Legendre-Zhang's Conjecture & Gilbreath's Conjecture and Proofs Thereof

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Abstract

If reduce limits which contain odd primes by a half for Legendre's conjecture, then there is at least an odd prime within the either half likewise, this is exactly the Legendre-Zhang's conjecture. We shall first prove the Legendre-Zhang's conjecture by mathematical induction with the aid of two number axes' positive half lines whose directions reverse from each other. Successively, prove the Gilbreath's conjecture by mathematical induction with the aid of the got result.

Keywords

Legendre-Zhang's conjecture, Mathematical induction, Number axis's positive half line, Odd prime points, PLS, $RPLS_P$, SCRP AB, Gilbreath's conjecture.

Basic Concepts

The Gilbreath's conjecture was first suggested in 1958 by the American mathematician and amateur magician Norman L. Gilbreath following some doodling on a napkin.

He started by writing down the first few primes.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,…

Under these he put their differences:

1, 2, 2, 4, 2, 4, 2, 4, 6, 2, …

Under these he put the unsigned difference of the differences.

1, 0, 2, 2, 2, 2, 2, 2, 4, …

And he continued this process of finding iterated differences:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, …

1, 2, 2, 4, 2, 4, 2, 4, 6, 2, …

1, 0, 2, 2, 2, 2, 2, 2, 4, …

1, 2, 0, 0, 0, 0, 0, 2, …

1, 2, 0, 0, 0, 0, 2, …

1, 2, 0, 0, 0, 2, …

1, 2, 0, 0, 2, …

1, 2, 0, 2, …

1, 2, 2, …

1, 0, …

1, …

The Gilbreath's Conjecture is that the numbers in the first column except for first number of first rank are all one.

The Legendre's conjecture is known as an age-old problem. Is there always a prime between n^2 and $(n+1)^2$ for every positive integer n?

If odd numbers between n^2 and $(n+1)^2$ are divided into two parts by n(n+1) except for n=1, then Legendre-Zhang's conjecture asserts that there is always an odd prime in either such part. Manifestly the Legendre-Zhang's conjecture is better than the Legendre's conjecture.

Please, see the sequence of integers in relation to the Legendre-Zhang's conjecture as listed below.

1(1+1), *3,* 2^2, *5,* 2(2+1), *7,* 3^2, *11,* 3(3+1), *13,* 4^2, *17, 19,* 4(4+1), *23,* 5^2, *29,* 5(5+1), *31,* 6^2, *37, 41,* 6(6+1), *43, 47,* 7^2, *53,* 7(7+1), *59, 61,* 8^2, *67, 71,* 8(8+1), *73, 79,* 9^2, *83, 89,* 9(9+1), *97,* 10^2, *101, 103, 107, 109,* 10(10+1), *113,* 11^2, *127, 131,* 11(11+1), *137, 139,* 12^2, *149, 151,* 12(12+1), *157, 163, 167,* 13^2, *173, 179, 181,* 13(13+1), *191, 193,* 14^2, *197, 199,* 14(14+1), *211, 223,* 15^2, *227, 229, 233, 239,* 15(15+1), *241, 251,* 16^2, …

Thus, the Legendre-Zhang's conjecture states concretely that there is at least an odd prime between n(n+1) and $(n+1)^2$; and there is at least an odd prime between $(n+1)^2$ and (n+1)(n+2), where n is a nature number. When n is an odd number, there is following a series of numbers.

There are *(n+1)/2* odd numbers between n(n+1) and $(n+1)^2$;

There are *(n+1)/2* odd numbers between $(n+1)^2$ and (n+1)(n+2);

There are *(n+1)/2* odd numbers between (n+1)(n+2) and $(n+2)^2$;

There are *(n+1)/2* odd numbers between $(n+2)^2$ and (n+2)(n+3) ;

There are *(n+3)/2* odd numbers between (n+2)(n+3) and $(n+3)^2$;

There are *(n+3)/2* odd numbers between $(n+3)^2$ and (n+3)(n+4);

There are *(n+3)/2* odd numbers between (n+3)(n+4) and $(n+4)^2$;

There are *(n+3)/2* odd numbers between $(n+4)^2$ and (n+4)(n+5) ;

There are *(n+5)/2* odd numbers between (n+4)(n+5) and $(n+5)^2$ …

Above has mentioned that odd numbers between n^2 and $(n+1)^2$ are divided into two parts by n(n+1), then we regard every part as a special segment. So begin with odd number 3 alone at №1 special segment between 1(1+1) and

2^2, afterwards, add an odd number at each special segment after per consecutive four special segments. For example, 1(1+1) $\underline{\ 1\ }$ 2^2 $\underline{\ 1\ }$ 2(2+1) $\underline{\ 1\ }$ 3^2 $\underline{\ 1\ }$ 3(3+1) $\underline{\ 2\ }$ 4^2 $\underline{\ 2\ }$ 4(4+1) $\underline{\ 2\ }$ 5^2 $\underline{\ 2\ }$ 5(5+1) $\underline{\ 3\ }$ 6^2 $\underline{\ 3\ }$ 6(6+1) $\underline{\ 3\ }$ 7^2 $\underline{\ 3\ }$ 7(7+1) $\underline{\ 4\ }$ 8^2 $\underline{\ 4\ }$ 8(8+1) $\underline{\ 4\ }$ 9^2 $\underline{\ 4\ }$ 9(9+1) $\underline{\ 5\ }$ 10^2 $\underline{\ 5\ }$ 10(10+1) $\underline{\ 5\ }$ 11^2 $\underline{\ 5\ }$ 11(11+1) $\underline{\ 6\ }$ 12^2 $\underline{\ 6\ }$ 12(12+1) $\underline{\ 6\ }$ 13^2 $\underline{\ 6\ }$ 13(13+1) $\underline{\ 7\ }$ 14^2 $\underline{\ 7\ }$ 14(14+1) $\underline{\ 7\ }$ 15^2 $\underline{\ 7\ }$ 15(15+1) …

In addition, we stipulate that odd number $(n+2)^2$ is added into special segment between $(n+2)^2$ and $(n+2)(n+3)$, where n is an odd number.

Obviously, all special segments both exist within the sequence of positive integers, and exist at the number axis's positive half line.

We number the ordinal number of seriate each and every special segment by seriate natural numbers ≥1, whether special segments exist in the series of positive integers, or they exist at the number axis's positive half line. Then, numbers of odd numbers at № (1+4t), at № (2+4t) and at № (3+4t) is all 1+t; yet the number of odd points at № (4+4t) is 2+t, where t ≥ 0.

Namely, numbers of odd numbers at №1, at №2 and at №3 special segment is all 1;

Numbers of odd points at №4, at №5, at №6 and at №7 special segment is all 2;

Numbers of odd points at №8, at №9, at №10 and at №11 special segment is all 3;

Numbers of odd points at №12, at №13, at №14 and at №15 special segment is all 4;

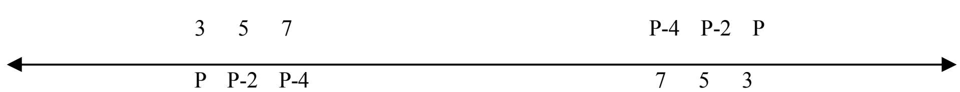
Numbers of odd points at №16, at №17, at №18 and at №19 special segment is all 5 …

We shall prove indirectly the Legendre-Zhang's conjecture with the aid of odd points at positive half line of the number axis, thereinafter.

On purpose of watching convenience, for the number axis's positive half line, we let it begins with odd point 3, and it is marked merely odd points, and the length between every two consecutive odd points is just the same.

We term a distance whereby each odd prime point and odd point 3 act as two endmost points "a prime length". "PL" is abbreviated from "prime length", and "PLS" denotes the plural of PL. From odd prime point 3 to odd point 3 according to the definition is a PL as well, nevertheless, its length is equal to zero.

We use two positive half lines of number axes, yet their directions reverse from each other, and endmost point 3 of either half line can coincide with any odd point of another. For example, endmost point 3 of either half line coincides with odd point P of another. Please, see first illustration below:



First Illustration

We term PLS at the leftward direction's half line "reverse PLS", and RPLS is abbreviated from reverse PLS.

RPLS whereby any odd point P at the rightward direction's half line acts as a common right endmost point are written as $RPLS_P$. One within $RPLS_P$ is written as a RPL_P.

When P=3, the RPL is written as RPL_3. Namely RPL_3 is under the case that odd point 3 at the leftward direction's half line coincides with odd point 3 at the rightward direction's half line.

The common right endmost point of $RPLS_P$ is odd point P, and part left endmost points of $RPLS_P$ coincide monogamously with part odd prime points at line segment 3P of the rightward direction's half line.

At line segment 3P, odd prime points at the rightward direction's half line and left endmost points of $RPLS_P$ at the leftward direction's half line assume always one-to-one bilateral symmetry whereby the center point of line segment 3P acts as the symmetric center.

At the rightward direction's half line, begin with an odd point B, leftwards take seriatim each odd point as a common right endmost point of $RPLS_{B-2f}$, then, part left endmost points of $RPLS_{B-2f}$ monogamously coincide with part odd prime points at line segment 3B of the rightward direction's half line, where f =0, 1, 2, 3, …c, … in proper order.

Suppose that f increases orderly to c, and left endmost points of $\sum RPLS_{B-2f}$ $[0\leq f\leq c]$ just monogamously coincide with all odd prime points at line segment 3B of the rightward direction's half line, then, we consider line segment B(B-2c) as shortest line segment of common right endmost points of $RPLS_{B-2f}$ at line segment 3B. "SCRP" is abbreviated from "shortest line segment of common right endmost points". So, such line segment B(B-2c) is written as SCRP B(B-2c) within line segment 3B too. Let A=B-2c, SCRP B(B-2c) is exactly SCRP AB.

If there is only a CRP at a shorter line segment which begins with odd point 3, then the odd point is an odd prime point surely, because odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point acts as the right endmost point.

At line segment 3B, we regard odd prime points which first coincide with left endmost points of $RPLS_{B-2f}$ as special odd prime points which left endmost points of $RPLS_{B-2f}$ coincide with, where $0 \le f \le c$. Manifestly, all odd prime points which monogamously coincide with left endmost points of $RPLS_B$ are namely special odd prime points which left endmost points of $RPLS_B$ coincide with. Yet special odd prime points which left endmost points of $RPLS_A$ coincide with are totally different from special odd prime points which left endmost points of $\sum RPLS_X$ $[B \ge X \ge A+2]$ coincide with.

Thus, all odd prime points at line segment 3B are namely special odd prime points which left endmost points of $\sum RPLS_{B-2f}$ $[0 \le f \le c]$ coincide with.

For example, when B=95, there are four odd points at SCRP AB, then A=89.

For the distribution of odd prime points which monogamously coincide with left endmost points of $RPLS_F$, where F=95, 93, 91 and 89, please, see following second illustration:

Odd prime point:	19	31	37	61	67	79								
Left end points of $RPLS_{95}$:	79	67	61	37	31	19								
Odd prime point:	7	13	17	23	29	37	43	53	59	67	73	79	83	89
Left end points of $RPLS_{93}$:	89	83	79	73	67	59	53	43	37	29	23	17	13	7
Odd prime point:	5	11	23	41	47	53	71	83	89					
Left end points of $RPLS_{91}$:	89	83	71	53	47	41	23	11	5					
Odd prime point:	3	13	19	31	61	73	79	89						
Left end points of $RPLS_{89}$:	89	79	73	61	31	19	13	3						

Second Illustration

Let us continue to leftwards take odd point A-2 as a common right endmost point of $RPLS_{A-2}$. Since left endmost points of $\sum RPLS_{B-2f}$ $[0 \le f \le c]$ just monogamously coincide with all odd prime points at line segment 3B of the rightward direction's half line, yet odd prime points which left endmost points of $\sum RPLS_{B-2e}$ [1≤e≤c] coincide with are constant, therefore, special odd prime points which left endmost points of $RPLS_{A-2}$ coincide with belong within odd prime points which left endmost points of $RPLS_B$ coincide with.

Since odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point at the rightward direction's half line acts as the right endmost point, therefore, provided B is an only odd prime point, and others are all odd composite points at SCRP AB, then A-2 is an odd prime point. In addition, there is SCRP (A-2)(B-2) within line segment 3(B-2).

Since odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point at the rightward direction's half line acts as the right endmost point, thus any SCRP contains at least an odd prime point. Provided there is only a CRP at a shorter line segment which begins with 3, then this CRP is an odd prime point too.

If A is an only odd prime point, and others are all odd composite points at SCRP AB, then B+2 is an odd prime point. So, there is SCRP A(B+2) within line segment 3(B+2). Here, SCRP A(B+2) within line segment 3(B+2) is an odd point more than SCRP AB within line segment 3B. Manifestly, the odd point is exactly odd prime point B+2.

Factual proof, that the aforesaid case is unique under these circumstances that numbers of odd points at two SCRP which share at least an odd point are inequality from each other.

Conversely, we may too begin with odd point A, rightwards take seriatim each odd point as a common right endmost point of $RPLS_{A+2f}$, when part left endmost points of $RPLS_{A+2f}$ monogamously coincide just with all odd prime points at line segment 3(A+2f), where $f \geq 0$, then there is SCRP A(A+2f) within line segment 3(A+2f). In reality, this is known as A+2f =B, so SCRP A(A+2f) is exactly SCRP AB. But, at here, special odd prime points which left endmost points of $RPLS_{A+2f}$ coincide with are determined according to from left to right odd points as common right endmost points of $RPLS_{A+2f}$.

Take seriatim each odd point as a common right endmost point of RPLS, whether the order is from right to left, or from left to right, there is always at least a special odd prime point which coincides with a left endmost point of RPLS whereby each odd point at SCRP AB acts as a common right endmost point.

Since the order of taking common right endmost points of RPLS can be each other's reverse directions, so special odd prime points which coincide with left endmost points of RPLS whereby an identical odd point acts as the common right endmost point are not all alike on two directions.

There are *(B-A+2)/2* odd points at SCRP AB. One complete SCRP must contain all odd points at the SCRP. Basically, a complete SCRP is able to be substituted by all odd points at the SCRP. Instance SCRP AB, SCRP AB is a complete SCRP within line segment 3B, and *(B-A+2)/2* odd points at SCRP AB can substitute for SCRP AB. If reduce any odd point at SCRP AB, then the remainder is not a SCRP.

The Proof of the Legendre-Zhang's Conjecture

Since there is at least an odd prime point at any complete SCRP, or when there is only a CRP at a shorter line segment which begins with 3, the CRP is an odd prime point according to preceding proof, thus let us first prove that there is a complete SCRP at №y special segment by mathematical induction, where y is any natural number.

(1). When $y=y_1=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19$ and 20, each number of odd points at №y_1 special segment in the proper order is orderly *1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5* and *6*. Also, each number of odd points at each SCRP which contains most right odd point within №y_1 special segment in the proper order is orderly *1, 1, 1, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 4, 4, 4* and *6*.

This shows that there is a CRP alone or a complete SCRP within №y_1 special segment.

(2). When y=k, suppose that there is a complete SCRP within №k special segment, where $k \geq 20$.

(3). When y=k+1, prove that there is a complete SCRP within №(k+1) special segment too.

Proof . Since there is a complete SCRP within №k special segment, then the complete SCRP contains at least an odd prime point.

Since odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point at the rightward direction's half line acts as the right endmost point, thus, there is at least an odd prime point at the complete SCRP within №k special segment.

Suppose that most right or unique odd prime point at №k special segment is P_m, then, let a complete SCRP within №k special segment contain P_m. Moreover, suppose that the left endmost point of the complete SCRP is odd point E, and its right endmost point is odd point F. Of course, P_m at the here may become to E or F.

There are *(F-E+2)/2* odd points at SCRP EF. Also let F+2=G, and 2F-E+2=H or 2F-E+4=H, then line segment GH is exactly SCRP GH within line segment 3H. Either the number of odd points at SCRP GH within line segment 3H is equal to the number of odd points at SCRP EF within line segment 3F, or SCRP GH within line segment 3H is an odd point more than SCRP EF within line segment 3F. But also, SCRP GH within line segment 3H contains at least an odd prime point. Undoubtedly, the odd prime point exists on the right side of №k special segment.

From the preceding exposition, we know that either the number of odd points at №k special segment is equal to the number of odd points at №(k+1) special segment, or №(k+1) special segment is an odd point more than №k special segment. Therefore, SCRP GH exists within №(k+1) special segment. Consequently, there is at least an odd prime point at №(k+1) special segment.

Start from a proven special segment to prove the next special segment which adjoins the proven special segment for each once, then after via infinite times, there are infinitely many proven special segments. Namely, let y is equal to each and every natural number, then inevitably reach a conclusion that there is at least an odd prime point at every special segment.

Or rather, there is at least an odd prime between n(n+1) and $(n+1)^2$; and there is at least an odd prime between $(n+1)^2$ and (n+1)(n+2), where n expresses each and every nature number.

Consequently the Legendre-Zhang's conjecture does hold water by proof.

The Proof of the Gilbreath's Conjecture

First, we number the ordinal number of each and every odd prime from small to great. Namely, we regard odd prime 3 as №1 odd prime, and write odd prime 3 down P_1; also regard odd prime 5 as №2 odd prime, and write odd prime 5 down P_2 … and so on and so forth, reckon odd prime P_m as №m odd prime.

On the supposition that m is the ordinal number of most right odd prime at a row of consecutive odd primes which begin 3, then the number of the consecutive primes from even prime 2 to odd prime P_m is m+1 altogether.

Let us review the set rule of the Gilbreath's Conjecture, namely after write down consecutive primes which began with 2 into a row, count out a difference of every two adjacent primes, and put each difference underneath the left prime, and continued this process of finding iterated unsigned differences.

After proven the Legendre-Zhang's conjecture, let us successively prove the Gilbreath's conjecture by mathematical induction with the aid of the result have gained, as follows.

(1). When m=20, P_m=73, all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture. Please, see following rows and columns.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73 …

1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 8, 6, 6, 2 …

1, 0, 2, 2, 2, 2, 2, 2, 4, 4, 2, 2, 2, 2, 0, 2, 2, 0, 4 …

1, 2, 0, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 2, 2, 0, 2, 4 …

1, 2, 0, 0, 0, 0, 2, 2, 2, 2, 0, 0, 2, 0, 2, 2 2 …

1, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 2, 2, 2, 0, 0 …

1, 2, 0, 0, 2, 2, 0, 0, 2, 2, 2, 0, 0, 2, 0 …

1, 2, 0, 2, 0, 2, 0, 2, 0, 0, 2, 0, 2, 2 …

1, 2, 2, 2, 2, 2, 2, 2, 0, 2, 2, 2, 0 …

1, 0, 0, 0, 0, 0, 0, 2, 2, 0, 0, 2 …

1, 0, 0, 0, 0, 0, 2, 0, 2, 0, 2 …

1, 0, 0, 0, 0, 2, 2, 2, 2, 2 …

1, 0, 0, 0, 2, 0, 0, 0, 0 …

1, 0, 0, 2, 2, 0, 0, 0 …

1, 0, 2, 0, 2, 0, 0 …

1, 2, 2, 2, 2, 0 …

1, 0, 0, 0, 2 …

1, 0, 0, 2 …

1, 0, 2 …

1, 2 …

1 …

When P_m=73, 73 exists at №14 special segment due to $1(1+1)\_2^2$ as №1 special segment, $2^2\_2(2+1)$ as №2 special segment, $2(2+1)\_3^2$ as №3 special segment… $8^2\_8(8+1)$ as №14 special segment, just right, 73 exists between 8^2 and 8(8+1).

(2). When m=c, suppose that all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture, where c≥20.

(3). When m=c+1, prove that all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture too.

Proof . Let us arrange 2 and from small to great odd numbers which begin with 3 into a row, and reckon the row as first row. Then put each difference of every two adjacent integers underneath each left integer. Furthermore, we continue this process of finding iterated unsigned differences. Evidently, except for first integer in first row, numbers in the first column are all one.

If delete odd composite numbers which contain prime factor 3 in first row, afterwards make operating and arranging unsigned differences like the aforementioned way of doing, then except for first integer in first row, numbers in the first column are all one.

If continue to delete odd composite numbers which contain prime factor 5 in first row, afterwards make operating and arranging unsigned differences like the aforementioned way of doing, then except for first integer in first row, numbers in the first column are all one.

…

If continue to delete odd composite numbers which contain prime factor P_a in first row, and suppose that except for first integer in first rank, numbers in the first column are all one.

Why numbers in the first column except for first integer in first row are all one? Because, the biggest difference of two adjacent odd numbers in first row after delete odd composite numbers which contain prime factors from 3 to P_a is always smaller than the sum of unsigned differences on the left of the biggest difference. If subtract the biggest difference from the right part of the all unsigned differences, then, the remainder is exactly a proven series of integers.

From proven the Legendre-Zhang's conjecture, we know that there is at least an odd prime at each and every special segment.

Suppose that odd prime P_c exists at №y special segment. If odd prime P_{c+1}exists at №y special segment too, then the number of odd numbers between p_c and p_{c+1} is less than the number of odd numbers at №y special segment.

If odd prime P_{c+1} exists at №(y+1) special segment, then the number of odd numbers between p_c and p_{c+1} is less than the number of odd numbers at №y special segment plus №(y+1) special segment.

From preceding basic concepts, we know that the most front and most behind two special segments within consecutive four special segments differ by one of odd number.

Thus, №y and №(y+1) special segments is one or two odd numbers more than №(y-1) and №(y-2) special segments.

Therefore, odd numbers between p_c and p_{c+1} are not more than odd numbers at №(y-1) and №(y-2) special segments.

In addition, there is at least an odd prime at every special segment, including

№(y-1) and №(y-2) special segments.

Thus, either the difference of p_c from $p_{c+1}$is less than the sum of unsigned differences of every two adjacent odd primes at №(y-1) special segment, or the difference of p_c from p_{c+1} is less than the sum of unsigned differences of every two adjacent odd primes at №(y-1) and №(y-2) special segments.

Therefore, when m=c+1, after above-mentioned either sum minus corresponding a difference, the remainder at first row is a proven series of primes.

Consequently, when m=c+1, all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture too.

Proceed from a proven conclusion to add a larger adjacent odd prime for each once, then via infinite times, namely let m to equal each and every natural number, or rather, let 2 and all odd primes are putting in first row, afterwards make all operational results and the arrangement thereof according to the set rule, then we reach inevitably a conclusion that except for first integer in first row, the numbers in the first column are all one. Namely the Gilbreath's conjecture holds water by proof.

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