Three conjectures about semiprimes inspired by a recurrent formula involving 2-Poulet numbers

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Abstract. Studying the relation between the two prime factors of a 2-Poulet number I found an interesting recurrent formula involving these numbers that seems to lead often to a value which is semiprime; based on this observation I made three conjectures about semiprimes.

Observation 1

We take a pair of 2-Poulet numbers which have a common prime factor, as for instance the pair [P_1 = 341 = 11*31, P_2 = 4681 = 31*151] or the pair [P_1 = 1387 = 19*73, P_2 = 2701 = 37*73] and apply on it the recurrent formula $P_n = P_{n-2} + \gcd((P_{n-2} - 1), (P_{n-1} - 1))$.

In the case [P_1, P_2] = [341, 4681] we have:

: P_3 = 341 + gcd(340, 4680) = 361;
: P_4 = 4681 + gcd(4680, 360) = 5041;
: P_5 = 361 + gcd(360, 5040) = 721;
: P_6 = 5041 + gcd(5040, 720) = 5761;
: P_7 = 721 + gcd(720, 5760) = 1441;
: P_8 = 5761 + gcd(5760, 1440) = 7201;
: P_9 = 1441 + gcd(1440, 7200) = 2881;
: P_{10} = 7201 + gcd(7200, 2880) = 8641;
: P_{11} = 2881 + gcd(2880, 8640) = 5761;
: P_{12} = 8641 + gcd(8640, 5760) = 11521;
: P_{13} = 5760 + gcd(5760, 11520) = 11521.

Starting from P_{14}, we will have $P_{14} = P_{15} = 2*(P_{12} - 1) + 1$, $P_{16} = P_{17} = 2*(P_{14} - 1) + 1$ and so on.

In the case [P_1, P_2] = [1387, 2701] we have:

: P_3 = 1387 + gcd(340, 4680) = 1405;
: P_4 = 2701 + gcd(4680, 360) = 2809;
: P_5 = 1405 + gcd(360, 5040) = 2809;

Starting from P_6, we will have $P_6 = P_7 = 2*(P_4 - 1) + 1$ = 5617, $P_8 = P_9 = 2*(P_{12} - 1) + 1$ = 11233, $P_{10} = P_{11} = 2*(P_8 - 1) + 1$ = 22465, $P_{12} = P_{13} = 2*(P_{10} - 1) + 1$ = 44929, $P_{14} = P_{15} = 2*(P_{12} - 1) + 1$ = 89857, $P_{16} = P_{17} = 2*(P_{14} - 1) + 1$ = 179713 and so on.

It can be seen that many of the values of the terms P_i are semiprimes: 361 = 19*19, 5041 = 71*71, 721 = 7*103, 5761 = 7*823, 1441 = 11*131, 7201 = 19*379, 2881 = 43*67, 11521 = 41*281, 1405 = 5*281, 2809 = 53*53, 5617 = 41*137, 11233 = 47*239, 22465 = 5*4493, 44929 = 179*251, 89857 = 59*1523, 179713 = 29*6197.

More than that, between the two distinct prime factors p and q from many of the semiprimes obtained above there exist the relation q – p + 1 = n, where n is a prime or a square of a prime:

: 103 – 7 + 1 = 97;
: 131 – 11 + 1 = 121 = 11^2;
: 379 – 19 + 1 = 361 = 19^2;
: 67 – 43 + 1 = 25 = 5^2;
: 281 – 41 + 1 = 41;
: 281 – 5 + 1 = 277;
: 67 – 43 + 1 = 25 = 5^2;
: 137 – 41 + 1 = 97;
: 239 – 47 + 1 = 193;
: 4493 – 5 + 1 = 4489 = 67^2
: 251 – 179 + 1 = 73.

A very interesting thing it happens even if between p and q there is not the relation from the preceding paragraph; in many of these cases q – p + 1 = n, where n is a semiprime whose two prime factors admit themselves the relation showed:

: 823 – 7 + 1 = 817 = 19*43 and 43 – 19 + 1 = 25 = 5^2;
: 1523 – 59 + 1 = 1465 = 5*293 and 293 – 5 + 1 = 17^2;
: 6197 – 29 + 1 = 6169 = 31*199 and 199 – 31 + 1 = 13^2.

Observation 2

We also observed that the iterative formula $a_{n+1} = 2*(a_n - 1) + 1$, where a_1 is a square of a prime minus nine, seems likewise to often conduct to primes, power of primes or semiprimes with the characteristics of those from Observation 1.

For a_1 = 7^2 – 9 = 40 we obtain the following sequence:

: 79, 157, 313, 625, 1249, 2497, 4993, 9985, 19969, 39937, 79873, 159745, 319489, 638977, 1277953 (...), where:
: 79, 157, 313, 1249, 4993, 39937, 79873, 319489, 638977 are primes;
: 625 = 5^4 is a power of prime;
: 2497 = 11*227; 227 – 11 + 1 = 217 = 7*31; 31 – 7 + 1 = 25 = 5^2;
: 9985 = 5*1997; 1997 – 5 + 1 = 1993 prime;
: 19969 = 19*1051; 1051 – 19 + 1 = 1033 prime.
: 1277953 = 101*12653; 12653 – 101 + 1 = 12553 prime.

Conjecture 1

For any odd prime n there exist an infinity of pairs of odd primes [p, q] such that q – p + 1 = n.

Conjecture 2

For any semiprime p_1*q_1, where p_1 and q_1 are odd distinct primes, there exist an infinity of pairs of odd primes [p_2, q_2] such that $q_2 - p_2 + 1 = p_1*q_1$.

Conjecture 3

For any odd prime n there exist an infinity of pairs of odd primes [p_i, q_i], for any i from 1 to infinite, such that:
: $q_1 - p_1 + 1 = n$;
: $q_2 - p_2 + 1 = p_1*q_1$;
: $q_3 - p_3 + 1 = p_2*q_2$;
(...)
: $q_i - p_i + 1 = p_{i-1}*q_{i-1}$.

Note:

This is an interesting way to construct (possible) infinite sequences of semiprimes p_i*q_i, starting from a given prime and considering, for instance, the smallest p_i for which the relations from Conjecture 3 are verified. For instance, in the conditions mentioned, we take n = 13. We have:

: $p_1 = 5$ because is the smallest prime such that $n - 1 + p_1 = q_1$ is prime, so $q_1 = 13 - 1 + 5 = 17$;
: $p_2 = 5$ because is the smallest prime such that $p_1*q_1 - 1 + p_2 = q_2$ is prime, so $q_2 = 5*17 - 1 + 5 = 89$;
: $p_3 = 5$ because is the smallest prime such that $p_2*q_2 - 1 + p_3 = q_3$ is prime, so $q_3 = 5*89 - 1 + 5 = 449$;
: $p_4 = 7$ because is the smallest prime such that $p_3*q_3 - 1 + p_4 = q_4$ is prime, so $q_4 = 5*449 - 1 + 7 = 2251$ (...).

We obtained the following sequence of semiprimes p_i*q_i: 85, 445, 2245, 15757 (...).