)$
2) $\dfrac{f(x_1)}{f(x_2)} = f(x_1 - x_2)$
3) $\left[f(x_1)\right]^c = f(cx_1)$
4) a) By raising a real number sub unitary (respectively higher than one) to a power with a rational positive exponent we'll obtain a sub unitary number (respectively higher than one)

   b)By raising a real number sub unitary (respectively higher than one) to a power with a rational negative we'll obtain a number higher than one (respectively sub unitary).

Exponential function

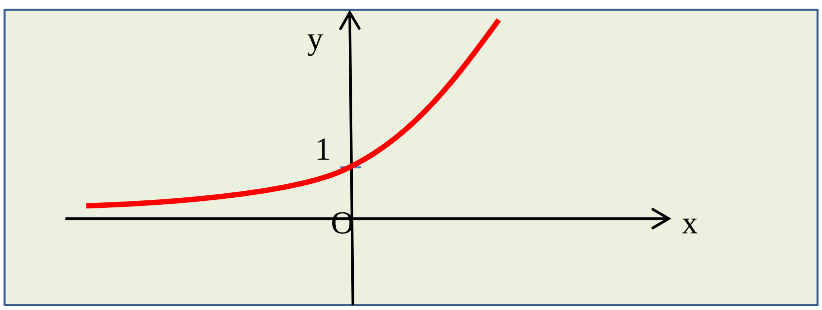
$f(x) = a^x$, where $a > 0$ and $a \neq 1$

$f: R \to R_+$

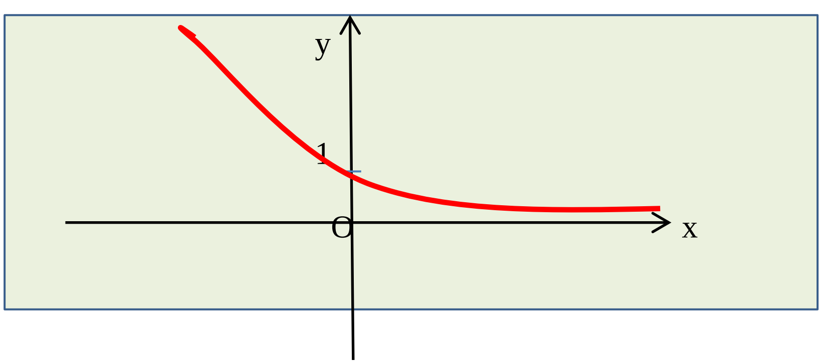
The exponential function is bijective

The monotony

a) If $a > 1$ then the function is strictly increasing on the whole domain



b) If $a < 1$ then the function is strictly decreasing on the whole domain



Deposits of money to the bank

$$f\left(x\right)=C\left(\frac{105}{100}\right)^{x}$$

C is the initial amount
x is the number of years

$f\left(x\right)$ represents the your sum of money in the bank after an x number of years from the initial deposit.

Exponential equations

1) Equations of the form $a^{x}=b$, where $b=a^{r}$; then
   $$a^{x}=a^{r}\Rightarrow x=r$$
   If $b\neq a^{r}$, then $a^{x}=b$,
   $x=log_{a}b$
2) Equations of the form $a^{f(x)}=b$, where $b=a^{r}$; then
   $a^{f(x)}=a^{r}\Rightarrow f\left(x\right)=r$
   If $b\neq a^{r}$, we use logarithms
   $a^{f(x)}=b\Rightarrow f\left(x\right)=log_{a}b$
3) Equations of the form $a^{f(x)}=b^{g(x)}$, where $b=a^{r}$; then
   $a^{f(x)}=a^{rg(x)}\Rightarrow f\left(x\right)=r\left(g\left(x\right)\right)$
   If $b\neq a^{r}$, we use logarithms
   $a^{f(x)}=b^{g(x)}\Rightarrow f\left(x\right)=\left(log_{a}b\right)-g\left(x\right)$
4) Equations of the form $a^{f(x)}=0$
   We are denote $a^{x}=y$ and we'll obtain an equation with the unknown y, easy to determine the solutions.
5) Equations which contain the unknown at the base of the powers as well as at the exponent:
   a) The case when the base $x=1$ (we'll verify in the equation)
   b) The case when the base is positive and different of 1.

Logarithms

The logarithm of a number real positive is the exponent of the power to which we must raise the base to obtain that number.

$$log_{a}A=x\Leftrightarrow a^{x}=A$$

The conditions for logarithm's existence

1) $log_{a}1=0$
2) $log_{a}a=1$

3) $log_a a^c = c$

4) $log_a AB = log_a A + log_a B$

5) $log_a \frac{A}{B} = log_a A - log_a B$

6) $log_a A^m = m\,log_a A$

7) $log_a \sqrt[n]{A} = \frac{1}{n} log_a A$

The collapse (grouping) of an expression containing algorithms

1) $log_a A + log_a B = log_a AB$

2) $log_a A log_a B = log_a \frac{A}{B}$

3) $log_a A = log_a A^c$

4) $k = log_a a^k$

Formulae for changing the base of a logarithm

1) $log_a A = \frac{1}{log_A a} \Rightarrow log_a A log_A a = 1$

2) $log_a A = log_{a^n} A^n$

3) $log_a A = \frac{log_b A}{log_b a}$

$a^{log_a b} = b$

The logarithmic function

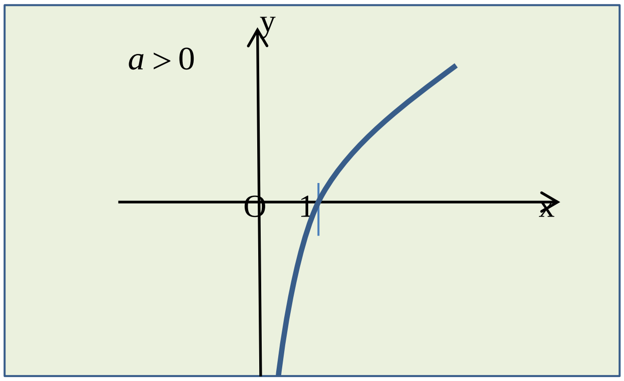
The logarithmic function is the invers of the exponential function (therefore these are symmetric in relation to the first bisector).

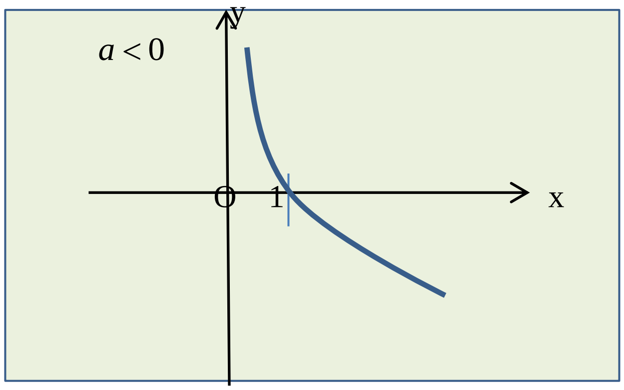
$f(x) = log_a x$

$f : R_+ \to R$

The function is bijective on R_+ if:

- $a > 0$, the function is strictly increasing
- $a < 0$, the function is strictly decreasing





Properties of the logarithmic functions

1) $f\left(x_1 \cdot x_2\right) = f\left(x_1\right) + f\left(x_2\right)$
2) $f\left(\frac{x_1}{x_2}\right) = f\left(x_1\right) - f\left(x_2\right)$
3) $f\left(x_1^{\,c}\right) = cf\left(x_1\right)$

The sign of a logarithm

1) $log_a b > 0$ if
   - $a > 1, b > 1$ or
   - $a < 1, b < 1$
2) $log_a b < 0$ if
   - $a > 1, b < 1$
   - $a < 1, b > 1$

Natural logarithms

The natural logarithms are defined as being the logarithms whose base is e=2.71828… (irrational number). The natural logarithms have been introduced by the mathematician Neper, and that's why they are called the neperieni logarithms $ln(A)$

Decimal logarithms

Decimal logarithms are called the logarithms whose base is 10: $lg(A)$. The decimal logarithms have been computed by Brigss. To use them one looks them up in the Mathematical Tables.

The logarithm of a number is formed by:

a) The characteristic, which is the whole part [it is equal to $m-1$, where m represents the number of digits of the given number, eliminating the digits that follow after the decimal point, in the case that the number is above unity].
   - If the number is <1 then its characteristic is negative and equal to the number of zeroes placed in front of the first digit different of zero including also the zero from the whole part. 45.042 = $1+\varepsilon_1$;

     $0.00345=-3+\varepsilon_2=\overline{3}+\varepsilon_2$

b) Mantissa (ε):

   $0<\varepsilon<1$

   Mantissa doesn't change if we multiply or divide a number with the powers of 10.

   $$log_{10} A = n+\varepsilon$$

   Sometimes we use the interpolation n method to compute the mantissa.

Operation with logarithms

Addition

$lg\,1534+lg\,2.23+lg\,0.022$

$$
\begin{array}{r}
+3.18583 \\
0.34830 \\
\overline{2}.34242 \\
\hline
1.87655
\end{array}
$$

Subtraction

$lg\,325-lg\,4116$

$lg\,325-lg\,4116=2.51188-3.61448=2.51188-(3+0.61448)=$

$=2.51188-3-1+1-0.61448=\overline{2}.89740$

Cologarithm

The Cologarithm (colog) of a number is equal to the opposite of the logarithm of the given number.

$co\,log\,A=-lg\,A$

$co\,log\,3126=-lg\,3126=-(3.49799)=-3-1+1-0.49499=\overline{4}.50501$

Therefore:

a) To the logarithm's characteristic we add 1 and change the sign of the given number, this way, we obtain the characteristic of the Cologarithm.

b) To obtain the mantissa of the cologarithm we subtract the last digit of the logarithm's mantissa from 10, and the rest of the digits (going to the left direction) from 9.

Multiplication with whole number

a) $8\cdot lg(38.23)=8\cdot 1.58240=12.65920$

b) $5\cdot lg(0.004798)=5\cdot\overline{3}.68106=5\cdot(-3+0.68106)=-15+3.40530=\overline{12}.40530$

Division with a natural number

a) $\frac{1}{3} lg\,486 = \frac{2.68664}{3} = 0.89554$

b) $lg\,\sqrt[3]{0.5942} = \frac{\bar{1}.77393}{3} = \frac{-1+0.77393}{3} = \frac{-1-2+2+0.77393}{3} = \bar{1}.92464$

Why are the logarithms as useful?

a) We can transform a product in a sum
b) We can transform a ratio in a difference
c) We can transform a power in a multiplication
d) We can transform a radical extraction in a division

Exponential equations which are resolved using logarithms

1) Exponential equations of the form $a^x = b$

$a^x = b \Rightarrow x = lg_a b$ or

$a^x = b \Rightarrow log_{10} a^x = log_{10} b \Rightarrow x = \frac{lg\,b}{log\,a}$

2) Equations of the form $a^{f(x)} = b$

$a^{f(x)} = b \Rightarrow f(x) lg\,a = lg\,b$

$f(x) = \frac{lg\,b}{lg\,a}$

Logarithmic equations

1) Equations of the form $lg_x = A$

$lg_x = A \Rightarrow x = 10^A$

2) Equations of the form $lg\ f(x) = A$

$lg\ f(x) = A \Rightarrow f(x) = 10^A$

3) Equations which contain more logarithms in the same base
   - We group all terms from each side and make an equality from the expressions under the logarithms.
   - To avoid computations with fractions we isolate all terms with the minus sign from the left side in the left side, and the rest in the right side.
   - Other equations are solved through transformations in other bases, and other substitutions.

Exponential and logarithmic systems

For solving these types of systems are used the same procedures as for solving the equations

Conditions imposed for the existence of logarithmic and exponential equations:

a) $log_a b \begin{cases} a > 0 \\ b > 0 \\ a \neq 1 \end{cases}$

b) $a^x \begin{cases} a > 0 \\ a \neq 1 \end{cases}$

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Permutations

A permutation is a bi-univocal application from a set A on itself.
Example:

$$\begin{pmatrix} a_1 a_2 a_3 \\ a_3 a_2 a_1 \end{pmatrix}$$

Inversions

An inversion is a per of natural numbers (i_k, i_l) (of a permutation φ) situated in the second line of

$$\begin{pmatrix} 1\ 2\ \dots\ n \\ i_1 i_2\ \dots\ i_n \end{pmatrix}$$

with the property $k < l$ and $(i_k > i_l)$
Example

$$\begin{pmatrix} 1\ \ 2\ 3 \\ 3\ 2\ 1 \end{pmatrix} \text{ with the inversions}$$

$(3,1)$ and $(3,2)$.

To determine the number of inversions in φ one counts how many numbers are before first digit (1); then 1 is cut off, one counts how many digits are before 2 (digit one being already excluded); then 2 is cut off, etc. until all digits have been cut off. The sum of all the numbers found before each number cut off.

Any permutation is a product of transpositions.

Permutations can be

- Even – if $Inv(\varphi)$ is an even number
- Odd – if $Inv(\varphi)$ is an odd number

The sign of the permutation

$$\varepsilon(\sigma) = \begin{cases} +1 \ if\, \sigma = even \\ -1 \ if\, \sigma = odd \end{cases}$$

Any permutation has an inverse

$$\varepsilon(\sigma) = \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j}; \ \ \varepsilon(\sigma \circ \tau) = \varepsilon(\sigma)\varepsilon(\tau)$$

Theorem

If two permutations are obtained one from another through a change of a position, then these are of different classes.

The permutation σ and σ^{-1} have the same sign.

Change of position - Transposition

A transposition is when in a permutation two elements are swapped between them and the rest remained unchanged.

Determinant of order n

A determinant of order n is the number associated to the sum of n! Products of the form $(-1)^{f(\varphi)} a_{1i_1} a_{2i_2} \dots a_{ni_n}$
that is:

$$det(A) = \sum_{\psi \in P_n} (-1)^{f(\psi)} a_{j_1 1} a_{j_2 2} \dots a_{j_n n}$$

Matrix

A matrix is table of n lines and m columns

$$A = \begin{bmatrix} a_{1,1} a_{1,2} a_{1,3} \dots \ a_{1,m} \\ a_{2,1} a_{2,2} a_{2,3} \dots a_{2,m} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n,1} a_{n,2} a_{n,3} \dots a_{n,m} \end{bmatrix}$$

The numbers $a_{i,j}$ are called the elements of matrix A

The numbers $a_{j,1}, a_{j,2}, a_{j,3}, \dots, a_{j,m}$ form the j line of matrix A

A matrix of type m,n is denoted $A(m,n)$ and it is the matrix with m lines and n columns

A matrix with only one line is

$$A_{1m} = \begin{bmatrix} a_{1,1} a_{1,2} a_{1,3} \dots \ a_{1,m} \end{bmatrix}$$

A matrix with only one column is

$$A_{n1} = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ \dots \\ a_{n,1} \end{bmatrix}$$

Square matrix

A square matrix is the matrix that has an equal number of lines as columns

$$A_{nn} = \begin{bmatrix} a_{1,1} a_{1,2} a_{1,3} \dots \ a_{1,n} \\ a_{2,1} a_{2,2} a_{2,3} \dots a_{2,n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n,1} a_{n,2} a_{n,3} \dots a_{n,n} \end{bmatrix}$$

Principal diagonal

The principal diagonal is made of elements of the form a_{ii} of a square matrix:

$$a_{11}, a_{22}, \dots, a_{nn}$$

The number of even permutations ν_1 and the number of odd permutations ν_2 of a set $\{1,2,...,n\}$ over the same set are in the following relation:

$$\nu_1 = \nu_2 = \frac{1}{2}n!$$

If φ is an identic application $\varphi(h) = h$; $h = 1,2,...,n$, then

$$Inv(\varphi) = 0;\ \begin{pmatrix} 12...n \\ 12...n \end{pmatrix}$$

The maximum number of inversions is:

$$Inv(\varphi) = \frac{n(n-1)}{2}\ \begin{pmatrix} 1 & 2 & ... & n \\ n & n-1 & ... & 1 \end{pmatrix}$$

therefore,

$$0 \le Inv(\varphi) \le \frac{n(n-1)}{2}$$

Singular matrix or degenerate

A singular matrix or degenerated is a matrix whose determinant is equal to zero.

Non-degenerate matrix

Anon-degenerate matrix is a matrix whose determinant is different of zero.

A determinant has n lines and n columns

The computation of a determinant of 3X3 matrix –
Sarrus' rule

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$
$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{matrix}$$

The first two lines are placed under the matrix, then we take with positive (+) sign the products determined by the principal diagonal and others parallel to it, and with negative (-) sign the products of the others.

The rule of triangles

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

The products that are taken with positive sign form triangles with the side parallel to the principal diagonal

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

The products that are taken with negative sign form triangles with the side parallel to the other diagonal.

Determinates' properties

The minor of element a_{ij} is a $n-1$ determinant which is obtained by suppressing in a determinant line i and column j $\left(\Delta_{ij}\right)$

The algebraic complement of the element a_{ij} is equal to $(-1)^{i+j}$ $\times$minor of the element: $(-1)^{i+j} \times \Delta_{ij}$.

1) If in a determinant the lines are swapped with the columns, the result is a determinant equal to the given determinant.
2) In in a determinant we swap two lines (or two columns) we obtain a determinant equal in absolute value to the given determinant but of opposite sign.
3) If in a determinant Δ two columns or two lines are equal, then the determinant $\Delta = 0$
4) To compute a determinant we use $\Delta = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \Delta_{ij}$
5) If we multiply a line (or a column) of a determinant with a number, we obtain a determinant equal to the product between the initial determinant and the number.
6) If two lines (or two columns) of a determinant are proportional, then $\Delta = 0$
7) If in a determinant a line (or column) has all elements expressed as a sum of two terms, then this determinant can be written as a sum of two determinants in which the rest of the lines remain unchanged, and line k is expressed as a sum of two terms.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots \\ a'_{k_1}+a''_{k_1} & a'_{k_2}+a''_{k_2} & \dots & a'_{k_n}+a''_{k_n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots \\ a'_{k_1} & a'_{k_2} & \dots & a'_{k_n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} + \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots \\ a''_{k_1} & a''_{k_2} & \dots & a''_{k_n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix}$$

8) If to a line (or a column)of a determinant we add another line (respectively column) multiplied by a number (in particular ± 1), we obtain a determinant equal to the given determinant.

If to a line or column we add the sum of the product of other lines with a number (or column) we obtain a determinant equal to the given determinant.

The sum of the product of a line (or column) with numbers is called a linear combination of the respective lines or columns.

The theorem (property) 8 is used to obtain some elements equal to zero on a certain line (or column) which helps to calculate the determinants easily.

Vandermonde determinant of order 3

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

This determinant is equal to the product of the differences of these numbers taken in pairs through circular permutations.

Generalization

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ a_1^3 & a_2^3 & \dots & a_n^3 \\ \dots\dots & & & \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \text{product of all the different } a_i - a_j\text{; } 1 \le j < i \le n\text{;}$$

$C_n^2 =$ number of factors of the product of differences.

Cramer's rule

This rule is applied to resolve systems of n equations with n unknowns.

If $\Delta \neq 0$, then the system is compatible determined and has unique solution:

$$\left(\frac{\Delta x_1}{\Delta}, \frac{\Delta x_2}{\Delta}, \dots, \frac{\Delta x_n}{\Delta}\right)$$

The Kronecker symbol

$$\delta : E \to \{0,1\}$$

$$\delta((p,q)) = \begin{cases} 1 \text{ if } p = q \\ 1 \text{ if } p \neq q \end{cases}$$

In a double sum the summation order can be changed

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n}\lambda_{ij}$$

Triangular determinant

A triangular determinant is a determinant whose elements above (or under) the principal diagonal are null. This determinant is equal to the product of the elements on the principal diagonal.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} = a_{1,1} \cdot a_{2,2} \cdot \dots \cdot a_{n,n}$$

Example: Solve the following system using Cramer's rule

$$\begin{cases} 2x + 3y + 4x = 16 \\ 5x - 8y + 2z = 1 \\ 3x - y - 2z = 5 \end{cases}$$

$$\Delta_s = \begin{vmatrix} 2 & 3 & 4 \\ 5 & -8 & 2 \\ 3 & -1 & -2 \end{vmatrix} = 32 + 18 - 20 + 96 + 4 + 30 = 160 \neq 0$$

Therefore, we can apply the Cramer's rule

$$\Delta_x = \begin{vmatrix} 16 & 3 & 4 \\ 1 & -8 & 2 \\ 5 & -1 & -2 \end{vmatrix} = 256 + 30 - 4 + 160 + 32 = 480$$

$$\Delta_y = \begin{vmatrix} 2 & 16 & 4 \\ 5 & 1 & 2 \\ 3 & 5 & -2 \end{vmatrix} = 320$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 16 \\ 5 & -8 & 1 \\ 3 & -1 & 5 \end{vmatrix} = 160$$

$$x = \frac{\Delta_x}{\Delta_s} = \frac{480}{160} = 3$$

$$y = \frac{\Delta_y}{\Delta_s} = \frac{320}{160} = 2$$

$$z = \frac{\Delta_z}{\Delta_s} = \frac{160}{160} = 1$$

Therefore, the solution is $(3,2,1)$

Δ_{x_n} is obtained by substituting in Δ_s (the system's determinant) the column of the coefficients of x_n with the column of the free (who don't contain unknown) coefficients

Operations with matrices

Addition

The matrices have to be of the same type and we add the corresponding elements.

$$\begin{vmatrix} a'_{1,1} & a'_{1,2} & \dots & a'_{1,n} \\ a'_{2,1} & a'_{2,2} & \dots & a'_{2,n} \\ \dots & \dots & \dots & \dots \\ a'_{m,1} & a'_{m,2} & \dots & a'_{m,n} \end{vmatrix} + \begin{vmatrix} a''_{1,1} & a''_{1,2} & \dots & a''_{1,n} \\ a''_{2,1} & a''_{2,2} & \dots & a''_{2,n} \\ \dots & \dots & \dots & \dots \\ a''_{m,1} & a''_{m,2} & \dots & a''_{m,n} \end{vmatrix} = \begin{vmatrix} a'_{1,1}+a''_{1,1} & a'_{1,2}+a''_{1,2} & \dots & a'_{1,n}+a''_{1,n} \\ a'_{2,1}+a''_{2,1} & a'_{2,2}+a''_{2,2} & \dots & a'_{2,n}+a''_{2,n} \\ \dots & \dots & \dots & \dots \\ a'_{m,1}+a''_{m,1} & a'_{m,2}+a''_{m,2} & \dots & a'_{m,n}+a''_{m,n} \end{vmatrix}$$

Theorem 1

The set $\mathcal{M}_{n,m}$ is a commutative group in relation to the addition in $\mathcal{M}_{n,m}$.

1) $A+(B+C)=(A+B)+C$ associative
2) $A+B=B+A$ commutative
3) $A+\mathcal{O}_{n,m}=A$ neutral element
4) $A+(-A)=\mathcal{O}_{n,m}$ symmetric element
5) The addition is defined

Subtraction

The matrices have to be of the same type and we subtract the corresponding elements.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} - \begin{vmatrix} a_1' & b_1' & c_1' \\ a_2' & b_2' & c_2' \end{vmatrix} = \begin{vmatrix} a_1-a_1' & b_1-b_1' & c_1-c_1' \\ a_2-a_2' & b_2-b_2' & c_2-c_2' \end{vmatrix}$$

Multiplication

a) Multiplication of a matrix with a number

$$\lambda \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} \lambda a_1 & \lambda b_1 & \lambda c_1 & \lambda d_1 \\ \lambda a_2 & \lambda b_2 & \lambda c_2 & \lambda d_2 \\ \lambda a_3 & \lambda b_3 & \lambda c_3 & \lambda d_3 \end{vmatrix}$$

The number multiplies each elements of the matrix, and we obtain a matrix of the same type.

b) Multiplication of two matrices

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{vmatrix} \cdot \begin{vmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,p} \\ b_{2,1} & b_{2,2} & \dots & b_{2,p} \\ \dots & \dots & \dots & \dots \\ b_{m,1} & b_{m,2} & \dots & b_{m,p} \end{vmatrix}$$

To obtain an element $a_{i,j}$ of the product $A_{n,m} \cdot B_{m,p} = C_{n,p}$ we will multiply the line i of the matrix A with the column j of the matrix B (the corresponding element are multiplied and then we perform the sum of the products).

The matrices which can be multiplied cannot be of any type. The condition is:

$A_{n,m} \cdot B_{m,p} = C_{n,p}$

The square matrices can be multiplied between them.

Theorem 2

The set $\mathcal{M}_{n,n}$ is an ring (non-commutative) with unitary element.

1) $\mathcal{M}_{n,n}$ is commutative group in relation to addition
2) The multiplication in $\mathcal{M}_{n,n}$ is associative $A\cdot(B\cdot C)=(A\cdot B)\cdot C$
3) The multiplication in $\mathcal{M}_{n,n}$ is distributive at the left and the right in relation with the addition in $\mathcal{M}_{n,n}$

$$(A+B)C=AC+BC$$

$$A(B+C)=AB+AC$$

4) The matrix unity U_n is a neutral element in $\mathcal{M}_{n,n}$

$$AU_n=U_nA=A \quad (\forall A\in\mathcal{M}_{n,n})$$

5) $AB\neq BA$ (in general)

The determinant of the product of two square matrices (of the same order) is equal to the product of the determinants of the respective matrices.

$$det(AB)=det(A)det(B)$$

If at least one of the matrices is singular, then the product $A\cdot B$ is singular; if the matrices A,B are non-singular. Then their product is a non-singular matrix.

Transposed matrix

A transpose matrix of another matrix A is the matrix obtained by changing the lines in columns (A^t)

The reciprocal matrix is the matrix obtained by substituting in A^t to each element its algebraic complement $det(A^t)$, denoted A^*.

A square matrix has an inverse only if it is non-singular.

The invers of a matrix

$$A^{-1}=\frac{1}{det(A)}A^*$$

$$AA^{-1}=U_n$$

Equations with matrices

1) Multiply with A^{-1} at left

$$X=B\,|\cdot A^{-1}$$

$$A^{-1}AX=A^{-1}B$$

$$U_nX=A^{-1}B$$

$$X=A^{-1}B$$

2) Multiply with A^{-1} at right

$$AX = B \mid \cdot A^{-1}$$
$$AXA^{-1} = BA^{-1}$$
$$XU_n = BA^{-1}$$
$$X = BA^{-1}$$

Resolving a system of equations using matrices

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \dots \ a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \dots \ a_{2,m}x_m = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 \dots \ a_{n,m}x_m = b_n \end{cases}$$

$$\begin{Vmatrix} a_{1,1}a_{1,2}a_{1,3} \dots \ a_{1,m} \\ a_{2,1}a_{2,2}a_{2,3} \dots \ a_{2,m} \\ \dots\dots\dots\dots\dots\dots \\ a_{n,1}a_{n,2}a_{n,3} \dots \ a_{n,m} \end{Vmatrix} \times \begin{Vmatrix} x_1 \\ x_2 \\ . \\ x_m \end{Vmatrix} = \begin{Vmatrix} b_1 \\ b_2 \\ . \\ b_m \end{Vmatrix}$$

The rang of a matrix

Theorem 1

If $A \neq O_{n,m}$ then there is a natural number r unique determined, $1 \leq r \leq min(n,m)$ such that at least one determinant of order r of A is not null, and in the case of $r < min(n,m)$, any determinant of order $r+q$, $q = 1,2,...,min(n,m) - r$ of A is null. (r is the rang of the matrix).

Or

The rang of a matrix is the order of the largest determinant different of zero, which is formed with the elements of A, $r \leq min(n,m)$.

Theorem 2

If all determinants of order p, $p < min(n,m)$ of A are null, then all determinants of order $p+q$ of A, $q = 1,2,...,min(n,m) - p$ are null.

$Pr.$ $r(A \cdot B) \leq min\{r(A), r(B)\}$

Systems of n linear equations with m unknown

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \dots \ a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \dots \ a_{2,m}x_m = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 \dots \ a_{n,m}x_m = b_n \end{cases}$$

The matrix of the system $S = \|a_{ij}\|$, where: $1 \leq i \leq n$, $1 \leq j \leq m$.

The rang of the system S rang (S), is the rang of matrix's S.

The complete matrix of the system S is the matrix which is obtained by adding the column of the free terms to matrix of the system.

$$\begin{Vmatrix} a_{1,1}a_{1,2}a_{1,3}\dots\ a_{1,m}b_1 \\ a_{2,1}a_{2,2}a_{2,3}\dots a_{2,m}b_2 \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{n,1}a_{n,2}a_{n,3}\dots a_{n,m}b_m \end{Vmatrix}$$

Determinant principal of a system is the largest determinant different of zero $r \le min(n,m)$ of the system's matrix.

The principal unknown: $x_1, x_2, ..., x_r$

The secondary unknown: $x_{r+1}, x_{r+2}, ..., x_m$

The principal equations are the equations in the principal determinant

The secondary equations are those which are not in the principal determinant.

The determinant obtained by bordering the determinant $|\alpha_{ij}|$ has an order greater by one than the initial determinant to which we add a line and a column.

The characteristic determinant is obtained by adding to the principal determinant a line and the column of the free terms.

Rouché's theorem

The necessary and sufficient condition that the system S of linear equations (with $r < m$) to be compatible is that all characteristic determinants of the system to be null.

Theorem

The necessary and sufficient condition that all characteristic determinants of a system to be null is that the rank of the system to be equal to the rank of the complete matrix of S.

Kronecker-Cappelli's theorem

The necessary and sufficient condition that the system S to be compatible is that its rang to be equal to the rang of the complete matrix of S.

The necessary and sufficient condition that a system of homogeneous equations to have also other solutions besides the banal one is that the system's rang to be smaller (strictly) than the number of unknowns.

The discussion of a system of n equations with n unknown (non-homogeneous).

1) $n = m$
   a) If $r = n \Rightarrow$ system compatible determined (we'll apply the Cramer's rule)

   b) If $r < n$, we'll apply the Rouché's theorem
      If all $\Delta_{car} = 0 \Rightarrow$ system compatible undetermined
      If at least one $\Delta_{car} \ne 0 \Rightarrow$ system incompatible
2) $n < m$
   a) $r = n \Rightarrow$ system compatible undetermined of the order $m - n$ (apply the Cramer's rule)

b) If $r < n$, we'll apply the Rouché's theorem
Id all $\Delta_{car} = 0 \Rightarrow$ the system is compatible undetermined of the order $m - r$
If at least one $\Delta_{car} \neq 0 \Rightarrow$ the system is incompatible

3) $n > m$
   a) If $r = m \Rightarrow$ we'll apply the Rouché's theorem.
   b) If $r < m \Rightarrow$ we'll apply the Rouché's theorem

Homogeneous systems

(The free term of each equation is equal to zero)
All homogeneous systems are compatible, having at list the banal solution (0,0,…,0).

1) $n = m$
   a) If $r = n \Rightarrow$ The system has only the banal solution
   b) $r < n \Rightarrow$ The system is undetermined
2) $n < m$
   a) If $r = n \Rightarrow$ The system is undetermined of order $m - n$
   b) $r < n \Rightarrow$ The system is undetermined of order $m - r$
3) $n > m$
   a) If $r = m \Rightarrow$ The system has only the banal solution
   b) If $r < m \Rightarrow$ The is undetermined of order $n - r$.

ALGEBRA GRADE 12TH

Problems

1) Show that the equation $2x^4-8x^3+13x^2+mx+n=0$, where $m,n\in\mathbb{R}$, cannot have all the solution real.

$$\left(x_1-x_2\right)^2+\left(x_1-x_3\right)^2+\left(x_2-x_3\right)^2+\left(x_2-x_4\right)^2+\left(x_3-x_4\right)^2=$$
$$=3\left(x_1^2+x_2^2+x_3^2+x_4^2\right)-2\left(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4\right)=$$
$$=3S_1^2-8S_2=3\cdot 4^2-8\cdot\frac{13}{2}=-4<0$$

Therefore the equation cannot have all the solution real.

2) Show that the equation with real coefficients $2x^4-4x^3+3x^2+ax+b=0$ has all the solution real, then all its solutions are equal.

$$\left(x_1-x_2\right)^2+\left(x_1-x_3\right)^2+...+\left(x_3-x_4\right)^2= \text{ (there are } C_4^2 \text{ parentheses)}$$
$$=3\left(x_1^2+x_2^2+x_3^2+x_4^2\right)-2\left(x_1x_2+x_1x_3+...+x_3x_4\right)=3S_1^2-8S_2=0$$

Therefore, $x_1=x_2=x_3=x_4=\frac{2}{4}=\frac{1}{2}$

3) The condition for $E\equiv az^3+bz+c=0$ and $E_1\equiv a'z^3+b'z+c'=0$ to have at least a common solution, we must have

$$aE_1-a'E=\left(ab'-a'b\right)z+\left(ac'-a'c\right)=0$$

The common solution being $z_1=\dfrac{a'c-ac'}{ab'-a'b}$

4) The primitive solution of an binomial equation (equation with two terms) is the solution which raised to a power 0,1,2,..., $(n-1)$ we'll obtain the n solutions of the given equation.
The primitive solutions are those in which k (from the complex numbers relation) takes prime values with n. The number of the solutions is given by the formula $\varphi(n)$ (Euler's function)

Polynomials

Polynomials with one variable

$$P(x)=2x^4+5x^3-7x+10$$

Polynomials with two variables

$$P(x,y)=x^2+y^2-3xy+x+y-6$$

Polynomials with three variables

$$P(x,y,z)=2x^2+3xyz+x-z+6$$

Definition

A polynomial is a sequence of operations (powers, additions, multiplications) which are computed between the values of the variables (unknowns) and with the coefficients.

The grade of a polynomial in relation to a variable is the exponent of the highest power of the variable:

Example:

$P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n$ is of grade n, with the essential condition that $a_n \neq 0$; a_0 is called the free term.

The null polynomial

$P^* = 0z^n + 0z^{n-1} + ... + 0z + 0$

In this case there is no grade.

The grade of the sum of two polynomials

$Q + P = S$

$gr.S \leq max(gr.P, gr.Q)$

The grade of the difference of two polynomials

$Q - P = D$

$gr.D \leq max(gr.Q, gr.P)$

The grade of the product of two polynomials

$A \cdot B = C$

$gr.C = gr.A + gr.B$

The grade of the ratio of two polynomials

$\frac{M}{N} = A$, and R

$gr.A = gr.M - gr.N$; $gr.R < gr.N$

Polynomial function

A polynomial function is a function $P : M \rightarrow M$, where M is one of the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, such that $P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n$ and $a_1, a_2, ..., a_n \in M$

The difference between a polynomial and a polynomial function is that the polynomial function is characterized by:

- The domain of the definition E
- The set in which the function takes values F
- A rule through which to any element x from E it associates an element unique $P(x) \in F$

The polynomial is characterized only by the third element.

Quantifications

- Universals $\forall$ (for any)
- Existential $\exists$ (exists, at least one)

Identical polynomials

Identical polynomials are two polynomials which define the same function.
Two equal polynomials are identical.

Polynomial identic null

Polynomial identic null is a polynomial which is nullified for any complex value of the variable.

A polynomial $P(z)$ is identical null if, and only if all its coefficients are null $P(z)=P^*$

If a polynomial $P(z)$ of grade n is nullified for $n+1$ values of z then it is identically null.

Two polynomials

$$P(z)=a_0z^n+a_1z^{n-1}+...+a_{n-1}z+a_n$$

$$Q(z)=b_0z^n+b_1z^{n-1}+...+b_{n-1}z+b_n$$

are identical if and only if their corresponding coefficients are equal:

$$a_0=b_0,\ a_1=b_1,...,\ a_n=b_n$$

The method of the undetermined coefficients

$$x^3+2x^2+1\equiv ax^4+bx^3+(b+c)x^2dx+e\Rightarrow\{a=0;b=1;b+c=2\Rightarrow c=1;d=0;e=1\}$$

The division of polynomials

Given two polynomials $D(z),I(z)$, there exists only one polynomial $C(z)$ and only one polynomial $R(z)$ such that we have the identity:

$$D(z)=I(z)\cdot C(z)+R(z)$$

and

$$gr.R(z)<gr.I(z)$$

Using the undetermined coefficients to find the quotient

$$(3z^4-7z^3+12z^2+3z-6)\div(z^2-2z+4)$$

$$3z^4-7z^3+12z^2+3z-6=(z^2-2z+4)(az^2+bz+c)+dz+e$$

$$3z^4-7z^3+12z^2+3z-6=az^4+(-2a+b)z^3+(4a-2b+c)z^2+(4b-2c+d)z+4c+e$$

$$a=3;b=-1;c=-2;d=3;e=2$$

Horner's Rule

$$\left(x^4+3x^2+x-5\right)\div\left(x+2\right)$$

$$\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & -5 & \\ \hline 1 & -2 & 7 & -13 & 21 & -2 \end{array}$$

The quotient is

$$x^3-2x^2+7x-13$$

and the remainder 21.

Algebraic equations with complex coefficients

If $P(a)=0$, then the number a is called solution of the equation $P(z)=0$.

The equations can be:

- Algebraic ($x^2-5x=0$)
- Transcendent ($x-2\sin x=0$)

Bezout's theorem

A polynomial $P(z)$ id divided by $z-a$ if, and only if the number a is a solution of the equation $P(z)=0$

A polynomial $A(z)$ is divisible by the polynomial $B(z)$ if there is a polynomial $C(z)$ such that takes place the following equality:

$$A(z)=B(z)\cdot C(z) \text{ for } \forall z,\ z\in C$$

Or a polynomial $A(z)$ is divisible by a polynomial $B(z)$ if, and only if the quotient of the division $A(z)\div B(z)$ is the null polynomial

The value of the polynomial $P(z)$ for $z=a$ is the remainder of the division between $P(z)$ and $z-a$. ($P(a)=R$).

If a and b are two solutions of the algebraic equation $P(z)=0$ and $P(z)\div(z-a)=C(z)$, then $C(z)$ is divisible by $(z-b)$

If a polynomial $P(z)$ is divisible by $z-a$ and $z-b$, then it is also divisible by $(z-a)(z-b)$.

The condition for two equations of second degree to have a common solution

$$\boxed{\begin{cases} ax^2+bx+c=0 \\ a'x^2+b'x+c'=0 \end{cases}}$$

$$\left(ab'-a'b\right)\left(bc'-b'c\right)=\left(ac'-a'c\right)^2 \Rightarrow x_1=x_1'$$

x_2 and x_2' arbitrary (these can be equal, or different)

Determining the remainder without performing the division of the polynomials

1) $P(z) \div (z-a)$; the remainder is $P(a)$

2) $P(z) \div (z-a)(z-b)$

$P(z) = Q(z)(z-a)(z-b) + A(z) + B \quad (1)$

For $z = a$ we substitute in (1) $\Rightarrow P(a) = Aa + B$

For $z = b$ we substitute in (1) $\Rightarrow P(b) = Ab + B$

We obtain the system

$$\begin{cases} P(a) = Aa + B \\ P(b) = Ab + B \end{cases} \qquad (2)$$

The remainder is $A(z) + B$ [A, B determined from system (2)]

3) $P(z) \div (z-a)(z-b)(z-c)$

$P(z) = Q(z)(z-a)(z-b)(z-c) + Az^2 + Bz + C$

For $z = a; z = b; z = c$, we obtain

$$\begin{cases} P(a) = Aa^2 + Ba + C \\ P(b) = Ab^2 + Bb + C \\ P(c) = Ac^2 + Bc + C \end{cases} \qquad (3)$$

The remainder is R= $Az^2 + Bz + C$, [A, B, C determined from system (2)]

The fundamental theorem of algebra

Any algebraic equation $P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = 0$, $a_0, a_1, ..., a_n \in C, n \rangle 0$ has at least a solution in C. (Gauss' theorem or d'Alembert's theorem).

The set of the complex numbers is algebraic closed (the enlargement of the notion of number ends when we introduce the complex numbers).

Any polynomial of n degree can be decomposed in n linear, different or equal, factors.

The polynomial decomposition in linear normalized factors is unique.

$$P(z) = a_0 (z - z_1)(z - z_2) \cdots (z - z_n)$$

where $z_1, z_2, ..., z_n$ are the solutions of the equation $P(z) = 0$.

The normalized linear factors are the factors of the form $z - p$ (the coefficient of z must be 1. (Example: $z - \frac{3}{2}$, and not $2z - 3$).

An algebraic equation of n degree has n solutions; each multiple solution being counted)

The solutions can be:

- Simple
- Multiple

If in a decomposition of a polynomial in linear factors we have $(z-a)^k$ and this is the highest power of $(z-a)$, we say that a is a multiple solution of order k for the equation $P(z)=0$.

A solution which is not multiple is called a simple solution.

If $P(z)$ has the solutions $z_1, z_2, ..., z_r$ with the orders $k_1, k_2, ..., k_r$ then

$$P(z)=a_0(z-z_1)^{k_1}(z-z_2)^{k_2}\cdots(z-z_n)^{k_r}$$

The condition for two equations of second degree to have a common solution

$$\boxed{\begin{cases} ax^2+bx+c=0 \\ a'x^2+b'x+c'=0 \end{cases}}$$

$$(ab'-a'b)(bc'-b'c)=(ac'-a'c)^2$$

The condition for two equations of n degree to have the same solutions

The condition for two equations of n degree to have the same solutions is that the quotients of the corresponding coefficients to be equal.

$$\frac{a_0}{b_0}=\frac{a_1}{b_1}=...=\frac{a_{n-1}}{b_{n-1}}=\frac{a_n}{b_n}=k$$

Or:

Two algebraic equations have the same solutions with the same order of multiplicity, if, and only if they differ by a constant factor different of zero.

Relations between coefficients and solutions (Viète's relations)

$$P(z)=a_0z^n+a_1z^{n-1}+...+a_{n-1}z+a_n$$

$$S_1=z_1+z_2+...+z_n=-\frac{a_1}{a_0}$$

$$S_2=z_1z_2+z_1z_3+...+z_{n-1}z_n=+\frac{a_2}{a_0}$$

$$S_3 = z_1 z_2 z_3 + z_1 z_2 z_4 + ... + z_{n-2} z_{n-1} z_n = -\frac{a_3}{a_0}$$

....

$$S_k = z_1 z_2 \cdots z_k + ... = (-1)^k \frac{a_k}{a_0}$$

....

$$S_n = z_1 z_2 \cdots z_n = (-1)^n \frac{a_n}{a_0}$$

S_1 has C_n^1 terms

S_2 has C_n^2 terms

………………..

S_k has C_n^k terms

……………….

S_n has C_n^n terms

n is the grade of the polynomial

To resolve an equation it is necessary to have also a relation between the solutions besides those of Viète.

Given a system of $n+1$equations with n unknown with parameters $a, b, ...$, to find the necessary and sufficient condition that the system would be compatible, we select n equations and we solve one of them. Then, we solve the system formed of n with n unknown, and the found values are substituted in the solved equation. The relation obtained between the parameters is the condition that we're looking for.

Multiple solutions

The derivative of a function of a complex variable

$$P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n$$
$$a_0, a_1, ..., a_n \in C$$
$$z \in C$$

is the polynomial $P'(z) = n a_0 z^{n-1} + (n-1) a_1 z^{n-1} + ... + a_{n-1}$

A number a is a multiple solution of the order k, $k \in \mathbb{N}$ of the algebraic equation $P(z) = 0$, if, and only if $P(z)$is divisible by $(z-a)^k$, and the quotient $Q(z)$ is not null for $z = a$; or

A number a is a multiple solution of order k of the equation $P(z) = 0$ if, and only if the polynomial $P(z)$ is divisible by $(z-a)^k$ and is not divisible by $(z-a)^{k+1}$.

Auxiliary proposition

A solution $z=a$ of the algebraic equation $P(z)=0$ is a multiple solution of order k, if, and only if it is a multiple solution of order $k-1$ of the derivative $P'(z)=0$

Theorem

A number is a multiple solution of order k of an algebraic equation $P(z)=0$ if and only it nullifies the polynomial $P(z)$ and the first $k-1$ derivative, and it does not nullify the derivative of the order k.

Complex numbers

$z=a+bi$, $a,b\in\mathbb{R},\ i=\sqrt{-1}$.

The conjugate of z is $\overline{z}=a-bi$

1) The conjugate of the sum of two complex numbers is the sum of their conjugate: $\overline{z+u}=\overline{z}+\overline{u}$
2) The conjugate of the product of two complex numbers is the product of their conjugate: $\overline{z\cdot u}=\overline{z}\cdot\overline{u}$
3) The natural powers of two imaginary numbers are imaginary conjugate: $\overline{z^n}=(\overline{z})^n$

The values of a polynomial with real coefficients for two imaginary conjugate values of the variable are imaginary conjugated: $P(\overline{z})=\overline{P(z)}$

If an algebraic equation with real coefficients has the solution $u+vi$, then it has also the solution $u-vi$, or:

The imaginary solutions of an algebraic equation with real coefficients are conjugated two by two.

Irrational squared solutions

The numbers of the form $m+n\sqrt{p}$ where $m,n\in\mathbb{Q},\ p\in\mathbb{N}$ and p does not contain any factor which can be a perfect square.

If an equation with rational coefficients $P(x)=0$ has a solution which is a squared irrational $m+n\sqrt{p}$, then it has also its conjugate $m-n\sqrt{p}$ as solution .

Proposition (SF)

If $m_1\sqrt{p_1}+m_2\sqrt{p_2}$ is a solution on $P(x)$ then $m_1\sqrt{p_1}-m_2\sqrt{p_2}$, $-m_1\sqrt{p_1}+m_2\sqrt{p_2}$, $-m_1\sqrt{p_1}-m_2\sqrt{p_2}$ are solutions for $p\in\mathbb{Q}[x]$. Similarly for three squares.

The transformation in $(-x)$

$$P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$$

$$P(-x)=a_0(-x)^n+a_1(-x)^{n-1}+\ldots+a_{n-1}(-x)+a_n$$

The limits of the solutions

The method of the grouping

The terms are arranged on several groups such that:

a) The coefficient of the first term from each group is positive;
b) Each group has just one variation or none;
c) Research for the smaller integer number, for which each group is positive and this is the superior limit of the solutions.
d) Calculate the transformation in $(-x)$ and find its limit; this will be the inferior limit of the solutions (taken with negative sign $x_i \in (l, L)$

If a polynomial with only one variation $(++++---)$ is positive for a positive value of x greater than a.

If an algebraic equation $P(x)=0$ has the solutions $x_1, x_2, \dots, x_n$ its transformation in $-x$ $P(-x)=0$ has the solutions $-x_1, -x_2, \dots, -x_n$.

Example

$$P(x) = x^6 - 3x^5 + 2x^4 + 3x^2 - 31x - 96 = 0$$

$$\left(\underbrace{x^6 - 3x^5}_{4}\right) + \left(\underbrace{2x^4 - 31x}_{3}\right) + \left(\underbrace{3x^2 - 96}_{6}\right) = 0 \text{ therefore } L = 6$$

$$P(-x) = x^6 + 3x^5 + 2x^4 + 3x^2 + 31x - 96 = 0$$

$$\left(\underbrace{x^6 - 96}_{3}\right) + \left(\underbrace{3x^5 + 2x^4 + 3x^2 + 31x}_{1}\right) = 0 \qquad \text{therefore } l = -3$$

Then

$x_i \in \{-3, 6\}$ where $i = 1,2,3,4,5,6$

If an algebraic equation

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$$

With integer coefficients has as solution the integer p, then a_n is divisible by p.

If an algebraic equation $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$ with integer coefficients has as solution the irreducible fraction $\frac{p}{q}$, then a_n is divisible by p and a_0 is divisible by q

$$a_0x^n + \dots + a_n$$
$$\frac{p}{q}$$

The exclusion of some fractions

Let consider the fraction $\frac{a}{b}$:

- If $P(1)$ is not divisible by $b-a$ then the fraction $\frac{a}{b}$ is not a solution

- $P(-1)$ is not divisible by $a+b$ then the fraction $\frac{a}{b}$ is not a solution

Rule

To find the rational solution of an algebraic equation with coefficient integer numbers we'll proceed as follows:

1) Determine the limits L, L' of the solutions
2) Find the integer solutions
3) Determine the fractional solutions (use the Horner' scheme)

A polynomial $P(x)$ is called irreducible in a numeric set ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$) if there exist two polynomials $A(x), B(x)$ with coefficients in that set such that $P(x) = A(x) \cdot B(x)$.

Any polynomial with real coefficients can be decomposed in irreducible factors of I and II degree with real coefficients.

To study the sign of a polynomial we decompose the polynomial in irreducible factors and we study their sign. Then we intersect the signs.

The separation of the solutions through the graphic method

1) The equations of the form $f(x) = 0$
   - Represent the function graphically; the abscises of the points in which the curve intersects the Ox axes are the solutions of the equation
2) Equations of the form $f(x) = g(x)$
   - Both functions are graphically represented; the abscises of the intersection points of the curves are the solutions of the equation.

To limit the interval in which the solutions are located, we give values to the variable and compute, using the Horner's scheme, ($f(a)$ being the remainder); between the contrary signs exits the value zero of the continue function.

The discussion of an equation dependent of a parameter

To discuss an equation we mean to find how many real solutions has the respective equation for different values of a parameter.

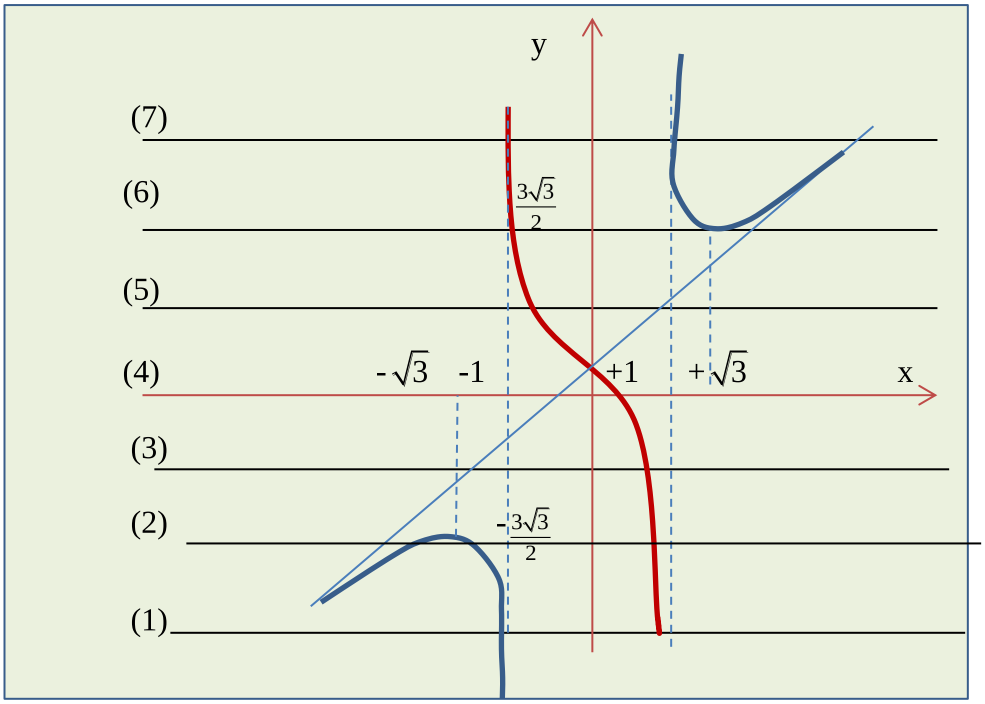
Example

$P(x) = x^3 - ax^2 + a = 0$

$a = \frac{x^3}{x^2 - 1}$

We'll denote $f(x) = \frac{x^3}{x^2 - 1}$ and $g(x) = a$; ($n \in \mathbb{R} \setminus \{-1, +1\}$)

Construct the curves graphs



The table of variation

The values of a	The intervals where the solutions are
$a < -\frac{3\sqrt{3}}{2}$	$(-\infty,-\sqrt{3}),(-\sqrt{3},-1),(0,1)$
$a = -\frac{3\sqrt{3}}{2}$	$x_1 = x_2 = -\sqrt{3}\quad (0,1)$
$-\frac{3\sqrt{3}}{2} < a < 0$	$(0,1)$
$a = 0$	$x_1 = x_2 = x_3 = 0$
$0 < a < \frac{3\sqrt{3}}{2}$	$(-1,0)$
$a = \frac{3\sqrt{3}}{2}$	$x_1 = x_2 = \sqrt{3}\quad (-1,0)$
$a > \frac{3\sqrt{3}}{2}$	$(-1,0),\ (1,\sqrt{3}),\ (\sqrt{3},+\infty)$

Rolle's theorem

Let f a function defined on an interval $\Im$ and a,b two points in $\Im$ with $a<b$.

If

1) f is continue on the close interval $[a,b]$
2) f is derivable on the open interval (a,b)
3) f has equal values in a,b, $f(a)=f(b)$,

Then there exist at least a point c between a and b, $a<c<b$, in which the derivative becomes null

$f'(c)=0$

Between two consecutive solutions of the equation $f(x)=0$ there exists at least a solution of the equation $f'(x)=0$

Let f a function continue on a closed interval $[x_1,x_2]$, $y_1=f(x_1)$ and $y_2=f(x_2)$ the values of the function at the extremities of the interval and y_3 a number between y_1 and y_2 ($y_1<y_3<y_2$), then there exist at least one point x_3 such that $y_3=f(x_3)$

Or

A continue function does not pass from a value to another value without passing through all intermediary values.

If a function f is continue on a closed interval $[x_1,x_2]$ and $f(x_1)\cdot f(x_2)<0$, then there exists at least a point x_3, $x_1<x_3<x_2$ such that $f(x_3)=0$

Or

A continue function cannot change its sign without becoming zero.

If α,β are two consecutive solutions of the derivative and $f(\alpha),f(\beta)$ are of contrary signs, then between (α,β) there exist only one solution of the equation; if $f(\alpha),f(\beta)$ have the same sign, or $f(\alpha)=0$ or $f(\beta)=0$ the equation does not have in the interval (α,β) any solution.

The equation has at most a solution smaller than the smaller solution of the derivative and at most a solution bigger than the biggest solution of the derivative.

The extremities (left and right) of the interval play the same role as the solutions of the derivative.

The sequence of Rolle

The solutions of the first derivative are written in order, the same the extremities a,b of the interval where the function is defined, and under are written the corresponding values of the function

x	a	x_1	x_2...........x_n	b
$f(x)$	$\lim\limits_{n\to a} f(x)$	$f(x_1)$	$f(x_2)$.....$f(x_2)$	$\lim\limits_{n\to\infty} f(x)$

The intervals at the end of which we see values of contrary signs contain a solution of the function $f(\alpha)=0$.

Discussion

Find the intervals in relation to the real parameter of the real solutions of the equation:

$$f(x)=3x^4-4x^3-12x^2+a=0$$

$$f'(x)=12x^3-12x^2-24x+a=0$$

$$\begin{cases} x_1=1 \\ x_2=0 \\ x_3=2 \end{cases}$$

We'll form the Rolle's sequence

x	-∞	-1	0	2	+∞
$f(x)$	+∞	$a-5$	a	a - 32	+∞
$a<0$	+	-	-	-	+
$a=0$	+	-	0	-	+
$0<a<5$	+	-	+	-	+
$a=5$	+	0	+	-	+
$5<a<32$	+	+	+	-	+
$a=32$	+	+	+	0	+
$a>32$	+	+	+	+	+

We determined the number of the real solutions and the intervals in which these reside.

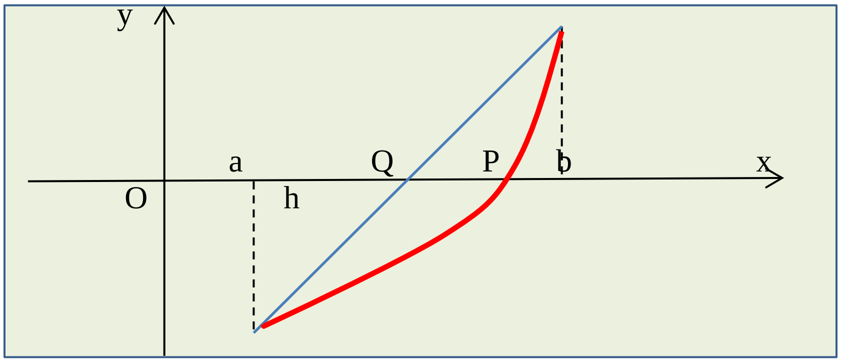
The approximation of the real solutions of an equation

1) The intervals' separation
   If is known that $x_i \in (a,a+1)$ we can reduce the interval trying the values: $a+0.1; a+0.2; ..., a+0.9$
   If $f(a)$ >0 and $f(a+1)$ <0, and $f(a+0.5)$ <0, then $x_i \in (a,a+0.5)$ and so on.
2) The cord method:
   We take the approximate value of the solution the abscise of the point P in which the line MN cuts the axis Ox.

$$MN: y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a)\text{, for } y=0$$

$$h = x - a = -\frac{(b-a)f(a)}{f(b)-f(a)}$$

$$x = a + h$$

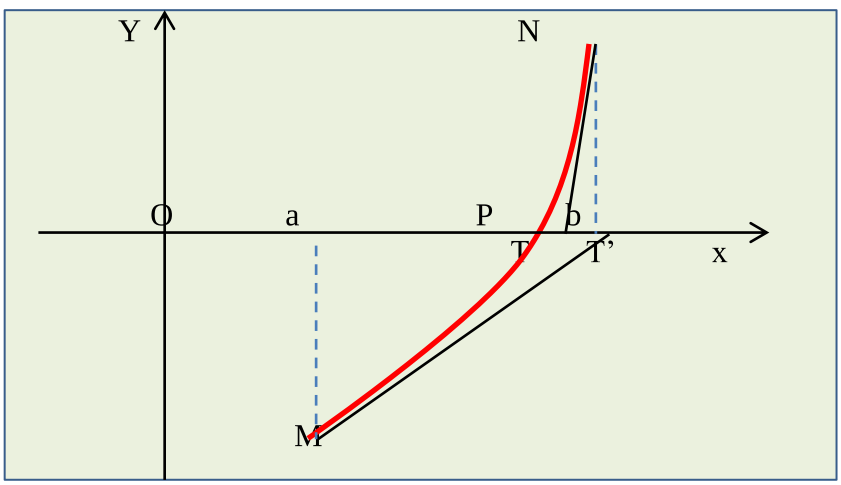


The sense of the error can be determined with the help of a graph or:

1) If $f'(x)\cdot f''(x) > 0$, we have approximation in minus
2) If $f'(x)\cdot f''(x) < 0$, we have approximation in excess

   x belonging to the interval in which we computed the solution: (a,b)

3) The method of tangent (Newton's method)



$$MT': \ y - f(a) = f'(a)(x-a)$$

For $y = 0$

$$h = x - a = \frac{f(a)}{f'(a)}$$

or

$$MT: \ y - f(b) = f'(b)(x-b)$$

For $y = 0$

$$k = x - b = -\frac{f(b)}{f'(b)}$$

- The sense of the error is given by the form of the curve (in minus or in excess)
- The approximations given by the cord method and by the Newton's method are of a contrary sense only when the second derivative maintains its sense in the interval (a,b)
- These methods can be applied repeatedly (to obtain a value more precise)

Laws of internal composition

1) Operations determined with determined elements (numbers) – makes the object of arithmetic
2) Operations determined with determined and undetermined elements – makes the objects of elementary algebra
3) Undetermined operations with undetermined elements makes the object of abstract algebra.

Definition

The law of internal composition between the elements of a set M is an application f defined on a part of the Cartesian product $M \times M$ with values in M (internal algebraic operation).

The corresponding element to a pair (x,y) through the function f is called the composite of x with y $(x \perp y,\ x\mathrm{T}y,\ x \circ y,\ x\Delta y,\ x * y)$

The addition in $\mathbb{C}$ is a function defined on the $\mathbb{C}\times\mathbb{C}$ with values in $\mathbb{C}$. $(f(x,y) = x + y)$ additive.

The subtraction in $\mathbb{C}$ is a function defined on the $\mathbb{C}\times\mathbb{C}$ with values in $\mathbb{C}$. $(f(x,y) = x - y)$.

The multiplication in $\mathbb{C}$ is a function defined on the $\mathbb{C}\times\mathbb{C}$ with values in $\mathbb{C}$. $(f(x,y) = x \cdot y)$

The division in $\mathbb{C}$ is a function defined on the $\mathbb{C}\times\mathbb{C}$ with values in $\mathbb{C}$. $\left(f(x,y) = \frac{x}{y}\right)$

A law of internal composition is defined everywhere if it associates to any pair of elements from M, only one element from M.

Associativity

A law of internal composition which is defined everywhere between the elements of a set E, denoted with the sign T, is called associative if:

$x\top(y\top z) = (x\top y)\top z$ for any elements x,y,z (distinct or not) form E:

$\forall x,y,z \in E : x\top(y+z) = (x\top y)\top z$

Generalized:

$(x_1\top x_2)\top\ldots\top(x_{n-1}\top x_n) = x_1\top x_2\top\ldots\top x_n$ Theorem of associativity

Commutatively

A law of internal composition, everywhere defined, between the elements of set E, denoted $\top$, is commutative if, for any elements x,y from E, we have $x \top y = y \top x$.

$\forall x,y \in E : x \top y = y \top x$

Commutative theorem

$a_1 \top a_2 \top ... \top a_n = a_k \top a_l \top ... \top a_s$, $1 \le k,l,s \le n$

Neutral element

Let E a set on which is given a composition law defined everywhere, denoted $\top$. If there is an element $e \in E$ such that $e \top x = x \top e = x$ for any $x \in E$, this element is called the neutral element for the law $\top$. (The neutral element is unique).

Symmetrical elements

Let E a set for which is defined a composition law denoted $\top$, and for which there exists the neutral element e. An element $x' \in E$ is called the symmetrical of x for the law $\top$, if

$x \top x' = x' \top x = e$

An element p from the set of the remainder's classes modulo m has a symmetrical element in relation to the multiplication, if $(m,p)=1$ (the two numbers are prime between them). If m is prime, then all elements p have an inverse

If the law is denoted additive, the symmetrical $-x$ is called the opposite of x;

If the law is denoted multiplication, the symmetrical x^{-1} is called the inverse of x; (each element of a set $x \in E$ admits only one symmetric).

Distributive

Let E a set on which are given two internal composition laws $\top$ and $\perp$. We say that the law $\top$ is distributive in relation to the law $\perp$ if for any x,y,z distinct or not from E we have:

$x \top (y \perp z) = (x \top y) \perp (x \top z)$ distributive to the left

$(y \perp z) \top x = (y \top x) \perp (z \top x)$ distributive to the right

Algebraic structure

1) Group
2) Ring
3) Field (a commutative ring)
4) Module
5) Vectorial space
6) Algebra

Group

A composition law forms a group if the following axioms are satisfied

1) The rule is defined everywhere
2) Associativity (A)
3) The existence of the neutral element (N)
4) Any element has a symmetrical (S)

If the law is commutative we called the group abelian (or commutative)

Immediate consequences of the group's axioms

a) The composite of three, four, or more elements is the same (A)
b) In a group exists a neutral element and only one (N)
c) Any element admits a symmetrical and only one (S)
d) In a group G whose law is denoted multiplication we have: $\left(x^{-1}\right)^{-1}=x$
   - In a group multiplicative $x\cdot x\cdot x\cdot\cdot x=x^n$ (the n^th power of x)
   - In a group additive $x+x+x+\cdot\cdot+x=nx$

$x^n x^m=x^{n+m}$

$\left(x^n\right)^m=x^{nm}$

$nx+mx=\left(n+m\right)x$

$m\left(nx\right)=\left(mn\right)x$

$\left(x\cdot y\right)^{-1}=x^{-1}\cdot y^{-1}$

$\left(x^{-1}\right)^n=\left(x^n\right)^{-1}\quad n\in\mathbb{N}$

Simplification at the left and at the right

$\left(a\top x=b\top x\right)\Rightarrow\left(a=b\right)$

$\left(x\top a=x\top b\right)\Rightarrow\left(a=b\right)$

$\left(a=b\right)\Leftrightarrow\left(a\top x=b\top x\right)$

$\left(a=b\right)\Leftrightarrow\left(x\top a=x\top b\right)$

Equations in a group

1) $a\top x=b$

$a'\top\left(a\top x\right)=a'\top b$

$\left(a'\top a\right)\top x=a'\top b$

$e\top x=a'\top b$

$x=a'\top b$

2) $x\top a=b$

$\left(x\top a\right)\top a'=b\top a'$

$x\top\left(a\top a'\right)=b\top a'$

$x\top e=b\top a'$

$x = b\top ba'$

Ring

Two composition rules, first denoted additive and the second multiplicative define on a set a structure ring I if:

1) A_P The addition is defined every where
2) A_A The addition is associative
3) A_C The addition is commutative
4) A_N There exists a neutral element (0)
5) A_S Any element admits a symmetrical
6) M_P The multiplication is defined everywhere
7) M_A The multiplication is associative
8) D_{MA} The multiplication is distributive in relation to addition

If the multiplication is commutative we call the ring commutative
If there is a neutral element in relation to the multiplication, we say that the ring has the unity element

Divisors of zero

An element $d \neq 0$ of a ring is called divisor of zero, if there exists an element $d' \neq 0$ such that $dd' = 0$ or $d'd = 0$.

The rule of signs in a ring

$$(-a)\cdot b = a\cdot(-b) = -(ab)$$
$$(-a)\cdot(-b) = ab$$

The simplification to the left and to the right in ring without divisors of zero

$$ax = ay \Rightarrow x = y \quad (a \neq 0)$$
$$xa = ya \Rightarrow x = y \quad (a \neq 0)$$

Field

Two composition rules, first denoted additive and the second multiplicative, determine on a set a structure of field if

1) A_P The addition is defined every where
1) A_A The addition is associative
2) A_C The addition is commutative
3) A_N There exists a neutral element (0)
4) A_S Any element admits a symmetrical
5) M_P The multiplication is defined everywhere

6) M_A The multiplication is associative

7) M_N There exists a neutral element (0)

8) D_{MA} The multiplication is distributive in relation to addition

9) M_S Any element admits a symmetrical

If the multiplication is commutative then the field is called commutative
In a field there are no divisors of zero.

Wedderburn's theorem

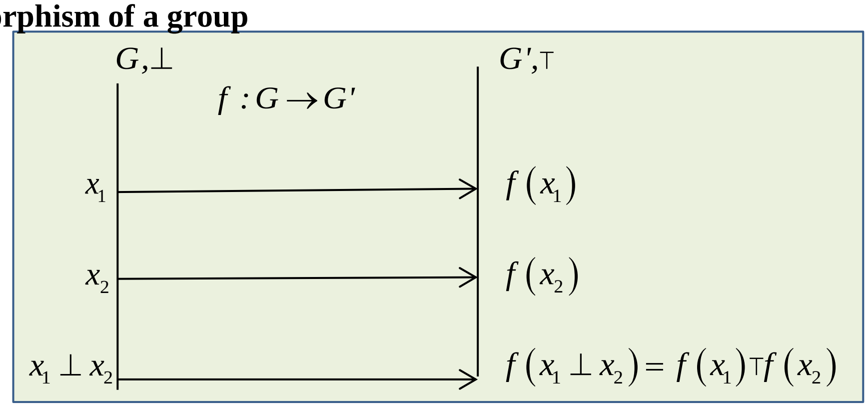
Any field finite is commutative.
The field of the remainders classes modulo p is commutative.

$$a=\frac{a}{1};\ \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd};\ \frac{a}{b}+\frac{c}{d}=\frac{ac}{bd}$$

It is admitted that a field contains at least two distinct elements

	Random function	Injective function	Surjective function	Bijective function
$E,*\to F,0$	morphism	Mono morphism	epimorphism	Isomorphism
$E,*\to F,0$	endomorphism			automorphism

Isomorphism of a group



Function $f:G\to G'$ is bijective

Definition

Let G a group whose rule is denoted $\perp$ and G' a group whose rule is $\top$. An application $f:G\to G'$ is called isomorphism if:

a) The function is bijective

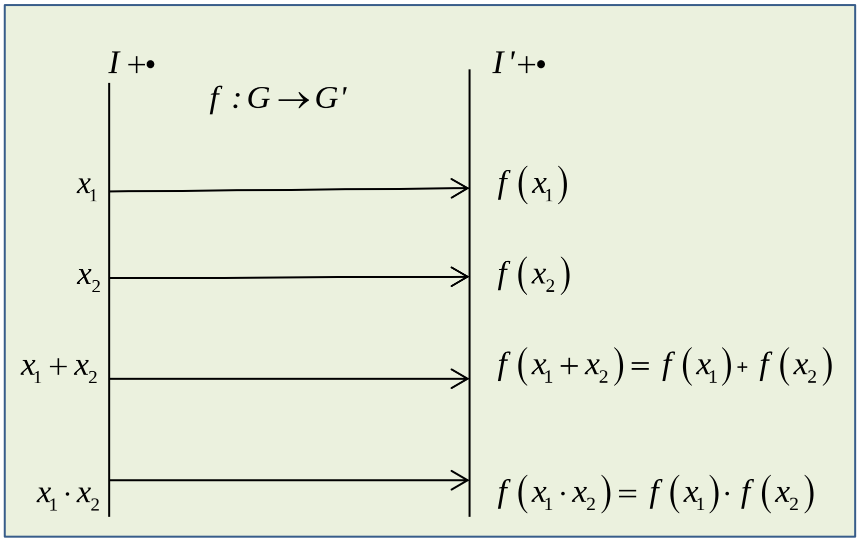
b) $f(x_1\perp x_2)=f(x_1)\top f(x_2)$

Then

$$f(e)=e'$$

$$f(x^{-1})=\left[f(x)\right]^{-1}$$

Isomorphism of ring



and the function $f:I\to I'$ is bijective then

- $f(0)=0$
- $f(x_1+x_2+...+x_n)=f(x_1)+f(x_2)+...+f(x_n)$
- $f(x_1+x_2+...+x_n)=f(x_1)+f(x_2)+...+f(x_n)$
- If I admits as unity element 1, then $f(1)$ is an unity element in I

Isomorphism of field

The same concept as for an isomorphism of a ring

The identical groups from algebraic point of view are two groups of a different nature, when their elements can be grouped two by two, such that any algebraic relation between the elements of the first group is true simultaneously with the relation obtained by substituting the elements of the first group with those of the second group.

Morphism of $E:F$

Given two sets E,F with the composition rules $*,\circ$, we call morphism of the set E in the set F, any application in which the result of the composite of any two elements from E using the rule $*$ has as image in F through f the composite through $\circ$ of the corresponding images of the elements considered in E

Table of a group G

$\perp$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Theorem (C. Ionescu-Tiu)

If $P(x) \vdots a_0x^n + ... + a_n$ and $P'(x) \vdots a_0x^n + ... + a_n$, then $P(x) \vdots (a_0x^n + ... + a_n)^2$

Stolg-Cesaro's Lemma

If b_n is strictly monotone, endless then the following are true:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_1a_2...a_n} = \lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n}$$

The formulae of the composed radicals

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

$$\sqrt{a-\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} - \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

Harmonic division

$$A,B,C,D: \text{ if } \frac{CA}{CB} \cdot \frac{DB}{DA} = 1$$

$$max(x,y) = \frac{x+y+|x-y|}{2};$$

$$min(x,y) = \frac{x+y-|x-y|}{2}$$

Jensen's inequality

$$f_{convex} \Rightarrow f\left(\frac{\sum_1^n x_i}{n}\right) \le \frac{1}{n}\sum_1^n x_i$$

$$f_{concave} \Rightarrow f\left(\frac{\sum_{1}^{n} x_i}{n}\right) \geq \frac{1}{n}\sum_{1}^{n} x_i$$

TRIGONOMETRY

Sexagesimal degrees
A Sexagesimal degree is equal to the 90^{th} part of a right angle
The Sexagesimal minute is equal to the 60^{th} part from a sexagesimal degree.
The sexagesimal second is equal to the 60thpart from a sexagesimal minute.

Centesimal degrees
A centesimal degree is equal to the 100^{th} part from a right angle
The centesimal minute is equal to the 100^{th} part from a centesimal degree
The centesimal second is the 100^{th} part of a centesimal minute.

The radian
The measure in radians is the ratio between the lengths of the arc of a circle, which corresponds to an angle in the center, and the length of the circle's radius $\frac{l}{r}$.
One radian is the angle whose vertex is in the center of a circle that corresponds to an arc whose length is equal to the length of the circle's radius.

How to transform a measure to another one:

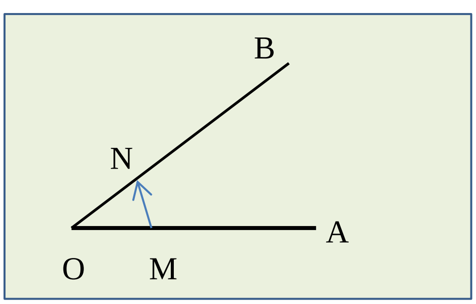
$$\frac{\alpha}{180^\circ} = \frac{a'}{200^g} = \frac{a}{\pi}$$

α°	0°	30°	45°	60°	90°	180°	270°	360°
a	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Oriented plane
An oriented plane is a plane in which it has been established the positive direction for rotations.

Oriented angle
An oriented angle is the angle for which has been established the rotation direction.

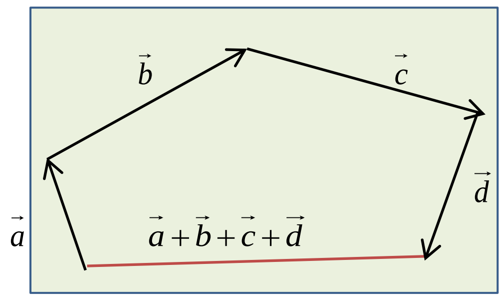
The rotation direction (sense) ca be:
- Positive (direct or trigonometric) – is the sense inverse to the clock rotation
- Negative (retrograde) – is the sense of the clock rotation.



OA - The initial side
OB - The final side
M - The origin of the arc
N - The extremity of the arc

The rotation sense from OA to OB is considered the positive sense

To sum several vectors is used the polygon rule: To construct the sum of several vectors we will construct equal vectors with the given ones, such that the origin of one vector will coincide with the extremity of the precedent one. The vector that coincides with the origin of the first term, and the extremity will coincide with the origin of the last term, represents the sum of the considered vectors.

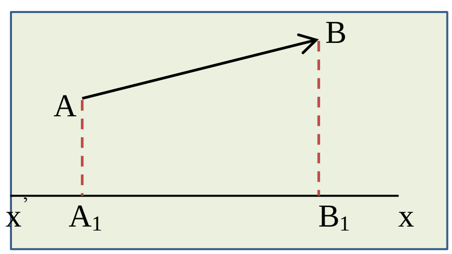


The product between a vector and a number

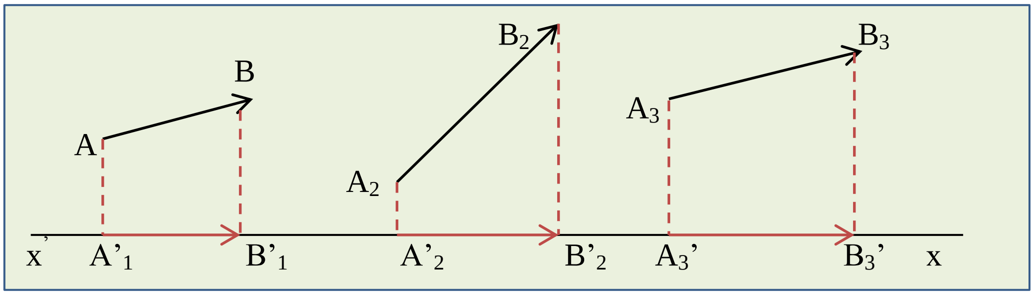
By multiplying a vector $\vec{a}$ with a real number λ, we obtain a vector which has its module equal to the product between the module of the number λ and the module of the vector $\vec{a}$, and the sense will be that of $\vec{a}$ or -$\vec{a}$, depending the sign of λ (λ>0, or λ<0); for λ=0, the product will be the null vector.

The projection of a vector on axis.

The projection of vector $\overrightarrow{AB}$ on axis x is the size of the vector $\overrightarrow{A_1B_1}$ and is denoted: $pr.x\overrightarrow{AB}$



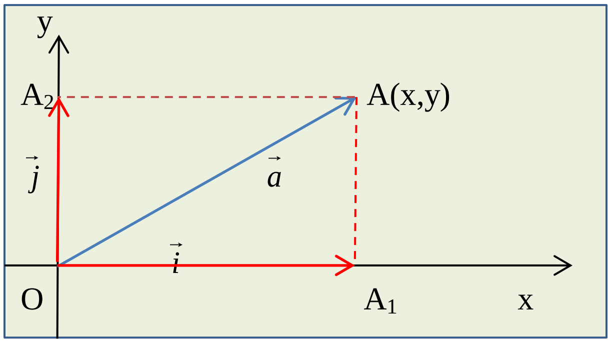
The projections of some vectors which have the same sense, on an axis, are proportional to their module.



$$\frac{\overrightarrow{A_1'B_1'}}{\overrightarrow{A_1B_1}} = \frac{\overrightarrow{A_2'B_2'}}{\overrightarrow{A_2B_2}} = \frac{\overrightarrow{A_3'B_3'}}{\overrightarrow{A_3B_3}}$$

The decomposition of vectors in the plane

$$\vec{a} = \overrightarrow{xi} + \overrightarrow{yj}$$



$\vec{i}$ and $\vec{j}$ are the versors of the axes.

A versor is the vector whose module is equal to the unity.

Two angles that have the same size are equal.

The sum of two angles is the angle whose size is equal to the sum of the sizes of the two angles.

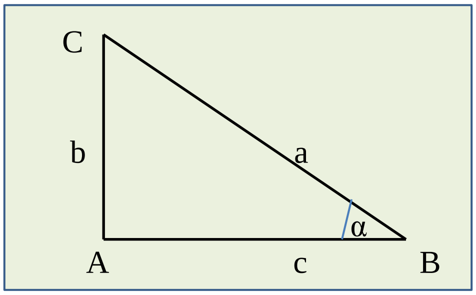
Between the set of real numbers and the set of oriented angles there is a bi-univocal correspondence.

Between the final position of an oriented angle and the set of the real numbers does not exist a bi-univocal correspondence; to a final position of the final side corresponds an infinity of angles that have various values.

Trigonometric functions of an acute angle

The sine

The sine of an acute angle is the ratio between the opposite cathetus and the hypothesis:



$$\sin\alpha = \frac{b}{a}$$

The cosine

The cosine of an acute angle is the ratio between the adjoin cathetus and the hypotenuse.

$$\cos\alpha = \frac{c}{a}$$

The tangent

The tangent of an acute angle is the ratio between the opposite cathetus and the adjoin cathetus of the angle:

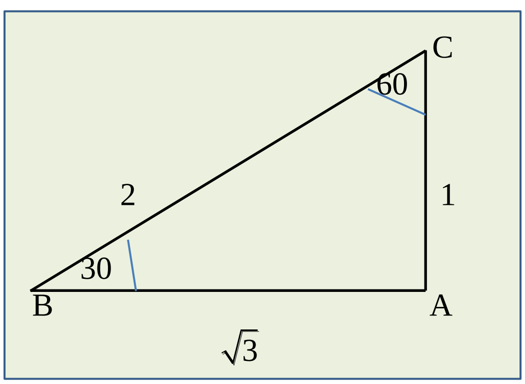
$$\tan\alpha = \frac{b}{c}$$

The cotangent

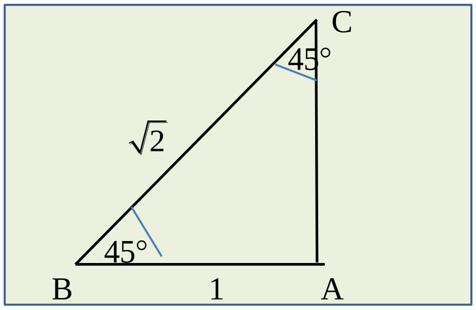
The cotangent of an acute angle is the ratio between the adjoin cathetus and the opposite cathetus of the angle:

$$\cot\alpha = \frac{c}{b}$$

The values of the trigonometric functions for angles: 30 , 45 , 60



$$\sin 30° = \frac{1}{2}$$
$$\cos 30° = \frac{\sqrt{3}}{2}$$
$$\tan 30° = \frac{1}{\sqrt{3}}$$
$$\cot 30° = \sqrt{3}$$
$$\sin 60° = \frac{\sqrt{3}}{2}$$
$$\cos 60° = \frac{1}{2}$$
$$\tan 60° = \sqrt{3}$$
$$\cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$



$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\tan 45° = 1$$
$$\cot 45° = 1$$

$$\frac{z_1}{z_2} = -\frac{\rho_1}{\rho_2}\left[\cos\left(\varphi_1 - \varphi_1\right) + i\sin\left(\varphi_1 - \varphi_1\right)\right]$$

The power of the complex numbers

The Moivre's formula

$$\left|z^n\right| = |z|^n;\ \arg\left(z^n\right) = \arg\left(nz\right)$$
$$z^n = \rho^n\left(\cos n\varphi + i\sin n\varphi\right);\left(\ n \in \mathbb{Z}\right)$$

The root of the n order from a complex number

$$\left|\sqrt[n]{z}\right| = |z|^{\frac{1}{n}};\ \arg\sqrt[n]{z} = \frac{\arg z + 2k\pi}{n};\ \left(k = 0,1,2,\ldots,n-1\right)$$

Binomial equations

A binomial equation is an equation of the form:

$$z^n + \alpha = 0$$

where α is a complex number and z is the unknown, $(z \in \mathbb{R})$.

Example"
Find the solution for the following binomial equation:

$$\left(\frac{1+iz}{1-iz}\right)^4 = \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\frac{1}{2}-\frac{\sqrt{3}}{2}i}$$

$$\rho_1 = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$tg\varphi_1 = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \sqrt{3} \Rightarrow \varphi_1 = \frac{\pi}{3}$$

$$z_1 = \frac{1}{2}+\frac{\sqrt{3}}{2}i = cos\frac{\pi}{3} + i\,sin\frac{\pi}{3}$$

$$\rho_2 = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$arg\, z_2 = 2\pi - arg\, z_1 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} = \varphi_2$$

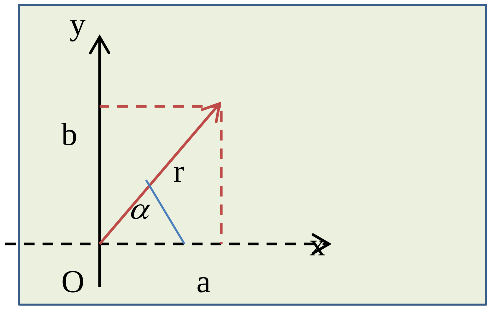
$$z_2 = \frac{1}{2}-\frac{\sqrt{3}}{2}i = cos\frac{5\pi}{3} + i\,sin\frac{5\pi}{3}$$

$$\frac{z_1}{z_2} = cos\left(\frac{-4\pi}{3}\right) + i\,sin\left(\frac{-4\pi}{3}\right) = cos\frac{4\pi}{3} - i\,sin\frac{4\pi}{3} = cos\frac{2\pi}{3} + i\,sin\frac{2\pi}{3}$$

$$\frac{1+iz}{1-iz} = \sqrt[4]{cos\frac{2\pi}{3} + i\,sin\frac{2\pi}{3}} = cos\frac{\frac{2\pi}{3}+2k\pi}{4} + i\,sin\frac{\frac{2\pi}{3}+2k\pi}{4}$$

The trigonometric functions of an orientated angle

The sine of an angle α is the ratio between the projection of a vector radius on the y-axis and its module: $sin\,\alpha = \frac{b}{r}$



The cosine of an angle α is the ratio between the projection of a vector radius on the x-axis and its module $cos\,\alpha = \frac{a}{r}$

The tangent of an angle α is the ratio between the projection of a vector radius on the y-axis and its projection on the x-radius: $tan\,\alpha = \frac{b}{a}$.

The cotangent of an angle α is the ratio between the projection of a vector radius on the x-axis and its projection on the y-radius:. $cot\,\alpha = \frac{a}{b}$

The secant of an angle α is the ratio between the vector radius and it projection on the x-axis: $sec\,\alpha = \frac{r}{a}$

The cosecant of an angle α is the ratio between the vector radius and its projection on the y-axis: $csc\,\alpha = \frac{r}{b}$

The signs of the trigonometric functions

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
sin α	+	+	-	-
cos α	+	-	-	+
tan α	+	-	+	-
cot α	+	-	+	-
sec α	+	-	-	+
csc α	+	+	-	-

Or

+	+
-	-

sin α
sec α

+	+
-	-

cos α
csc α

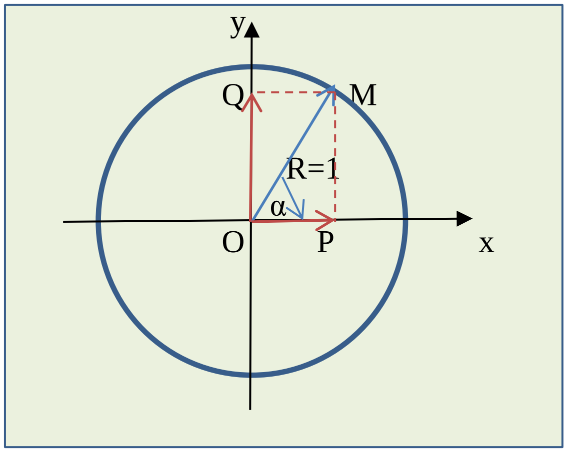
+	+
-	-

tan α
cot α

Trigonometric circle

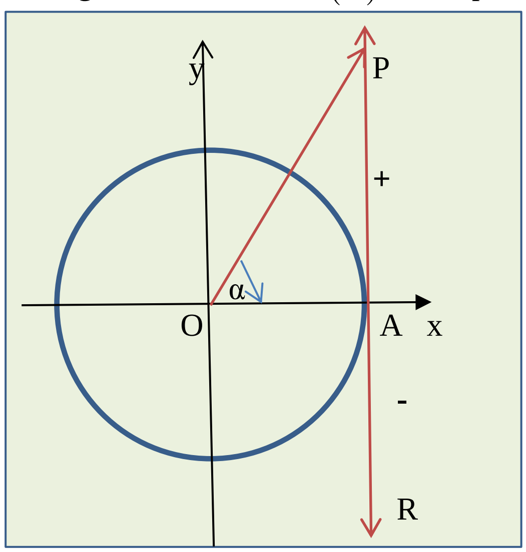
A trigonometric circle is a circle whose radius $R = 1$ (the unity) and with the center in the origin of the coordinate axes.

1. The sine of an angle is equal to the projection of the vector radius on the x-axis
2. The cosine of an angle is equal to the projection of the vector radius on the y-axis

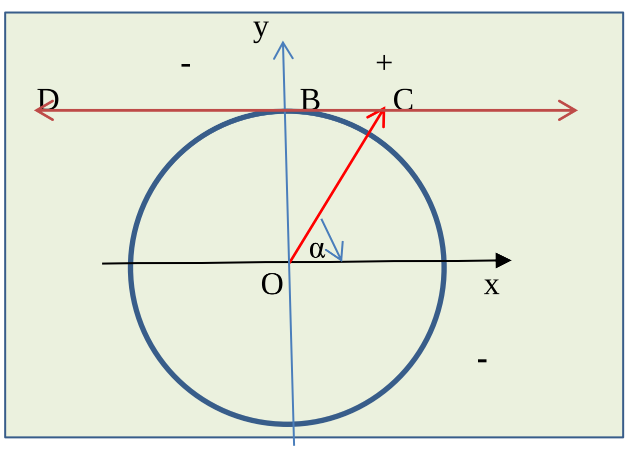


3. The tangent of an angle is equal to the projection of the vector radius on the tangent axis

PR is the tangent axis (the tangent to the circle $C(O)$ in the point A)



4. The cotangent of an angle is equal to the projection of the vector radius on the cotangent axis.



CD is the cotangent axis (is the tangent to the circle $C(O)$ in the point B)

$sin\,90° = 1 \quad cos\,90° = 0 \quad tan\,90°$ there is none

$sin\,180° = 0 \quad cos\,180° = -1 \quad tan\,180° = 0$

$sin\,270° = -1 \quad cos\,270° = 0 \quad tan\,270°$ there is none

$sin\,360° = 0 \quad cos\,360° = 1 \quad tan\,360° = 0$

$cot\,90° = 0$

$cot\,180°$ there is none

$cot\,270° = 0$

$cot\,360°$ there is none

The reduction to an acute angle $\alpha \le 45°$

$$sin(90° - \alpha) = cos\,\alpha \qquad sin(90° + \alpha) = cos\,\alpha$$
$$cos(90° - \alpha) = sin\,\alpha \qquad cos(90° + \alpha) = -sin\,\alpha$$
$$tan(90° - \alpha) = cot\,\alpha \qquad tan(90° + \alpha) = -cot\,\alpha$$
$$cot(90° - \alpha) = tan\,\alpha \qquad cot(90° + \alpha) = -tan\,\alpha$$

$$sin(180° - \alpha) = sin\,\alpha \qquad sin(180° + \alpha) = -sin\,\alpha$$
$$cos(180° - \alpha) = -cos\,\alpha \qquad cos(180° + \alpha) = -cos\,\alpha$$
$$tan(180° - \alpha) = tan\,\alpha \qquad tan(180° + \alpha) = tan\,\alpha$$
$$cot(180° - \alpha) = -cot\,\alpha \qquad cot(180° + \alpha) = cot\,\alpha$$

$$sin(270° - \alpha) = -cos\,\alpha \qquad sin(270° + \alpha) = -cos\,\alpha$$
$$cos(270° - \alpha) = -sin\,\alpha \qquad cos(270° + \alpha) = sin\,\alpha$$
$$tan(270° - \alpha) = cot\,\alpha \qquad tan(270° + \alpha) = -cot\,\alpha$$
$$cot(270° - \alpha) = tan\,\alpha \qquad cot(270° + \alpha) = -tan\,\alpha$$

$$sin(360° - \alpha) = -sin\,\alpha \qquad sin(-\alpha) = -sin\,\alpha$$
$$cos(360° - \alpha) = cos\,\alpha \qquad cos(-\alpha) = cos\alpha$$
$$tan(360° - \alpha) = -tan\,\alpha \qquad tan(-\alpha) = -tan\,\alpha$$
$$cot(360° - \alpha) = -cot\,\alpha \qquad cot(-\alpha) = -cot\,\alpha$$

Periodic fractions

A function $f(x)$ is called periodic if there is a number $T \neq 0$, such that for any x belonging to its domain of definition, the function's values in the $x+T$ and x are equal: $f(x+T) = f(x)$.

The number T is called the period of the function $f(x)$.

Theorem

If the number T is the period for the function $f(x)$, then the numbers kT $(k = \pm 1, \pm 2, ...)$ also are periods for the given function.

Principal period

The principal period is the smaller positive period of a period function (frequently the principal period is called shortly: the period).

The functions $\sin x$ and $\cos x$ have 2π as principal period, and the functions $\tan x$ and $\cot x$ have π as principal period.

The principal period of function $f(x) = a \sin(\omega x + \varphi)$, where a and ω are constants non-null, is equal to $\frac{2\pi}{\omega}$

The harmonic simple oscillation: $|a|$ is the amplitude, ω is the pulsation and φ is the phase difference of the oscillation.

Periodic functions odd and even

The function $f(x)$ is even if for any x belonging to its domain of definition, is satisfied the following relation:

$$f(x) = f(x)$$

The function $f(x)$ is odd if for any x belonging to its domain of definition, is satisfied the following relation:

$$f(-x) = -f(x)$$

Theorem

The graph of an even function is symmetric relative to the x-axis, and the graph of the odd functions is symmetric relative to the y-axis.

$\cos(-x) = \cos x$ -even

$$\left.\begin{aligned} \sin(-x) &= -\sin x \\ \tan(-x) &= -\tan x \\ \cot(-x) &= -\cot x \end{aligned}\right\} \text{odd}$$

There are functions which are neither odd nor even.

Strict monotone functions

1. If $f(x_1) < f(x_2)$ for any $x_1 < x_2$, then the function is called strictly increasing on the interval $[a,b]$.
2. If $f(x_1) > f(x_2)$ for any $x_1 < x_2$, then the function is called strictly decreasing on the interval $[a,b]$.

A function strictly increasing or decreasing on an interval is called a strictly monotone function on that interval.

The monotony intervals of the trigonometric functions

Function $sin\,x$

0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
0	$\nearrow\nearrow$	1	$\searrow\searrow$	0	$\searrow\searrow$	-1	$\nearrow\nearrow$	0

Function $cos\,x$

0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
1	$\searrow\searrow$	0	$\searrow\searrow$	-1	$\nearrow\nearrow$	0	$\nearrow\nearrow$	1

Function $tan\,x$

0		$\frac{\pi}{2}$		π
0	$\nearrow\nearrow$	$+\infty$	$\searrow\searrow$	0

Function $cot\,x$

0		$\frac{\pi}{2}$		π
$+\infty$	$\searrow\searrow$	0	$\searrow\searrow$	$-\infty$

The domains of definition and the set of values for the trigonometric functions

$sin\,x : \mathbb{R} \to [-1,+1]$

$cos\,x : \mathbb{R} \to [-1,+1]$

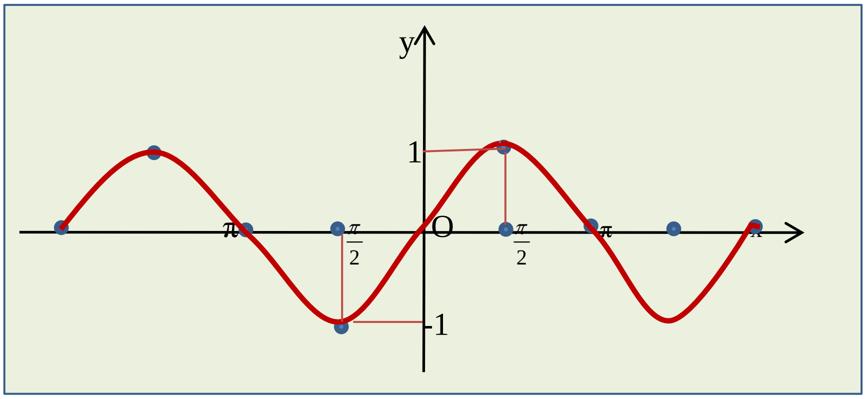
$tan\,x : \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\} \to \mathbb{R}$

$cot\,x : \mathbb{R} - \{ k\pi, k \in \mathbb{Z} \} \to \mathbb{R}$

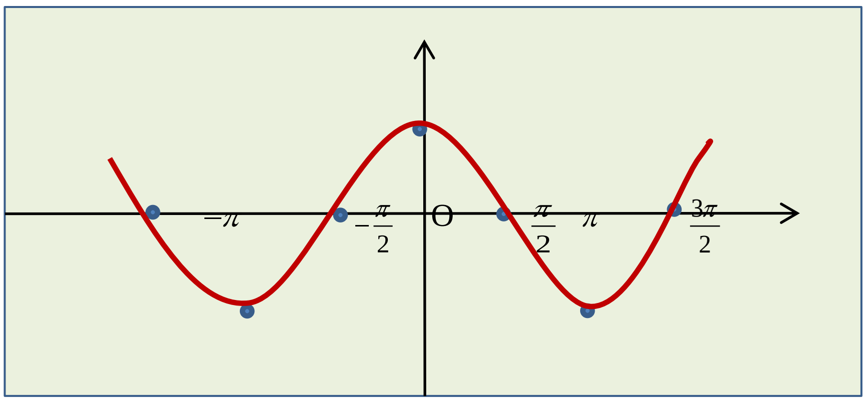
The graphics of the trigonometric functions

The graphics of $sin\,x$, $cos\,x$ are called sinusoidal

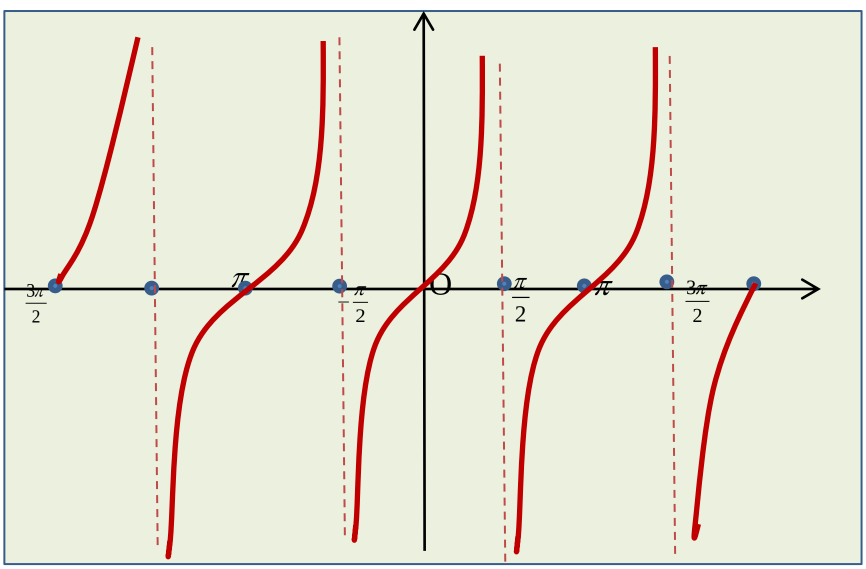
1. $f(x) = sin\,x$



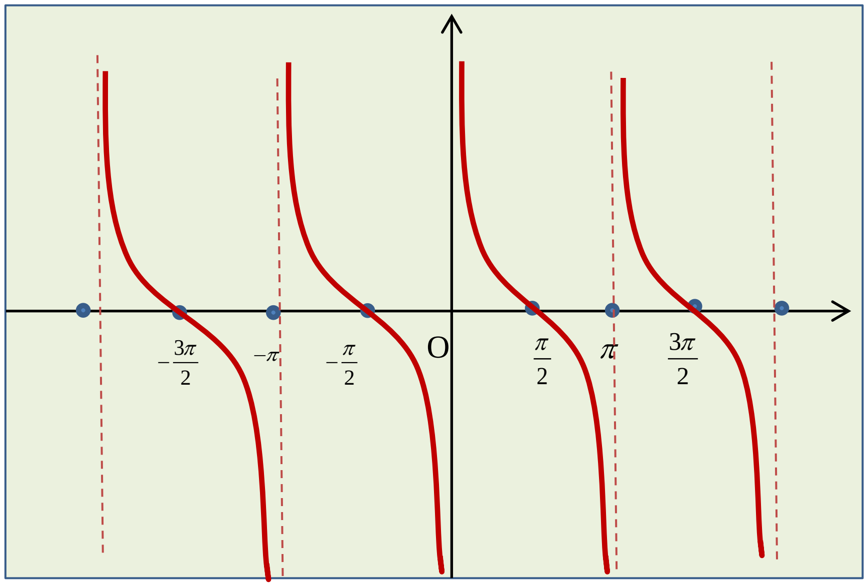
2. $f(x) = cos\,x$



3. $f(x) = tan\,x$



4. $f\left(x\right)=cot\,x$

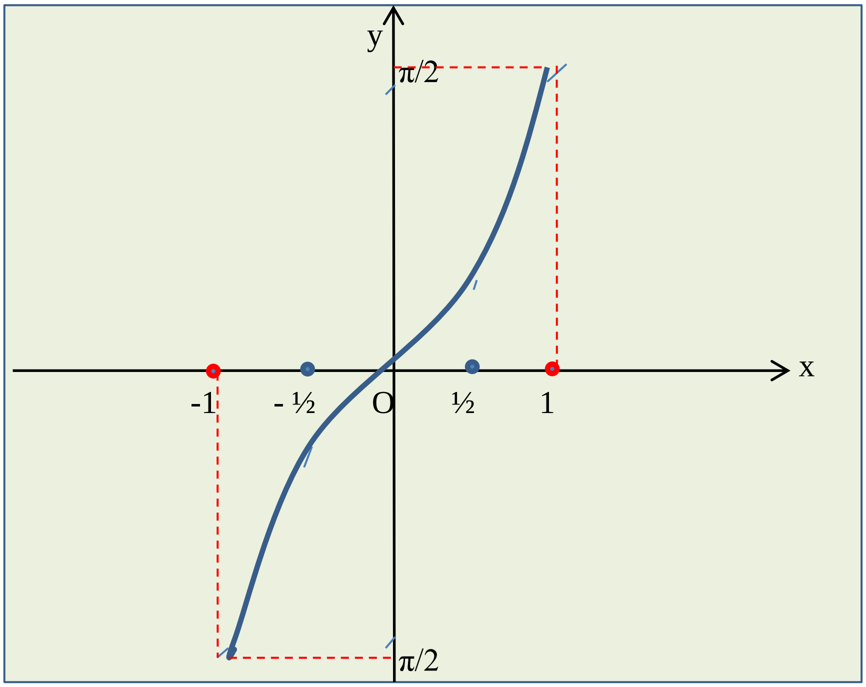


Trigonometric functions inverse

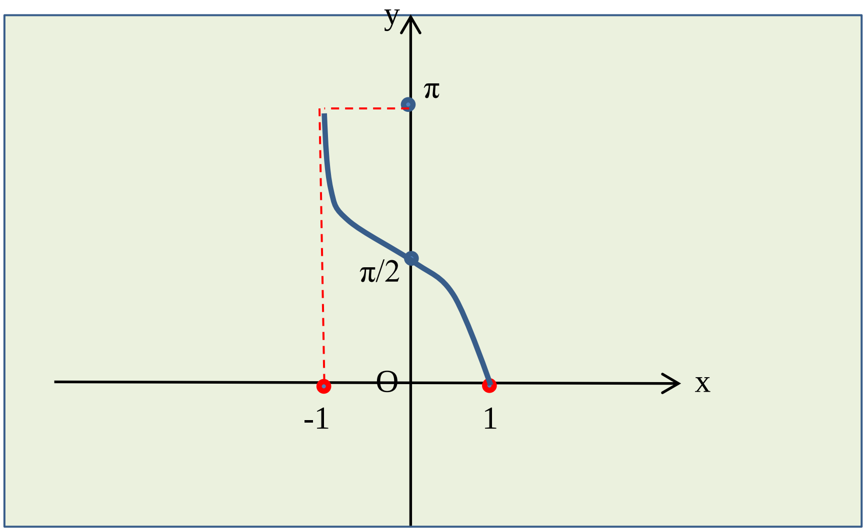
Any function which is strictly monotone on an interval can be inversed; the domain of definition and the set of the values of the function will be respectively the set of the values and the domain of definition of the inverse function.

If a function is strict increasing (strict decreasing) on an interval, then the inverse function will be also strict increasing (strict decreasing) on that interval.

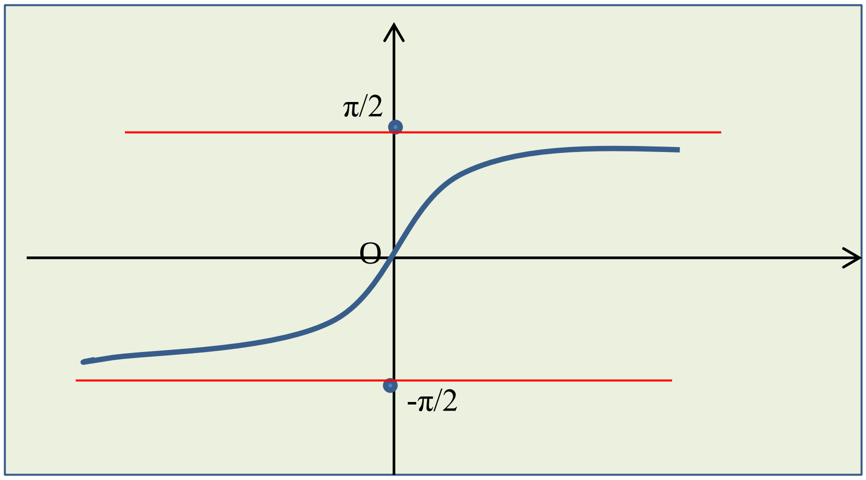
Function *arcsin x*



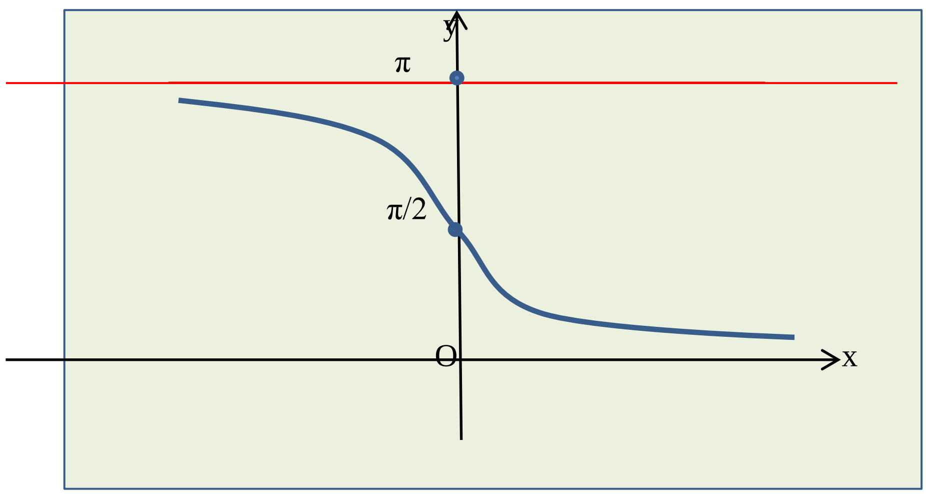
Function *arccos x*



Function $arctan\, x$



Function $arc\,cot\, x$



Fundamental formulae

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\csc\alpha$
$\sin\alpha$		$\pm\sqrt{1-\cos^2\alpha}$	$\frac{\tan\alpha}{\pm\sqrt{1+\tan^2\alpha}}$	$\frac{1}{\pm\sqrt{1+\cot^2\alpha}}$	$\pm\sqrt{1-\frac{1}{\sec^2\alpha}}$	$\frac{1}{\csc\alpha}$
$\cos\alpha$	$\pm\sqrt{1-\sin^2\alpha}$		$\frac{1}{\pm\sqrt{1+\tan^2\alpha}}$	$\frac{\cot\alpha}{\pm\sqrt{1+\cot^2\alpha}}$	$\frac{1}{\sec\alpha}$	$\pm\sqrt{1-\frac{1}{\csc^2\alpha-1}}$
$\tan\alpha$	$\frac{\sin\alpha}{\pm\sqrt{1-\sin^2\alpha}}$	$\frac{\pm\sqrt{1-\cos^2\alpha}}{\cos\alpha}$		$\frac{1}{\cot\alpha}$	$\pm\sqrt{\sec^2\alpha-1}$	$\pm\frac{1}{\sqrt{\csc^2\alpha-1}}$
$\cot\alpha$	$\frac{\pm\sqrt{1-\sin^2\alpha}}{\sin\alpha}$	$\frac{\cos\alpha}{\pm\sqrt{1-\cos^2\alpha}}$	$\frac{1}{\tan\alpha}$		$\pm\sqrt{\frac{1}{\sec^2\alpha-1}}$	$\pm\sqrt{\csc^2\alpha-1}$
$\sec\alpha$	$\frac{1}{\pm\sqrt{1-\sin^2\alpha}}$	$\frac{1}{\cos\alpha}$	$\pm\sqrt{1+\tan^2\alpha}$	$\frac{\pm\sqrt{1+\csc^2\alpha}}{\csc\alpha}$		$\frac{1}{\pm\sqrt{1-\frac{1}{\csc^2\alpha-1}}}$
$\csc\alpha$	$\frac{1}{\sin\alpha}$	$\frac{1}{\pm\sqrt{1-\cos^2\alpha}}$	$\frac{\pm\sqrt{1+\tan^2\alpha}}{\tan\alpha}$	$\pm\sqrt{1+\cot^2\alpha}$	$\frac{1}{\pm\sqrt{1-\frac{1}{\sec^2\alpha-1}}}$	

$$\sin^2\alpha+\cos^2\alpha=1$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$$

$$\cot\alpha=\frac{\cos\alpha}{\sin\alpha}$$

$$\tan\alpha\cdot\cot\alpha=1$$

$$1-\cos\alpha=2\sin^2\frac{\alpha}{2}$$

$$1+\cos\alpha=2\cos^2\frac{\alpha}{2}$$

$$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}$$

$$\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$$

$$1 - \sin 2\alpha = \left(\sin\alpha - \cos\alpha\right)^2$$

$$1 + \sin 2\alpha = \left(\sin\alpha + \cos\alpha\right)^2$$

$$\sec\alpha = \frac{1}{\cos\alpha}$$

$$\csc\alpha = \frac{1}{\sin\alpha}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$$

$$\tan\alpha\tan\beta = \frac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta + 1}{-\cot\alpha + \cot\beta}$$

$l = ar$
l is the lengths of the circle's arc
a is the measure in radian of the angle with the vertex in the center of the circle
r is the circle's radius.

$\omega = \frac{2\pi}{T}$; $\alpha = \omega t$
ω is he angular speed (in the circular and uniform movement)
t is the time
α is the angle

The trigonometric functions of a sum of three angles

$$sin\left(\alpha_1+\alpha_2+\alpha_3\right)=sin\,\alpha_1\,cos\,\alpha_2\,cos\,\alpha_3+sin\,\alpha_2\,cos\,\alpha_1\,cos\,\alpha_3+sin\,\alpha_3\,cos\,\alpha_1\,cos\,\alpha_2-sin\,\alpha_1\,sin\,\alpha_2\,sin\,\alpha_3$$

$$cos\left(\alpha_1+\alpha_2+\alpha_3\right)=cos\,\alpha_1\,cos\,\alpha_2\,cos\,\alpha_3-sin\,\alpha_1\,sin\,\alpha_2\,cos\,\alpha_3-sin\,\alpha_1\,sin\,\alpha_3\,cos\,\alpha_2-sin\,\alpha_3\,sin\,\alpha_2\,cos\,\alpha_1$$

$$tan\left(\alpha_1+\alpha_2+\alpha_3\right)=\frac{tan\,\alpha_1+tan\,\alpha_2+tan\,\alpha_3-tan\,\alpha_1\,tan\,\alpha_2\,tan\,\alpha_3}{1-tan\,\alpha_1\,tan\,\alpha_2-tan\,\alpha_1\,tan\,\alpha_3-tan\,\alpha_2\,tan\,\alpha_3}$$

$$cot\left(\alpha_1+\alpha_2+\alpha_3\right)=\frac{cot\,\alpha_1\,cot\,\alpha_2\,cot\,\alpha_3-cot\,\alpha_1-cot\,\alpha_2-cot\,\alpha_3}{cot\,\alpha_1\,cot\,\alpha_2+cot\,\alpha_1\,cot\,\alpha_3+cot\,\alpha_2\,cot\,\alpha_3-1}$$

The trigonometric functions of a double-angle

$$sin\,2\alpha=2\,sin\,\alpha\,cos\,\alpha$$

$$cos\,2\alpha=cos^2\,\alpha-sin^2\,\alpha=1-2\,sin^2\,\alpha=2\,cos^2\,\alpha-1$$

$$tan\,2\alpha=\frac{2\,tan\,\alpha}{1-tan^2\,\alpha}$$

$$cot\,2\alpha=\frac{cot^2\,\alpha-1}{2\,cot\,\alpha}$$

The trigonometric functions of a triple-angle

$$sin\,3\alpha=sin\,\alpha\left(3-4\,sin^2\,\alpha\right)=3\,sin\,\alpha-4\,sin^3\,\alpha$$

$$cos\,3\alpha=cos\,\alpha\left(4\,cos^2\,\alpha-3\right)=4\,cos^3\,\alpha-3\,cos\,\alpha$$

$$tan\,3\alpha=\frac{3\,tan\,\alpha-tan^3\,\alpha}{1-3\,tan^2\,\alpha}$$

$$cot\,3\alpha=\frac{cot^3\,\alpha-3\,cot\,\alpha}{3\,cot^2\,\alpha-1}$$

The trigonometric functions of a half-angle

$$sin\,\frac{\alpha}{2}=\pm\sqrt{\frac{1-cos\,\alpha}{2}}$$

$$cos\,\frac{\alpha}{2}=\pm\sqrt{\frac{1+cos\,\alpha}{2}}$$

$$an\,\frac{\alpha}{2}=\pm\sqrt{\frac{1-cos\,\alpha}{1+cos\,\alpha}}=\frac{sin\,\alpha}{1+cos\,\alpha}=\frac{1-cos\,\alpha}{sin\,\alpha}$$

$$cot\,\frac{\alpha}{2}=\pm\sqrt{\frac{1+cos\,\alpha}{1-cos\,\alpha}}=\frac{1+cos\,\alpha}{sin\,\alpha}=\frac{sin\,\alpha}{1-cos\,\alpha}$$

The trigonometric functions of an angle α in function of $tan\,\frac{\alpha}{2}=t$

$$sin\,\alpha=\frac{2\,t}{1+t^2}$$

$$cos\,\alpha = \frac{1-t^2}{1+t^2}$$

$$tan\,\alpha = \frac{2t}{1-t^2},$$

$$cot\,\alpha = \frac{1-t^2}{2t}$$

Transformation of sums of trigonometric functions in products

$$sin\,p + sin\,q = 2\,sin\frac{p+q}{2}cos\frac{p-q}{2}$$

$$sin\,p - sin\,q = 2\,sin\frac{p-q}{2}cos\frac{p+q}{2}$$

$$cos\,p + cos\,q = 2\,cos\frac{p+q}{2}cos\frac{p-q}{2}$$

$$cos\,p - cos\,q = 2\,sin\frac{q+p}{2}sin\frac{q-p}{2}$$

$$tan\,p + tan\,q = \frac{sin(p+q)}{cos\,p\,cos\,q}$$

$$tan\,p - tan\,q = \frac{sin(p-q)}{cos\,p\,cos\,q}$$

$$cot\,p + cot\,q = \frac{sin(p+q)}{cos\,p\,cos\,q}$$

$$cot\,p - cot\,q = \frac{sin(q-p)}{sin\,p\,sin\,q}$$

Transformation of products of trigonometric functions in sums

$$sin\,\alpha\,sin\,\beta = \frac{cos(\alpha-\beta) - cos(\alpha+\beta)}{2}$$

$$sin\,\alpha\,cos\,\beta = \frac{sin(\alpha+\beta) + sin(\alpha-\beta)}{2}$$

$$cos\,\alpha\,cos\,\beta = \frac{cos(\alpha+\beta) + cos(\alpha-\beta)}{2}$$

Relations between the arc functions

$$arc\,sin\,x + arccos\,x = \frac{\pi}{2}$$

$$arctan\,x + arc\,cot\,x = \frac{\pi}{2}$$

$$sin\left(arc\,sin\,x\right)=x$$

$$sin\left(arccos\,x\right)=\sqrt{1-x^2}$$

$$sin\left(arctan\,x\right)=\frac{x}{\sqrt{1+x^2}}$$

$$sin\left(arc\,cot\,x\right)=\frac{1}{\sqrt{1+x^2}}$$

$$cos\left(arccos\,x\right)=x$$

$$cos\left(arcsin\,x\right)=\sqrt{1-x^2}$$

$$cos\left(arctan\,x\right)=\frac{1}{\sqrt{1+x^2}}$$

$$cos\left(arc\,cot\,x\right)=\frac{x}{\sqrt{1+x^2}}$$

$$tan\left(arctan\,x\right)=x$$

$$tan\left(arcsin\,x\right)=\frac{x}{\sqrt{1-x^2}}$$

$$tan\left(arccos\,x\right)=\frac{\sqrt{1-x^2}}{x}$$

$$tan\left(arc\,cot\,x\right)=\frac{1}{x}$$

$$cot\left(arc\,cot\,x\right)=x$$

$$cot\left(arcsin\,x\right)=\frac{\sqrt{1-x^2}}{x}$$

$$cot\left(arccos\,x\right)=\frac{x}{\sqrt{1-x^2}}$$

$$cot\left(arctan\,x\right)=\frac{1}{x}$$

$$arcsin\left(sin\,\alpha\right)=\alpha$$

$$arccos\left(cos\,\alpha\right)=\alpha$$

$$arctan\left(tan\,\alpha\right)=\alpha$$

$$arc\,cot\left(cot\,\alpha\right)=\alpha$$

$$arcsin(cos\,\alpha) = \frac{\pi}{2} - \alpha$$

$$arccos(sin\,\alpha) = \frac{\pi}{2} - \alpha$$

$$arctan(cot\,\alpha) = \frac{\pi}{2} - \alpha$$

$$arc\,cot(tan\,\alpha) = \frac{\pi}{2} - \alpha$$

$$tan\,\alpha\, tan\,\beta = \frac{tan\,\alpha + tan\,\beta}{cot\,\alpha + cot\,\beta}$$

$$cot\,\alpha\, cot\,\beta = \frac{cot\,\alpha + cot\,\beta}{tan\,\alpha + tan\,\beta}$$

$$cot\,\alpha\, tan\,\beta = \frac{cot\,\alpha + tan\,\beta}{tan\,\alpha + cot\,\beta}$$

The computation of a sum of arc-functions

$$arcsin\,x + arcsin\,y = arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$arcsin\,x - arcsin\,y = arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$$

$$arccos\,x + arccos\,y = arccos\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$arccos\,x - arccos\,y = arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$arctan\,x + arctan\,y = arctan\frac{x+y}{1-xy}$$

$$arctan\,x - arctan\,y = arctan\frac{x-y}{1+xy}$$

$$arc\,cot\,x + arc\,cot\,y = arc\,cot\frac{xy-1}{x+y}$$

$$arc\,cot\,x - arc\,cot\,y = arc\,cot\frac{xy+1}{x-y}$$

Trigonometric sums

$$S_1 = sin\,a + sin(a+h) + ... + sin\left[a + (n-1)h\right] = \frac{sin\frac{nh}{2}\,sin\left[a + (n-1)\frac{h}{2}\right]}{sin\frac{h}{2}}$$

$$S_2 = cos\,a + cos\left(a+h\right) + ... + cos\left[a+\left(n-1\right)h\right] = \frac{sin\frac{nh}{2}cos\left[a+\left(n-1\right)\frac{h}{2}\right]}{sin\frac{h}{2}}$$

How is computed

$$S_2 + iS_2 = cos\,a + cos\left(a+h\right) + ... + cos\left[a+\left(n-1\right)h\right] +$$

$$+ i\,sin\,a + i\,sin\left(a+h\right) + ... + i\,sin\left[a+\left(n-1\right)h\right] =$$

$$= \left(cos\,a + i\,sin\,a\right) + \left[cos\left(a+h\right) + i\,sin\left(a+h\right)\right] + ... +$$

$$+ \left\{cos\left[a+\left(n-1\right)\frac{h}{2}\right] + i\,sin\left[a+\left(n-1\right)\frac{h}{2}\right]\right\} =$$

$$= \left(cos\,a + i\,sin\,a\right)\left\{1 + \left(cosh + i\,sinh\right)^1 + ... + \left(cosh + i\,sinh\right)^{n-1}\right\} =$$

$$= \left(cos\,a + i\,sin\,a\right)\frac{\left(cosh + i\,sinh\right)^n - 1}{\left(cosh + i\,sinh\right) - 1} = \left(cos\,a + i\,sin\,a\right)\frac{-2\,sin^2 n\frac{h}{2} + i\,sin\,n\frac{h}{2}}{-2\,sin^2\frac{h}{2} + i2\,sin\frac{h}{2}cos\frac{h}{2}} =$$

$$= \left(cos\,a + i\,sin\,a\right)\frac{+2i\,sin\,n\frac{h}{2}\left(i\,sin\,n\frac{h}{2} + cos\,n\frac{h}{2}\right)}{+2i\,sin\frac{h}{2}\left(i\,sin\frac{h}{2} + cos\frac{h}{2}\right)} =$$

$$= \left(cos\,a + i\,sin\,a\right)\frac{sin\frac{nh}{2}}{sin\frac{h}{2}}\left(cos\frac{n-1}{2}h + sin\frac{n-1}{2}h\right) =$$

$$= \frac{sin\frac{nh}{2}}{sin\frac{h}{2}}\left[cos\left(a + \frac{n-1}{2}h\right) + i\,sin\left(a + \frac{n-1}{2}h\right)\right] =$$

$$= \frac{sin\frac{nh}{2}}{sin\frac{h}{2}}cos\left(a + \frac{n-1}{2}h\right) + i\frac{sin\frac{nh}{2}}{sin\frac{h}{2}}sin\left(a + \frac{n-1}{2}h\right)$$

Then,

$$\begin{cases} S_2 = \dfrac{sin\frac{nh}{2}}{sin\frac{h}{2}}cos\left(a + \dfrac{n-1}{2}h\right) \\ S_1 = \dfrac{sin\frac{nh}{2}}{sin\frac{h}{2}}sin\left(a + \dfrac{n-1}{2}h\right) \end{cases}$$

$$S_3 = cos^2 x + cos^2 2x + ... + cos^2 nx = \frac{n}{2} + \frac{sin\, nx\, cos(n+1)x}{2\, sin\, x}$$

$$S_4 = sin^2 x + sin^2 2x + ... + sin^2 nx = \frac{n}{2} - \frac{sin\, nx\, cos(n+1)x}{2\, sin\, x}$$

$$S_5 = sin\, x + sin\, 2x + ... + sin\, nx = \frac{sin\frac{nx}{2}}{sin\frac{x}{2}} sin\left(\frac{n+1}{2}x\right)$$

$$S_6 = cos\, x + cos\, 2x + ... + cos\, nx = \frac{sin\frac{nx}{2}}{sin\frac{x}{2}} cos\left(\frac{n+1}{2}x\right)$$

Trigonometric functions of multiple angles

$$sin\, nx = C_n^1 cos^{n-1} x\, sin\, x - C_n^3 cos^{n-3} x\, sin^3 x + ...$$

$$cosnx = C_n^0 cos^n x - C_n^2 cos^{n-2} x\, sin^2 x + ...$$

$$tan\, nx = \frac{C_n^1 tan\, x - C_n^3 tan^3 x + ..}{1 - C_n^2 tan^2 x + ...}$$

$$cot\, nx = \frac{C_n^0 cot^n x - C_n^2 cot^{n-2} x + ..}{C_n^1 cot^{n-1} x - C_n^3 cot^{n-3} x + ...}$$

The power of the trigonometric functions

I)

1) $sin^{2p} x = \frac{(-1)^p}{2^{2p-1}}$.

$$.\left[cos\, 2px - C_{2p}^1 cos\, 2(p-1)x - C_{2p}^2 cos\, 2(p-2)x + ... + (-1)^{p-1} C_{2p}^{p-1} cos\, 2x\right] +$$

$$+ \frac{1}{2^{2p}} C_{2p}^p;$$

2) $sin^{2p+1} x = \frac{(-1)^p}{2^{2p}} \cdot$

$$\cdot\left[sin(2p+1)x - C_{2p+1}^1 sin(2p-1)x + C_{2p+1}^2 sin(2p-3)x + ... + (-1)^p C_{2p+1}^p sin\, x\right]$$

II)

1) $cos^{2p} x = \frac{1}{2^{2p-1}} \cdot$

$$\cdot\left[cos\, 2px + C_{2p}^1 cos\, 2(p-1)x + C_{2p}^2 cos\, 2(p-2)x + ... + C_{2p}^{p-1} cos\, 2x\right] +$$

$$+ \frac{1}{2^{2p}} C_{2p}^p$$

2) $$cos^{2p+1} x = \frac{1}{2^{2p}} \cdot$$
$$\cdot \left[cos\left(2p+1\right)x + C_{2p+1}^{1} cos\left(2p-1\right)x + C_{2p+1}^{2} cos\left(2p-3\right)x + ... + C_{2p+1}^{p} cos\, x \right]$$

Trigonometric identity

A trigonometric identity is the equality which contains the trigonometric functions of one or more angles and it is true for all their admissible values.

Conditional identities

Conditional identities are two trigonometric expressions which are not equal for the whole set of admissible values of their angles, but for a subset of it (which will satisfy certain conditions).

Solved Problem

Prove that the identity: $sin\left(arccos\, x\right) = \sqrt{1-x^2}$.

The equality takes place for $|x| \le 1$.

We denote: $arccos\, x = \alpha$, then $0 \le \alpha \le \pi$, and therefore $sin\,\alpha \ge 0$.

But $|sin\,\alpha| = \sqrt{1-cos^2\alpha} = \sqrt{1-cos^2\alpha\left(arccos\, x\right)} = \sqrt{1-x^2}$, therefore $sin\left(arccos\, x\right) = \sqrt{1-x^2}$.

Trigonometric equations

$$arcsin\, a = \left(-1\right)^k arcsin\, a + k\pi,\ \ k \in \mathbb{Z}$$
$$arccos\, a = \pm arccos\, a + 2k\pi,\ \ k \in \mathbb{Z}$$
$$arctan\, a = arctan\, a + k\pi,\ \ k \in \mathbb{Z}$$
$$arc\,cot\, a = arc\,cot\, a + k\pi,\ \ k \in \mathbb{Z}$$

(The $arcsin\, a$ is the set of all angles whose sin is equal to a.

$arcsin\, a$ is the angle $\alpha \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ whose sin is equal to a)

A trigonometric equation is the equality which contains the unknown only under the sign of the trigonometric function and which is true only for certain values of the unknown.

I. Trigonometric equations elementary

$$sin\, x = a \Rightarrow x = \left(-1\right)^k arcsin\, a + k\pi \quad \left(k \in \mathbb{Z}\right)$$
$$cos\, x = a \Rightarrow x = \pm arccos\, a + 2k\pi \quad \left(k \in \mathbb{Z}\right)$$
$$tan\, x = a \Rightarrow x = arctan\, a + k\pi \quad \left(k \in \mathbb{Z}\right)$$
$$cot\, x = a \Rightarrow x = arc\,cot\, a + k\pi \quad \left(k \in \mathbb{Z}\right)$$

II. The equality of two trigonometric functions of the same name

$sin\,u = sin\,v \Rightarrow u = (-1)^k \cdot v + k\pi \quad (k \in \mathbb{Z})$

$cos\,u = cos\,v \Rightarrow u = \pm v + 2k\pi \quad (k \in \mathbb{Z})$

$tan\,u = tan\,v \Rightarrow u = v + k\pi \quad (k \in \mathbb{Z})$

$cot\,u = cot\,v \Rightarrow u = v + k\pi \quad (k \in \mathbb{Z})$

III. Trigonometric equations of the form $sin\,f(x) = sin\,g(x)$

$sin\,f(x) = sin\,g(x) \Rightarrow f(x) = (-1)^k g(x) + k\pi \quad (k \in \mathbb{Z})$

IV. Trigonometric equations reducible to equations which contain the same function of the same angle.

- The trigonometric equation is reduced to an algebraic equation in which the unknown is a trigonometric function of the angle which needs to be determined.

V. Homogeneous equations in $sin\,x$ and $cos\,x$

$a_0\,sin^n\,x + a_1\,sin^{n-1}\,x\,cos\,x + a_2\,sin^{n-2}\,x\,cos^2\,x + ... + a_n\,cos^n\,x = 0$

- We divide the equation by $cos^n\,x$ and obtain an equation of n degree in $tan\,x$:

$$a_0\,tan^n\,x + a_1\,tan^{n-1}\,x + a_2\,tan^{n-2}\,x + ... + a_n = 0$$

(Dividing by $cos^n\,x$ there are no solutions lost).

VI. The linear equation in $sin\,x$ and $cos\,x$

1) The method of the auxiliary angle

$a\,sin\,x + b\,cos\,x = c \quad |:a$

$sin\,x + \frac{b}{a}cos\,x = \frac{c}{a}$

Denote: $sin\,x + \frac{b}{a} = tan\,\varphi \quad -\frac{\pi}{2} < \varphi < +\frac{\pi}{2}$

$sin\,x + tan\,\varphi\,cos\,x = \frac{c}{a}$

$sin\,x\,cos\,\varphi + sin\,\varphi\,cos\,x = \frac{c}{a}cos\,\varphi$

$sin(x+\varphi) = \frac{c}{a}cos\,\varphi$

But $cos\,\varphi = \frac{1}{\sqrt{1+tan^2\,\varphi}} = \frac{|a|}{\sqrt{a^2+b^2}}$

Therefore $sin(x+\varphi) = \frac{c|a|}{a\sqrt{a^2+b^2}} = \frac{c}{\pm\sqrt{a^2+b^2}}$

We take the positive sign if $a > 0$ and the negative sign, when $a < 0$.

The equation has solutions if $\left|\frac{c}{\pm\sqrt{a^2+b^2}}\right| \le 1$, that is if $c^2 \le a^2 + b^2$

$x = (-1)^k\,arcsin\frac{c}{\pm\sqrt{a^2+b^2}} + k\pi - arctan\frac{b}{a} \quad (k \in \mathbb{Z})$

2) The substitution method

We substitute $sin\,\alpha = \frac{2t}{1+t^2}$; $cos\,\alpha = \frac{1-t^2}{1+t^2}$

$a\,sin\,x + b\,cos\,x = c$

$a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2} = c$

$(b+c)t^2 - 2at + (c-b) = 0$

Conditions: $c+b \neq 0$ and $\Delta \geq 0$

$\Delta = a^2 + b^2 - c^2 \geq 0 \Rightarrow c^2 \leq a^2 + b^2$

$\Delta = a^2 + b^2 - c^2 \geq 0 \Rightarrow c^2 \leq a^2 + b^2 \Rightarrow$

$\Rightarrow t = \frac{a \pm \sqrt{a^2 + b^2 -} c^2}{b+c} = tan\frac{\alpha}{2}$

$\Rightarrow x = 2\left(arctan\frac{a \pm \sqrt{a^2 + b^2 -} c^2}{b+c} + k\pi\right) \quad (k \in \mathbb{Z})$

Performing these substitutions we can lose solutions (values of angle x for which $tan\frac{x}{2}$ doesn't have sense: $\frac{\pi}{2} + k\pi \ (k \in \mathbb{Z})$, for this reason, after computation, we need to verify these values as well; if they verify, we add them to the set of solutions.

3) We can solve the system:

$$\begin{cases} a\,sin\,x + b\,cos\,x = c \\ sin^2 x + cos^2 x = 1 \end{cases}$$

We eliminate the foreign solution by verification, $sin\,x$ and $cos\,x$ are the unknown.

VII. Equation that can be resolved by decomposition in factors.

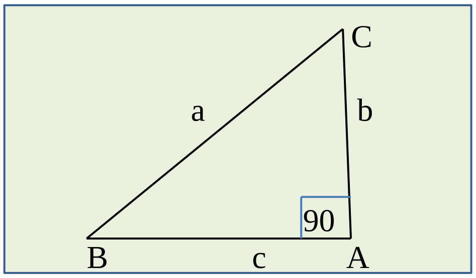
In some situations, the equations are resolved by appealing to the squaring process, but care has to be taken not to introduce foreign solutions.

Systems of trigonometric equations

- We do certain substitutions
- We decompose the products in sums or vice versa
- We use the fundamental trigonometric formulae.

Trigonometric applications in geometry

1. **Rectangle triangle**



$\sphericalangle B + \sphericalangle C = 90°$

$\sphericalangle A = 90°$

2. **Random triangle**

The sine theorem

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

The cosine theorem

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The tangent theorem

$$\frac{a-b}{a+b} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}$$

The trigonometric functions of the angles of a triangle expressed using the sides of the triangle.

$$\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}},...$$

$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}},...$$

$$\tan\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}},...$$

$$\cot\frac{A}{2} = \sqrt{\frac{p(p-a)}{(p-b)(p-c)}},...$$

$$\cos A = \frac{a^2+b^2-c^2}{2bc}$$

Relations in a random triangle

$$R = \frac{abc}{4S}$$

S is the aria of the triangle

Euler's elation: $OI^2 = R(R-2r)$; O the center of the circumscribed circle, I is the center of the inscribed circle.

$$r = \frac{S}{p}$$

$$a = b \cdot cos\, C + c \cdot cos\, B, \dots$$

$$r = 4R\, sin\frac{A}{2} sin\frac{B}{2} sin\frac{C}{2} = p\, tan\frac{A}{2} tan\frac{B}{2} tan\frac{C}{2} = \frac{S}{p} = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}}$$

$$r_a = 4R\, sin\frac{A}{2} sin\frac{B}{2} sin\frac{C}{2} = p\, tan\frac{A}{2} = \frac{S}{p-a} = \sqrt{\frac{p(p-b)(p-c)}{(p-a)}}, \dots$$

r_a is the radius of the triangle's ex-inscribed circle.

$$p = 4R\, cos\frac{A}{2} cos\frac{B}{2} cos\frac{C}{2}$$

$$r_a + r_b + r_c = 4R + r$$

$$b_a = \frac{2bc}{b+c} cos\frac{A}{2} = \frac{2R\, sin\, B\, sin\, C}{cos\frac{B-C}{2}} = \frac{2\sqrt{bcp(p-a)}}{b+c}$$

$$b_a = \frac{c\, sin\, B}{cos\frac{B-C}{2}}, \dots$$ b_a is the bisector of $\measuredangle A$

$$m_a = \frac{1}{2}\sqrt{2(b^2+c^2)+a^2}, \dots$$

$$S = \sqrt{p(p-a)(p-b)(p-c)}$$ - Heron's formula

$$S = \frac{abc\, sin\, C}{2} = \frac{a^2\, sin\, B\, sin\, C}{2\, sin(B+C)}, \dots$$

Aria of a quadrilateral

$$S_{\square} = \frac{d_1 d_2\, sin\,\varphi}{2}$$ φ is the angle between the diagonals d_1, d_2

$$d_1 d_2 \le ac + bd$$

Trigonometric tables

To find the values of the trigonometric functions of various angles one uses the trigonometric tables, where these are listed.

Interpolation

Interpolation is an operation through which we can determine the trigonometric function's values of angles which cannot be found in the trigonometric tables.

Example:

$sin\,34°20' = 0.56401$

$sin\,34°30' = 0.56641$

How much is $sin\,34°22' = ?$

$sin\,34°20' < sin\,34°22' < sin\,34°30'$

$34°30' - 34°20' ....0.56641 - 0.56401$

$34°22' - 34°20' ....x$ $\qquad \Rightarrow sin\,34°22' = 0.56401 + 0.00048 = 0.56449$

$$x = \frac{2 \cdot 0.00240}{10} = 0.00048$$

Logarithmic tables of trigonometric functions

Example $ln\,sin\,17°34' = \bar{1}.47974$

Complex numbers under a trigonometric form

$i = \sqrt{-1}$

A complex number is a number of the format: $z = x + iy$, $x, y \in \mathbb{R}$

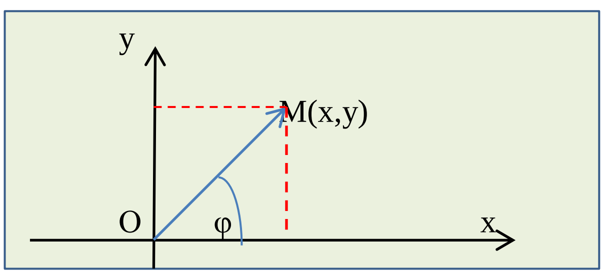
$i^{4m} = 1;\ i^{4m+1} = i;\ i^{4m+2} = -1;\ i^{4m+3} = -i$, $m \in \mathbb{Z}$

x is the real part

iy is the imaginary part

y is the coefficient of the imaginary part.

There is bi-univocal correspondence between the set of complex numbers $z = x + iy$ and the set of points $M(x, y)$ from plane.



The module of a complex number is the length of the segment OM, $|z|$

The argument of a complex number is the angle formed by the vector OM the x axis; $arg\,z$.

A complex plane is the plane in which we represent the complex numbers $z = x + iy$.

$|z| = \sqrt{x^2 + y^2}\ \ (= \rho)$

$tan\,\varphi = \frac{y}{x}$ or $sin\,\varphi = \frac{x}{\rho}; cos\,\varphi = \frac{y}{\rho}$

$z = \rho(cos\,\varphi + i\,sin\,\varphi)$.

The conjugate of the complex number $z = x + iy$ is the complex number $\bar{z} = x - iy$; $\bar{z} = |z|$; $arg\,\bar{z} = 2\pi - arg\,z$.

The sum and the difference of the complex numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = x_1 + x_2 + i\left(y_1 + y_2\right)$$

$$\left(x_1 - x_2\right) + i\left(y_1 - y_2\right)$$

The multiplication of complex numbers

$$z_1 \cdot z_2 = \left(x_1 x_2 - y_1 y_2\right) + i\left(x_1 y_2 + x_2 y_1\right)$$

The multiplication is

- Commutative
- Associative
- Distributive relative to the sum of the complex numbers.

The module of the product of two complex numbers is equal to the product of their modules, and the product argument is equal to the sum of the arguments:

$$\left|z_1 \cdot z_2\right| = \left|z_1\right| \cdot \left|z_2\right|;$$

$$arg\left(z_1 \cdot z_2\right) = arg\, z_1 + arg\, z_2$$

$$z_1 = \rho_1\left(cos\,\varphi_1 + i\,sin\,\varphi_1\right)$$

$$z_2 = \rho_2\left(cos\,\varphi_2 + i\,sin\,\varphi_2\right)$$

$$z_1 \cdot z_2 = \rho_1 \rho_2\left[cos\left(\varphi_1 + \varphi_2\right) + i\,sin\left(\varphi_1 + \varphi_2\right)\right]$$

Generalization:

$$z_1 \cdot z_2 ... z_n = \rho_1 \rho_2 ... \rho_n\left[cos\left(\varphi_1 + \varphi_2 + ... + \varphi_n\right) + i\,sin\left(\varphi_1 + \varphi_2 + ... + \varphi_n\right)\right]$$

The division of the complex numbers

$$\left|\frac{z_1}{z_2}\right| = \frac{\left|z_1\right|}{\left|z_2\right|}$$

$$arg\left(\frac{z_1}{z_2}\right) = arg\, z_1 - arg\, z_2$$

$$\frac{z_1}{z_2} = -\frac{\rho_1}{\rho_2}\left[cos\left(\varphi_1 - \varphi_2\right) + i\,sin\left(\varphi_1 - \varphi_2\right)\right]$$

The power of the complex numbers (Moivre relation)

$$\left|z^n\right| = \left|z\right|^n$$

$$arg\left(z^n\right) = arg\left(nz\right)$$

$$z^n = \rho^n\left(cos\,n\varphi + i\,sin\,n\varphi\right)\ n \in \mathbb{Z}$$

The root of n order from a complex number

$$\left|\sqrt[n]{z}\right| = \left|z\right|^{\frac{1}{n}}$$

$$arg\sqrt[n]{z}=\frac{arg\,z+2k\pi}{n};k=0,1,2,...,n-1$$

$$\sqrt[n]{\rho\left(cos\,\varphi+i\,sin\,\varphi\right)}=\rho^{\frac{1}{n}}\left(cos\frac{\varphi+2k\pi}{n}+i\,sin\frac{\varphi+2k\pi}{n}\right);k=0,1,2,...,n-1$$

Binomial equations

A binomial equation is an equation of the form $z^n+\alpha=0$, where α is a complex number and z is the unknown $z\in\mathbb{R}$.

Example: Solve the following equation

$$\left(\frac{1+iz}{1-iz}\right)^4=\frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\frac{1}{2}-\frac{\sqrt{3}}{2}i}$$

$$\rho_1=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$tan\,\varphi_1=\frac{\sqrt{3}}{2}\cdot\frac{2}{1}=\sqrt{3}\Rightarrow\varphi_1=\frac{\pi}{3}$$

$$z_1=\frac{1}{2}-\frac{\sqrt{3}}{2}i=cos\frac{\pi}{3}+i\,sin\frac{\pi}{3}$$

$$\varphi_2=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$arg\,z_2=2\pi-arg\,z_1=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}=\rho_2$$

$$\frac{z_1}{z_2}=cos\left(\frac{-4\pi}{3}\right)+i\,sin\left(\frac{-4\pi}{3}\right)=cos\frac{4\pi}{3}-i\,sin\frac{4\pi}{3}=cos\frac{2\pi}{3}+i\,sin\frac{2\pi}{3}$$

$$\frac{1+iz}{1-iz}=\sqrt[4]{cos\frac{2\pi}{3}+i\,sin\frac{2\pi}{3}}=cos\frac{\frac{2\pi}{3}+2k\pi}{4}+i\,sin\frac{\frac{2\pi}{3}+2k\pi}{4}$$

$$\frac{1+iz}{1-iz}=cos\,\alpha+i\,sin\,\alpha\,;\ \alpha=\left(\frac{2\pi}{3}+2k\pi\right)\cdot\frac{1}{4}$$

$$cos\,\alpha+i\,sin\,\alpha-iz\,cos\,\alpha-T-iz=0$$

$$z=\frac{cos\,\alpha+i\,sin\,\alpha-1}{i\left(cos\,\alpha+1\right)-sin\,\alpha}=\frac{-2\,sin^2\frac{\alpha}{2}+i-2\,sin\frac{\alpha}{2}cos\frac{\alpha}{2}}{-2\,sin\frac{\alpha}{2}cos\frac{\alpha}{2}+2\,cos^2\frac{\alpha}{2}}=$$

$$=\frac{2i\,sin\frac{\alpha}{2}\left(cos\frac{\alpha}{2}+i\,sin\frac{\alpha}{2}\right)}{2i\,cos\frac{\alpha}{2}\left(cos\frac{\alpha}{2}+i\,sin\frac{\alpha}{2}\right)}=tan\frac{\alpha}{2}$$

$$z = tan\frac{\frac{2\pi}{3}+2k\pi}{2\cdot 4} = tan\frac{\frac{\pi}{3}+k\pi}{4} \qquad k=0,1,2,3$$

$$\begin{cases} k=0 \Rightarrow z_1 = tan\frac{\pi}{12} \\ k=1 \Rightarrow z_2 = tan\frac{\pi}{3} \\ k=2 \Rightarrow z_3 = tan\frac{7\pi}{12} \\ k=3 \Rightarrow z_4 = tan\frac{5\pi}{6} \end{cases}$$

Cebyshev polynomials

$T_n(x) = cos\, n\, arccos\, x$, where $n \in \mathbb{N}$, are polynomials of n degree, defined on $[-1,+1]$; these satisfy the relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$arctan\frac{1}{2} + arctan\frac{1}{2\cdot 2^2} + ... + arctan\frac{1}{2\cdot 2^n} = arctan\frac{n}{n+1}$$

$$\frac{1+tan\,\alpha}{1-tan\,\alpha} = tan\left(\alpha+\frac{\pi}{4}\right)$$

- If $\alpha+\beta+\gamma=\pi$ then $sin\,\alpha + sin\,\beta + sin\,\gamma = 4cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\gamma}{2}$
- If $\alpha+\beta+\gamma=\pi$ then $tan\,\alpha + tan\,\beta + tan\,\gamma = tan\,\alpha\, tan\,\beta\, tan\,\gamma$
- If $\alpha+\beta+\gamma=\pi$ and are positive, then $cos\,\alpha + cos\,\beta + cos\,\gamma = 1 + 4sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}$.

Sometimes the powers of the trigonometric fractions are transformed in linear equations. Example:

$sin^4 x = \left(sin^2 x\right)^2 = \left(\frac{1-cos^2 x}{2}\right)^2$, etc.

$\left(sin\,x - cos\,x\right)^2 - 1 - sin\,2x$.

Sometimes we multiply both members of a relation with the same factor, but one has to make sure that foreign solutions are not introduced.

$$\left(sin\,x - cos\,x\right)^2 - 1 - \frac{1}{2}sin^2 2x$$

Since childhood I got accustomed to study with a pen in my hand.
I extracted theorems and formulas, together with the definitions, from my textbooks.
It was easier, later, for me, to prepare for the tests, especially for the final exams at the end of each semester.

I kept (and still do today) small notebooks where I collected not only mathematical but any idea I read from various domains.

These two volumes reflect my 1973-1974 high school studies in mathematics at the Pedagogical High School of Rm. Vâlcea, Romania.
Besides the textbooks I added information I collected from various mathematical books of solved problems I was studying at that time.

The first volume contains: *Arithmetic*, *Plane Geometry*, and *Space Geometry*.
The second volume contains: *Algebra (9th to 12th grades)*, and *Trigonometry*.

The Author

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