On the Failure of Weyl's 1918 Theory

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Abstract

In 1918 the German mathematician Hermann Weyl developed a non-Riemannian geometry in which electromagnetism appeared to emerge naturally as a consequence of the non-invariance of vector magnitude. Although an initial admirer of the theory, Einstein declared the theory unphysical on the basis of the non-invariance of the line element ds, which is arbitrarily rescaled from point to point in the geometry. We examine the Weyl theory and trace its failure to its inability to accommodate certain vectors that are inherently scale invariant. A revision of the theory is suggested that appears to refute Einstein's objection.

As is well known, in early 1918 the noted German mathematician Hermann Weyl proposed a generalization of Riemannian geometry that appeared to unify gravitation and electromagnetism, the only forces of nature known at the time. Weyl accomplished this by abandoning one of the tenets of Riemannian geometry, the invariance of vector magnitude or length under physical transport. In doing so, Weyl was obligated to introduce a new vector quantity into his geometry that he subsequently identified as the electromagnetic four-vector. Weyl's theory gave rise to the geometry known as *Weyl space*, characterized by a non-metricity tensor along with invariance of the geometry under a local rescaling of the symmetric metric tensor $g_{\mu\nu} \rightarrow e^{\pi(x)} g_{\mu\nu}$. Weyl's theory was later recognized as the origin of *gauge theory*.

Although an initial admirer of the theory, Einstein argued that it was non-physical, noting that the invariant line element ds, viewed either as a local clock or measuring stick, would itself be rescaled from point to point in accordance with $ds \rightarrow e^{\pi/2} ds$. Einstein claimed that the spacing of atomic spectral lines and related phenomena would vary with time according to Weyl's theory, in obvious contradiction to experience. Unable to defend his theory from Einstein's criticism, Weyl ultimately abandoned it, though he successfully resurrected the gauge idea in 1929 when he and others applied it to quantum mechanics.

Despite its early failure, the beauty of Weyl's idea and its underlying mathematics struck a chord in the physics community, and from the theory's inception to the present day it has been explored by scores of researchers, including Dirac and Eddington. Today, Weyl's notions of conformal invariance have potential application to modern cosmological problems like dark matter.

In this brief and very elementary discussion we examine what appears to be the real underlying cause of the theory's failure and offer a revision that eliminates one obvious problem with the theory, though it is debatable as to whether it adequately addresses Einstein's criticism.

1. Notation

Following Adler et al., we denote ordinary partial differentiation with a single subscripted bar, while covariant differentiation is denoted using double subscripted bars. The coefficient of connection in Weyl space is representated by the symbol Γ, while its representation in Riemannian space is the usual Christoffel bracket. Thus, the Weyl covariant derivative of the mixed tensor $F^{\lambda}_{\ \alpha}$ is given as

$$F^{\lambda}_{\ \alpha||\beta} = F^{\lambda}_{\ \alpha|\beta} - F^{\lambda}_{\ \mu}\Gamma^{\mu}_{\alpha\beta} + F^{\mu}_{\ \alpha}\Gamma^{\lambda}_{\mu\beta}$$

while for Riemannian space we just make the substitution

$$\Gamma^{\mu}_{\alpha\beta} \rightarrow \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}$$

where

$$\left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu|\beta} + g_{\nu\beta|\alpha} - g_{\alpha\beta|\nu} \right)$$

Like the metric tensor $g_{\mu\nu}$, the connection is taken to be symmetric in its lower indices.

2. Review of Weyl's 1918 Theory

With Cartan, Weyl pioneered the concept of parallel transfer in differential geometry by assuming that the infinitesimal change in an arbitrary contravariant vector ξ^μ under physical transport is proportional to the transport interval dx and the vector itself, or

$$d\xi^\mu = -\Gamma^\mu_{\alpha\beta}\,\xi^\alpha dx^\beta \tag{2.1}$$

(The differential quantity $d\xi^\mu$ should not be viewed as a total differential, but in what follows the difference is not important.) Weyl noted that the magnitude L of a vector, expressed as $L^2 = g_{\mu\nu}\xi^\mu\xi^\nu$, might change under parallel transport, and he wrote

$$2L\,dL = g_{\mu\nu|\alpha}\xi^\mu\xi^\nu dx^\alpha dx^\alpha - g_{\mu\nu}\xi^\mu d\xi^\nu dx^\alpha - g_{\mu\nu}\xi^\nu d\xi^\mu dx^\alpha$$

After using the Weyl transport law (2.1) and relabeling indices, this becomes

$$\begin{aligned} 2L\,dL &= \left(g_{\mu\nu|\alpha} - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha}\right)\xi^\mu\xi^\nu dx^\alpha \\ &= g_{\mu\nu||\alpha}\xi^\mu\xi^\nu dx^\alpha \end{aligned}$$

The quantity $g_{\mu\nu||\alpha}$ is the *nonmetricity tensor*, which vanishes identically in Riemannian space. We will see that the entirety of Weyl's theory is based on a non-zero $g_{\mu\nu||\alpha}$, as it alone is responsible for $dL \neq 0$.

Since Weyl had no idea what comprised his connection, he reasonably assumed that, like the transport law, the differential change in length dL was also linear with respect to the vector and the distance. He thus wrote

$$dL = L\,\phi_\alpha dx^\alpha \tag{2.2}$$

where ϕ_α was some as yet undefined vector quantity. With the use of (2.2) Weyl was able to express the non-metricity tensor as

$$g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha \tag{2.3}$$

He also noted that the resulting expression for the change in length $dL = L\phi_\alpha dx^\alpha$ could be immediately integrated to give

$$L = L_0\,e^{\int \phi_\alpha dx^\alpha} \tag{2.4}$$

where L_0 is the initial vector length.

3. Gauge Invariance in Weyl's Theory

Weyl's assumed identity for the non-metricity tensor (2.3) allowed him to identify the components of his connection. By taking cyclic permutations of the expanded form

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha}$$

Weyl was able to show that

$$\Gamma^\lambda_{\nu\alpha} = \left\{ \begin{matrix} \lambda \\ \nu\alpha \end{matrix} \right\} - \delta^\lambda{}_\nu\phi_\alpha - \delta^\lambda{}_\alpha\phi_\nu + g_{\nu\alpha}g^{\lambda\beta}\phi_\beta \tag{3.2}$$

It was at this point that Weyl made a critical observation. If the metric tensor is made to undergo the infinitesimal local rescaling

$$\hat{g}_{\mu\nu} = e^{\epsilon\pi(x)}g_{\mu\nu} = (1+\epsilon\pi)g_{\mu\nu}$$

then the Weyl connection is invariant to the rescaling provided the associated ϕ-field transforms according to

$$\hat{\phi}_\alpha = \phi_\alpha + \frac{1}{2}\epsilon\pi_{|\alpha}$$

That is, if we write $\delta g_{\mu\nu} = \hat{g}_{\mu\nu} - g_{\mu\nu}$ and $\delta\phi_\alpha = \hat{\phi}_\alpha - \phi_\alpha$, then

$$\delta\Gamma^\lambda_{\mu\nu} = 0 \quad \text{for} \quad \delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu} \quad \text{and} \quad \delta\phi_\alpha = \frac{1}{2}\epsilon\pi_{|\alpha}$$

The resemblance of the change in the ϕ_α vector under a local metric rescaling with the gauge transformation property of the electromagnetic four-potential A_α, coupled with the scale invariance of the connection itself, clearly motivated Weyl to identify his ϕ-field with the four-potential. The seemingly miraculous appearance of electromagnetism in the guise of variable vector length further motivated Weyl to believe he had indeed unified electromagnetism with gravitation.

Weyl proceeded to develop an alternative to Einstein's theory of gravitation based on scale (or, more properly, *conformal*) invariance. The Ricci tensor $R_{\mu\nu}$ is conformally invariant in Weyl's geometry, but the free-space Einstein-Hilbert Lagrangian $\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$ is not. Since $\delta\sqrt{-g} = 2\epsilon\pi\sqrt{-g}$ (for four spacetime dimensions), the simplest Lagrangian that Weyl could use was $\sqrt{-g}\, R^2$ where $R = g^{\mu\nu} R_{\mu\nu}$. This and many other conformally-invariant Lagrangians have been extensively investigated by others, and the interested reader is encouraged to see the Mannheim and Kazanas reference for additional information.

4. Einstein's Objection

Historically, it was (2.4) that Einstein had difficulty with, for no matter how small the Weyl field ϕ_α is taken the length of a vector would change continuously and arbitrarily from place to place and from time to time. For example, the free-space and Klein-Gordon four-momentum vectors

$$p^\mu = mc\,\frac{dx^\mu}{ds},$$
$$p^\mu = \frac{mc}{\hbar}\,\frac{dx^\mu}{ds}$$

would vary in Weyl's theory, in contradiction to experience. In fact, the length of *any* vector would undergo a length change in Weyl's theory, the simplest example being the unit tangent vector dx^α/ds itself. Given the invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, the vector's length (unity) would thus change in the presence of Weyl's ϕ-field — a nonsensical prediction.

London later noted that if ϕ_α in (2.4) is purely imaginary, one could effectively set the exponential term equal to unity, thus avoiding Einstein's argument. Indeed, a more complete analysis shows that for a particle of charge q in a Weyl field the expression is actually

$$\phi_\alpha = \frac{iq}{\hbar c} A_\alpha$$

However, if the length of a vector (or its differential change) is to be considered a real quantity, an imaginary potential field is problematic. More importantly, if Weyl's theory is to have any relevance to reality, then it must provide a means for accommodating vector quantities whose magnitudes are indeed fixed.

5. Revising the Theory

We again consider the tangent vector, whose change under parallel transport is given by

$$2L dL = g_{\mu\nu||\alpha} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} dx^\alpha \tag{5.1}$$

If the tangent vector is to have a fixed length, then the right-hand side of this expression must be consistent with $dL = 0$. There are only two ways to accomplish this. One is to set $g_{\mu\nu||\alpha} = 0$, but this just takes us back to Riemannian geometry. The other is to recognize that for $dL = 0$, (5.1) is equivalent to

$$g_{\mu\nu||\alpha} dx^\mu dx^\nu dx^\alpha = 0$$

which implies the peculiar symmetry condition

$$g_{\mu\nu||\alpha} + g_{\alpha\mu||\nu} + g_{\nu\alpha||\mu} = 0 \tag{5.2}$$

This identity has in fact been seen before. Schrödinger was apparently the first to show that the most general connection (symmetric or otherwise) possible is

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta} T_{\mu\nu\beta}$$

where the T-tensor is an arbitrary collection of symmetric and antisymmetric functions having the same symmetry properties as $g_{\mu\nu||\alpha}$ in (5.2). To show that Schrödinger's tensor and $g_{\mu\nu||\alpha}$ are the same, we need only add the expanded form of $g_{\mu\nu||\alpha}$ with its two cyclic permutations, which gives

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta} g_{\mu\nu||\beta} \tag{5.3}$$

Thus, Schrödinger's T-tensor is indeed the nonmetricity tensor. This is interesting, but we still have no idea what $g_{\mu\nu||\alpha}$ is. Following Weyl, we might assume that it also consists of the metric tensor and a vector field ϕ_α. Since the Weyl identity in (2.3) does not satisfy the symmetry condition in (5.2), the closest identity we can write is

$$g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha - g_{\alpha\mu}\phi_\nu - g_{\nu\alpha}\phi_\mu \tag{5.4}$$

which indeed satisfies the symmetry condition.

6. Scale-Invariant Vectors in the Revised Weyl Formalism

It is easy to see that, in addition to any four-vector that is proportional to the unit tangent vector dx^α/ds, $dL = 0$ is also satisfied for the vector ϕ_μ itself, since

$$(2g_{\mu\nu}\phi^\mu\phi^\nu\phi_\alpha - g_{\alpha\mu}\phi^\mu\phi^\nu\phi_\nu - g_{\nu\alpha}\phi^\mu\phi^\nu\phi_\mu)\, dx^\alpha = 0$$

Although it is not clear if the true electromagnetic four-potential A_μ should be invariant with respect to parallel transfer, there is one vector that we should absolutely demand to be of fixed magnitude, and that is the conjugate momentum four-vector for a particle of charge q in an electromagnetic field. We therefore expect that the four-vector

$$p^\mu = mc\,\frac{dx^\mu}{ds} + \frac{q}{c}A^\mu$$

be scale invariant if we identify Weyl's ϕ^μ vector with the four-potential (yet another electromagnetic quantity we would expect to be invariant is the source vector $j^\mu = \rho\, dx^\mu/ds$). Whether this small collection of fixed-length vectors serves to refute Einstein's objection to the revised Weyl theory remains to be seen, but such a claim at least seems plausible now.

7. Gauge Invariance in the Revised Theory

Unfortunately, the connection $\Gamma^\lambda_{\mu\nu}$ in the revised theory is not conformally invariant, as it is in Weyl's original theory. This observation appears to rob the revised theory of a particularly beautiful aspect of Weyl's idea. While a conformally-invariant Lagrangian can still be constructed from the revised connection, it is far more complicated than Weyl's $\sqrt{-g}R^2$ Lagrangian. However, the contracted connection $\Gamma^\lambda_{\mu\lambda}$ is fully scale invariant, and using this fact it is an easy exercise to show that the antisymmetric quantity $R_{\mu\nu} - R_{\nu\mu}$ reduces to what appears to be the electromagnetic stress-energy tensor $F_{\mu\nu} = \phi_{\mu||\nu} - \phi_{\nu||\mu}$ (which also obtains in Weyl's theory). Weyl viewed this as additional evidence that electromagnetism was indeed embedded in his non-Riemannian geometry.

8. Does Weyl's Theory Really Involve Electromagnetism?

It is interesting to note that Weyl's 1918 theory can be succinctly summarized by proposing a new geometry in which the metric tensor is made scale invariant with a suitable scale factor:

$$\hat{g}_{\mu\nu} = e^{-2\int \phi_\lambda dx^\lambda} g_{\mu\nu}$$

Here, the ϕ-field is transformed under an infinitesimal gauge variation in accordance with $\delta\phi_\mu = \frac{1}{2}\epsilon\pi_{|\mu}$ and $\delta\hat{g}_{\mu\nu} = 0$. It is then an easy matter to show that the revised Christoffel symbol associated with this metric is just the Weyl connection Γ, or

$$\widehat{\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}} = \Gamma^\alpha_{\mu\nu} = \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} + g_{\mu\nu} g^{\alpha\beta}\phi_\beta - \delta^\alpha_\mu \phi_\nu - \delta^\alpha_\nu \phi_\mu$$

Indeed, all geometric quantities constructed from this metric are then automatically scale invariant, including the metric determinant $\sqrt{-\hat{g}}$, the Riemann-Christoffel tensor $\hat{R}^{\ \alpha}_{\mu\nu\beta}$ and the Ricci tensor $\hat{R}_{\mu\nu}$. Similarly, the revised line element

$$d\hat{s} = e^{-\int \phi_\lambda dx^\lambda} ds$$

is also scale invariant. However, Weyl's mathematics still requires that the length of an arbitrary vector change in accordance with $dL = \phi_\mu dx^\mu L$ and, as we have explained, this inability to accommodate fixed-length vectors is the primary reason that Weyl's theory failed. While Weyl's original theory produced a purely geometric field ϕ_μ strongly resembling the four-potential A_μ, it is very doubtful that it describes any electromagnetic phenomena. Nevertheless, the theory's non-Riemannian formalism remains compelling.

We can, however, show that Schrödinger's connection may in fact have some relevance to electrodynamics. Consider the geodesic equations, which for a general connection are simply

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

Using Schrödinger's connection, we then have

$$\frac{d^2x^\alpha}{ds^2} + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -g^{\alpha\beta} g_{\mu\nu||\beta}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

The term on the right-hand side would appear to be a Lorentz force term f^α. For a particle of mass m and charge q, the Lorentz force is known to be

$$f^\alpha = \frac{q}{m} F^\alpha_{\ \nu}\frac{dx^\nu}{ds}$$

This implies that

$$\frac{q}{m} F^\alpha_{\ \nu}\frac{dx^\nu}{ds} = -g^{\alpha\beta} g_{\mu\nu||\beta}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds},$$

which is equivalent to

$$\frac{q}{m} F_{\alpha\nu}\frac{dx^\nu}{ds} = -g_{\mu\nu||\alpha}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{8.1}$$

Using (5.2), this can be simplified to

$$\frac{q}{m} F_{\alpha\nu} = \frac{2}{3}\left(g_{\mu\alpha||\nu} - g_{\mu\nu||\alpha}\right)\frac{dx^\mu}{ds}$$

This indicates that the nonmetricity tensor should be associated not with the four-potential A_μ as suggested in (5.4) but with the electromagnetic tensor $F_{\alpha\nu}$. Interestingly, the symmetry properties of $g_{\mu\nu||\alpha}$ and $F_{\alpha\nu}$ guarantee that both sides of (8.1) vanish under multiplication by dx^α/ds.

9. Conclusions

Scale-invariant vectors such as the relativistic four-momentum p^μ of course existed in Weyl's time, and in hindsight it seems indeed strange that he did not recognize the need to accommodate such vectors in his 1918 theory. If he had, it seems plausible that he would have arrived at something similar to the revised formalism described here. Perhaps, at the very least, it would have forestalled Einstein's objections or motivated another round of spirited correspondence between Weyl and Einstein, who were friends as well as colleagues. At the same time, it must be recognized that neither Weyl's 1918 theory nor the revised version presented here provides any meaningful additional insight into Einstein's brilliant achievement of November 1915. As for the failure of Weyl's own notable effort of 1918, it must also be remembered that he was in fact striving for a unified theory of gravity and electromagnetism, a perhaps hopeless goal that has now frustrated legions of physicists for nearly one hundred years.

To his enduring credit, Weyl had the courage to abandon his theory when all hope was lost, although he managed to salvage his dream of gauge invariance in Nature when it was successfully applied to quantum mechanics as the *phase invariance* concept. At the same time, it seems ironic that Einstein successfully refuted Weyl's theory even as he was consigning himself to thirty years of useless effort on a failed quest for a unified field theory.

References

1. Adler, R., Bazin, M. and Schiffer, M. 1975 *Introduction to general relativity.* McGraw-Hill, Inc.
2. Eddington, A. 1975 *The mathematical theory of relativity.* Chelsea Publishing Co.
3. London, F. 1927 Z. Physik **42**, 375-389.
4. Mannheim, P. and Kazanas, D. 1989 Astrophys. J. **342**, 635-638.
5. O'Raifeartaigh, L. 1997 *The dawning of gauge theory.* Princeton University Press.
6. Rosen, N. 1982 Found. Phys. **12**, 213-248.
7. Schrödinger, E. 1947 Proc. Roy. Irish Acad. **51**, 163-171, 205-216.
8. Schrödinger, E. 1950 *Space-time structure.* Cambridge University Press.
9. Weyl, H. 1918 Sitzber. Preuss. Akad. Wiss. Berlin. **2**, 465-480.
10. Weyl, H. 1922 *Space-time-matter.* Methuen, London.