A Wave-centric View of Special Relativity

William M. Nelson*

(Dated: May 13, 2013)

Abstract

An approach to special relativity is outlined which emphasizes the wave and field mechanisms which physically produce the relativistic effects, with the goal of making them seem more natural to students by connecting more explicitly with prior studies of waves and oscillators.

I. INTRODUCTION

Special Relativity developed simultaneously with the first fundamental theory based on fields and waves, namely electromagnetism, but it is usually taught to students in the context of particle kinematics, using an axiomatic approach in which spacetime is implicitly presented as a causative agent. This can be quite confusing as it seems to place relativity on a different footing from other physical theories, even theories of symmetry principles.

Rotational symmetry, for example, is not typically introduced in terms of axioms or properties of spacetime, but rather in terms of concrete properties of force laws and potentials. Non-symmetric examples are readily produced, allowing the student to understand the content of the symmetry principle. With special relativity, by contrast, one typically only discusses the two cases of Lorentzian and Galilean symmetry, attributing both to inherent properties of spacetime, and students may not even realize that there exists a vast range of easily constructible theories which do not satisfy either principle.

Approaches to relativity which place a greater emphasis on mechanism have been discussed periodically over the years, and lately seem to be gaining some momentum under the rubric of "Lorentzian" or "constructive" approaches.[1–4] In the following we attempt to advance this program by tracing the major relativistic effects as directly as possible to their origins, which lie most often in the elementary behavior of waves or oscillators.

Many relativistic phenomena which seem strange or opaque in the context of particle kinematics, or in an axiomatic derivation, are quite natural in the context of the field- and wave-based theories which actually appear to underlie nature. Indeed they are often connected to intuitions so commonplace that it is a bit difficult to accept that they could be relevant to recondite matters such as the "flow of time"; yet, any physical effect must come from somewhere, and the more generic the effect the simpler the underlying cause ought to be.

Incorporating such material into standard courses would, we believe, have at least the following benefits:

1. Allow students to build from things they already understand, e.g. transverse waves on a string, rather than being confronted immediately with abstract principles.

2. Provide a better historical context for the theory, connecting with partial results which

were obtained by Lorentz, Poincarè and others before Einstein's formulation.[1,2]

3. Provide tools for reasoning about non-Lorentz-invariant theories, which have been increasingly under discussion.[5,6]

4. Provide better preparation for the deeper study of relativistic quantum mechanics or quantum field theories.

Incorporating constructive explanations does not at all mean ignoring the Lorentz transformation, because the simplest and most natural examples are, in fact, Lorentz invariant, and non-invariant examples quickly become intractable. Exposure to constructive ideas should, rather, increase and deepen the student's appreciation of the Lorentz symmetry, by revealing the numerous mechanisms which are organized and brought into a coherent pattern when the symmetry holds.

The discussion is largely qualitative, with few precise calculations, however those that are carried out employ natural units $c = \hbar = 1$. The author claims no originality for the calculations or demonstrations, some of which are standard; references have been cited when known.

II.   WAVES AND RIGIDITY

Perhaps the simplest qualitative consequence of the wave paradigm, but one which is rarely emphasized, is the non-rigidity of objects. Rigidity is probably the last thing one would associate with waves, hence the simple realization that wave theories underlie all forces (and matter as well) should help to make Lorentz contraction a more inuitively reasonable possibility.

Ideal rigid objects can exist in Newtonian theory because the forces between constituents propagate at infinite speed, allowing a perturbation in one part of the object to be immediately reacted to by all other parts of the object. In reality, any change to the forces inside an ordinary object is propagated via electromagnetic (EM) waves, the complete basis of non-static solutions to Maxwell's equations away from charge centers. But waves inherently move at finite speed, hence the forces within an object cannot be transmitted instantaneously, and the object cannot respond rigidly to perturbations. It will deform under an applied force, simply because one part starts to move before the other parts experience any

change to their local EM fields. This fundamental non-rigidity is independent of the strength or organization of molecular bonds.

The object necessarily deforms during acceleration, but what happens when the acceleration stops? Does the object resume its original shape, as happens with ordinary elastic deformations? To see that it does not requires further discussion of the internal EM fields (Sec.VII), but it is clear that wave propagation introduces a generic non-rigidity which is not present in the Newtonian paradigm.

III.   RETARDATION EFFECTS IN CLOCKS

Simple aspects of wave propagation also affect dynamic processes such as the cyclic processes occurring in clocks. Any process built from constituent parts relies on wave signals traveling between the parts, and the travel time of the waves forms part of the duration of the process. If the overall object is set into motion, the travel time of the waves between the constituents will generically change, causing the clock rate to change.

This effect can be seen most clearly within the venerable "light clock" example, which is a model clock whose tick period is determined entirely by wave travel time. The light clock consists of a light pulse bouncing back and forth between two mirrors (Fig. 1A).
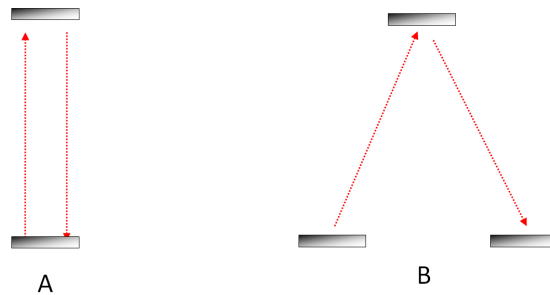

FIG. 1. A. Stationary light clock. B. Moving light clock (bottom mirror is drawn twice).

If we now imagine the clock placed on board a spaceship with glass walls and flown by us at high speed, it will look as in Fig. 1B. The light pulse now travels a longer distance for each cycle, hence the tick rate is slower.

The vertically oriented clock shown in Fig. 1 is the most striking example, but other orientations also change rate, as can be seen by simple algebra. One doesn't get the cor-

rect relativistic answer without also considering length contraction, but it is nevertheless clear qualitatively that the rate must change. One may also consider other complications, for example direction-dependence of light speed (as was originally expected to result from motion through an aether) and the qualitative result remains: the rate must change, except possibly for a few particular velocities and orientations.

Two generic aspects of wave propagation contribute to this effect. First, waves propagate at finite speed, which both enables the construction of a light clock and also makes retardation a factor in virtually every other kind of clock. And second, the propagation speed of the waves does not depend on the speed of the mirrors from which they reflect; this prevents the moving clock from automatically adjusting to restore its original rate. Source-independence of wave velocity should be familiar and intuitive to students, as it holds for all the waves studied in the basic physics curriculum.

We note the strong contrast between source-independence, which is intuitively straightforward and determines the qualitative fact of motion-dependent clock rates, and observer-independence, the much stronger assumption which is usually made. Observer-independence implies the precise Lorentz symmetry, but it requires a large intuitive leap because it depends on the whole range of motion-dependent phenomena occurring within the observers themselves and their measuring devices. Students should be in a better position to understand this postulate if they are first brought to believe that such phenomena *must* occur, as a consequence of simpler principles they have already learned.

Source-independence is less clear-cut for dispersive waves, e.g. relativistic matter waves (discussed below) or the Schrödinger waves of quantum mechanics, for which the motion of the source can affect the frequency of emitted waves and hence their speed; however, this sort of complication need not deter students from developing basic intuitions using familiar non-dispersive waves. Even for dispersive waves the indirect nature of the source's influence on speed makes it clear that, in the vast majority of cases, they will not respond to a moving source in just the right way to counteract retardation effects inside an arbitrarily-moving clock.

IV. CLOCK SYNCHRONIZATION

Clocks which one observer synchronizes according to a reasonable procedure do not match those synchronized by a differently-moving observer, using the exact same procedure. From the constructive point of view this phenomenon has a simple, yet subtle cause: synchronization of separated clocks requires additional motion to carry out. Two clocks must be synchronized at one location, and then one of the clocks must be physically carried to the remote location. Carrying the clock, however, creates additional time dilation beyond that caused by the observer's baseline motion.[7]

On first glance it seems that the extra effect can be made negligible simply by making the extra motion extremely slow. However, this fails because the slower movement takes longer to execute, hence the extra time dilation accumulates over a longer time.

As with time dilation, the synchronization effect can be visualized using a vertically-oriented light clock. We assume the clock is carried very slowly from back to front, i.e. in the direction of the ship's velocity, covering a distance L along the the ship (L is measured in the *stationary* frame). We assume that it is carried at some constant velocity, with the initial and final accelerations having no significant effect.

To analyze this situation geometrically, note first that the mirror reflections have no effect on the calculation, or equivalently one may give the light clock a height H such that it completes exactly one upward bounce during the time taken to carry it the distance L. The situation is then as shown in Fig. 2.

The line tilted at angle θ_0 shows the light pulse of a clock aboard the ship which stays in the same place (is not carried). It completes one vertical pulse of height H, traversing a distance d_0 as seen from the stationary frame. The line tilted by the additional angle θ shows the light pulse of the carried clock, which begins at the same location as the non-carried clock, but covers a horizontal distance L, as measured in the stationary frame, and traverses total distance d, also as seen the stationary frame.

With this setup, the "slow transport" limit is the limit $H \to \infty$ with θ_0 and L held constant, and the question is whether the difference in pulse times goes to zero, or equivalently whether the distance $\Delta d \equiv d - d_0$ goes to zero. From the figure one sees that the limiting value is in fact $\Delta d = L \sin \theta_0$, so it approaches zero only when the ship's speed is also zero. The carried light clock gets out of synch with the non-carried clock, as seen by the stationary
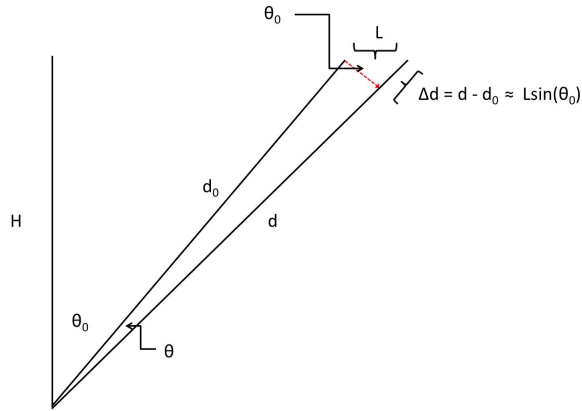
FIG. 2. Transported light clock.

observer, even when the extra movement is arbitrarily slow.

The vertically oriented light clock again provides a particularly clear demonstration because wave travel times are visualized as geometric distances. For light clocks of other orientations, and for other forms of clock, the synchronization effect is best demonstrated algebraically; however, the qualitative explanation in terms of extra motion remains the same.

It is, of course, much more usual to consider methods of clock synchronization using light signals. These methods, such as the famous "Einstein train" thought experiment, are confusing in our view because they conceal the fundamental physical origin of the effect. Synchronization by signalling gets its validity from prior knowledge that light speed is measured to be the same by all observers, but this knowledge can only be gained through experiments involving clocks and rulers which are physically transported. Moreover, observer-independence of the speed of light depends critically on the synchronization effect, so explanations of the latter in terms of the former have a circular character. We note also that, in field-based theories which are not Lorentz-invariant, synchronization by signalling becomes invalid but the physical mechanisms affecting carried clocks are still qualitatively the same.

An essential sublety is that the synchronization discrepancy *does* vanish when the observer's velocity goes to zero ($\theta_0 = 0$ on Fig. 2). This is important since otherwise there would be no reasonable way to synchronize clocks, even within a single reference frame. In

an arbitrary field theory, in which Figs. 1,2 may be complicated by different wave speeds in different directions and for different fields, there is, in fact, no guarantee that observers can construct useful reference frames by carrying clocks or any other means; the motion-dependent effects may simply become too complex. (Of course, it is also unlikely that any observers would exist in such a universe.)

V. WAVE PACKETS, MASSIVE PARTICLES, AND THE RELATIVISTIC MASS

The relativistic mass increase

$$m_{rel} = \gamma m,$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

seems rather ad-hoc when introduced in the context of particle mechanics. It is difficult to understand why an elementary and indivisible piece of matter should become harder to accelerate when moving faster.

This effect is, however, quite natural if a particle is conceived as a wave packet, which moreover is the way particles actually appear in modern physics. Given the central nature of the concept and the fact that students should already be familiar with most of the necessary wave mathematics, we suggest that students might benefit from seeing a wave packet description of particles during their study of relativity. A treatment along these lines is sketched here.

Working in one spatial dimension for simplicity, we suppose first that matter is actually described by a wave ξ obeying the simplest possible second-order wave equation, that of transverse waves on a string:

$$-\partial_t^2 \xi + \partial_x^2 \xi = 0, \tag{2}$$

where we have set all the constants to one for simplicity. All waves in this model travel with fixed speed $c = 1$ regardless of frequency, hence the model corresponds to "massless" waves such as light. To model massive particles, which is our main interest here, one adds a linear restoring force at every point, which can be visualized as placing Hooke's law springs on either side of the the string at frequent intervals:

$$-\partial_t^2 \xi + \partial_x^2 \xi - m^2 \xi = 0. \tag{3}$$

Here the restoring force is labeled as m^2 in anticipation that m will, in fact, be understandable as the mass of a "particle" in this system.

The $m > 0$ case differs qualitatively from the massless case in two important ways. First, the presence of the "springs" slows down the waves, since they cannot pass through a region of space without first exciting the springs. Second, the massive waves are able to sit still, at least approximately, oscillating in place under the restoring force provided by the mass term. The massless waves, by contrast, cannot sit still, since the only restoring force comes from neighboring parts of the string.

The elementary wave solutions take the form $\cos(kx \pm \omega t), \sin(kx \pm \omega t)$, where the angular velocity ω and wavenumber k satisfy

$$\omega^2 = k^2 + m^2. \tag{4}$$

The elementary sine and cosine solutions don't look much like particles, since they extend over all space, but this can be remedied by building a "wave packet", which is a superposition of waves having similar k values, for example:[8]

$$\xi(x,t) = \int_{\bar{k}-\Delta k}^{\bar{k}+\Delta k} dk \, \cos(kx - \omega t). \tag{5}$$

At $t = 0$, the waves are all in phase at $x = 0$, but interfere destructively elsewhere, creating a localized packet having approximate location $\bar{x} = 0$ and approximate wavenumber $\bar{k}$ (and corresponding angular velocity $\bar{\omega}$). As t changes, the location of the (approximate) in-phase maximum moves, and with it the wave packet. By substituting

$$\omega \approx \bar{\omega} + \left.\frac{\partial \omega}{\partial k}\right|_{\bar{k}} (k - \bar{k}), \tag{6}$$

one sees that the packet moves with approximate velocity given by the "group velocity"

$$v = \left.\frac{\partial \omega}{\partial k}\right|_{\bar{k}} = \frac{\bar{k}}{\bar{\omega}}. \tag{7}$$

We henceforth drop the bar notation and use k, ω, x for the wave packet's central values.

This "waves on a string" model is not so far from the real description of particles in modern theories. The non-particle-like extended solutions are also found in real quantum theories, but they are quickly converted to more localized wave packets by interaction with other clumps of matter.[9] Quantization also prevents the waves from simply dissipating away to zero.

A wave packet can be subjected to a force and made to accelerate by placing it in an external potential field, e.g. through a coupling such as $\phi(x)\xi^2$, which is equivalent to having the "spring tension" m^2 vary with position (and is essentially a Higgs field coupling), or more complex couplings such as $\phi(x)\xi\dot{\xi}$, which resembles the coupling of an electric potential.

The qualitative origin of relativistic mass can be seen immediately, since the group velocity satisfies $v < 1$ for all values of k. The packets cannot attain this "spring-free" limiting speed because their propagation is hindered by the presence of the "springs"; hence, they must become more and more difficult to accelerate as they approach $v = 1$. In other words, their effective mass must grow with v! As the limiting speed is approached, the energy applied to accelerate the packet goes instead into internal vibrations.

To see this in more detail, we start with the standard expressions for energy and momentum of a vibrating string (including the less-standard m term, whose contribution is simply a Hooke's law energy):

$$\mathcal{E} = \int dx \, \frac{1}{2} \left\{ (\dot{\xi})^2 + (\xi')^2 + m^2\xi^2 \right\}$$
$$\mathcal{P} = \int dx \, \dot{\xi}\xi'. \tag{8}$$

Evaluating these for a wave packet gives

$$E = \omega^2 N^2$$
$$P = \omega k N^2 \tag{9}$$

where N^2 is the squared norm, $N^2 = \int dx\xi^2$. (These equations are actually approximate, depending on the width of the wave packet in $k-$space, but for simplicity we will often write such equations as equalities.)

To go further in the program of constructing particles out of wave packets, one has to decide what value of N^2 constitutes a single particle. Not just any arbitrary convention will do, but rather it must be one which is preserved, at a minimum, under conditions of slowly-varying external forces ("adiabatic processes"), because such processes should simply move particles around without creating or destroying them. Hence, we need to know the behavior of N^2 as the wave packet moves in a slowly-varying potential field.

Similar questions were famously debated in the early days of quantum mechanics, and it turns out that classical mechanics does provide a suitable adiabatic invariant, which for the

harmonic oscillator takes the form E/ω.[10] This invariant also applies to the wave packets, because the vibrating string is essentially a collection of harmonic oscillators, one for each k, as can be seen by fourier-transforming Eq.(3) in the spatial variable.

In other words, the wavepacket norm N will evolve such that

$$\frac{E}{\omega} \approx \omega N^2 \approx const \tag{10}$$

and the normalization convention must be consistent with this. We will choose the simplest option,

$$N(\omega) = \frac{1}{\sqrt{\omega}}, \tag{11}$$

which is also the normalization arrived at through quantization.

The $\frac{1}{\sqrt{\omega}}$ dependence of the normalization plays an important role in both relativity and quantum mechanics, and we emphasize that it is not arbitrary, nor introduced ad-hoc in order to get relativistically correct results, but rather is determined by the basic physics of harmonic oscillators. We note also that, for complex fields, in many cases it is elevated to an exact result, the conservation of the electric charge Q, defined as

$$Q = \int dx \, Im(\xi^* \dot{\xi}). \tag{12}$$

It is easy to verify that Q is invariant, $\dot{Q} = 0$, for waves ξ satisfying Eq.(3), even if m is taken to be a Higgs potential $\phi(x, t)$ which varies in both position and time.

Applying this to Eq.(9), one finds for the single-particle energy and momentum

$$E = \omega$$
$$P = k, \tag{13}$$

which are the well-known relations proposed by Einstein for photons, and then by de Broglie for matter waves (in units with $\hbar = 1$). These relations are seen to be naturally contained in the elementary physics of waves on a string.

A force F, by definition, acts to change the wave packet momentum by

$$\dot{P} = F \tag{14}$$

and hence one has

$$F = \dot{k} \tag{15}$$

which is exactly as expected for a force acting on the relativistic mass Eq.(1), since Eqs.(4,7) imply

$$\omega = m\gamma \tag{16}$$

and

$$k = m\gamma v. \tag{17}$$

Hence, the relativistic mass and its associated force law are embedded in the simple physics of a vibrating string, which also (with many additional complications) is the physics of the real fields describing particles in nature. It is interesting also to go further and derive the standard relativistic action for a point particle; this is done in Sec.XIII.

We note for future reference that factors of ω in any expression involving wave packets are essentially equivalent to relativistic γ factors, as seen in Eq.(16).

VI. INERTIALLY-DOMINATED TIME DILATION

The relativistic mass increase, Eq.(1) provides another mechanism of time dilation, since a potential gradient which accelerates a wave packet in one direction will also reduce its speed in orthogonal directions, decreasing the frequency of cyclical motions. This can be seen in a simple "clock" consisting of a circular orbiting system, accelerated slowly along the axis perpendicular to the orbit.

The tangential force applied to the orbiting particle is zero, hence the orbital momentum doesn't change, but since the effective mass does increase one finds that the orbital speed is reduced:

$$0 = \frac{d}{dt}(m\gamma v_\perp) = m\frac{d}{dt}(\gamma v_\perp) \tag{18}$$

implying $\gamma v_\perp = const$, or

$$v_\perp \propto \frac{1}{\gamma}. \tag{19}$$

Assuming that the orbit radius does not change (as also implied by conservation of angular momentum), this implies that the orbital period increases by the time dilation factor γ.

Actually we have been slightly imprecise, since the particle's γ-factor differs from that of the overall system. Denoting the system's overall velocity as $\vec{\beta}$, one has $\vec{\beta} \perp \vec{v_\perp}$, and

$$\bar{\gamma} = (1 - (\vec{\beta} + \vec{v}_\perp)^2)^{-\frac{1}{2}}$$
$$= (1 - \beta^2 - v_\perp^2)^{-\frac{1}{2}}$$
$$= \gamma(1 - \gamma^2 v_\perp^2)^{-\frac{1}{2}}. \tag{20}$$

However, this leads to the same conclusion as before, since the force equation $0 = \frac{d}{dt}(m\bar{\gamma}v_\perp)$ still depends on γ and $v_\perp$ only through the combination $\gamma v_\perp$.

Similar reasoning applies to other clocks using inertial mechanisms, for example a vertically-oriented "bouncing ball" clock, the massive analog of the light clock. As the ship holding the clock slowly accelerates transverse to the bounce directions, the same acceleration must be applied to the ball to keep it bouncing between the two plates. The clock's caretakers on board the ship must do this without applying any vertical force, because that invalidates the system's timekeeping function even within their own reference frame. Hence they will maintain the clock through small impulses perpendicular to its bounce direction, leading to the same slowing effect seen with the accelerated orbit.

For general orientations (non-transverse), the calculations are much more complex and involve not just the relativistic mass, but also changes to the potential fields, as described in the following section. One must certainly resort to Lorentz invariance to prove precise time dilation for an arbitrary system (and conversely, the precise time-dilation formula will not be true for systems lacking Lorentz invariance). However, the qualitative mechanisms can be understood from the simpler configurations.

VII.   CHANGES TO POTENTIAL FIELDS

We have not yet addressed length contraction directly, and it is indeed difficult to demonstrate constructively, at least in a quantitatively correct fashion. The problem is that lengths of objects are determined by the shapes of quantum orbitals, whose calculation about a moving center of force is very difficult without resorting to Lorentz invariance. Even in the classical limit of particle orbits the only tractable example appears to be the transversely-oriented circular orbit discussed above.

The qualitative origin of length contraction is, however, very easy to see, and has been

recognized since Fitzgerald's famous "letter to Science" in 1889.[11] It is caused by the change in shape of potential fields about a moving source, a change which is made inevitable by basic aspects of the wave propagation by which the field is generated. Rather than rigorously derive the shape change, which is done in standard textbooks,[12] or attempt to explicitly construct length contraction from it, we will be content to describe how the moving-source potential connects to familar aspects of wave propagation. It is hoped that this will make it more intuitively obvious that the shape of the potential field must change, from which it is fairly inescapable that the shape of any object built using such fields will also change.

We consider first a scalar field, which is essentially a Higgs field, and produces a standard $1/r$ potential in three dimensions. The field generated by a source moving with constant velocity $\vec{\beta}$ has the following form:

$$\phi = \frac{1}{\gamma R(1 - \vec{\beta} \cdot \hat{R})}. \tag{21}$$

The three factors in the denominator come about as follows. The γ factor reflects the decrease in amplitude of the source wave packet as it is accelerated to higher speed, Eq.(11) (note that ϕ will typically be sourced by the wavepacket norm N^2 through a coupling such as $\phi(x)\xi^2(x)$). The R factor is the familiar $1/r$ falloff for a potential field in three dimensions, but measured from the retarded emission point along the trajectory of the source; this retardation correction ultimately reflects the fact that the potential is generated by emission of spherical waves from the source, and waves inherently move at finite speed. The final factor, $1 - \vec{\beta} \cdot \hat{R}$, can be thought of as an amplitude component of the Doppler effect, which persists for zero-frequency fields (here $\hat{R}$ is a unit vector in the retarded direction, along which R is measured). This factor can be seen most clearly in the simplest of all wave equations, the first-order equation in one dimension $\partial_t \phi + \partial_x \phi = 0$, for which the solution for a right-moving source is $\phi = \frac{\theta(x - \beta t)}{1 - \beta}$.

Taking motion in the $\hat{x}$ direction, $\vec{\beta} = \beta \hat{x}$, one can work out the geometry of retarded distance and verify that the potential is just the Lorentz-transformed $1/r$ potential,

$$\phi(\vec{x}, t) = \frac{1}{\sqrt{\gamma^2(x - \beta t)^2 + y^2 + z^2}}. \tag{22}$$

Hence one sees that this potential arises through generic wave mechanisms, in particular finite speed and the Doppler effect.

The EM fields about a moving source can be derived from the scalar field solution. The EM scalar ϕ_E has nearly the same form as the Higgs (which we will call ϕ_H). The only

difference is that the EM scalar couples to the adiabatically conserved quantity ωN^2 (which for charged particles becomes the exactly-conserved electric charge Eq.12), hence is not diminished by the extra $\frac{1}{\gamma}$ factor seen in Eq.(21). We can write then

$$\phi_E = \gamma \phi_H. \tag{23}$$

The EM vector potential $\vec{A}$ is generated in exactly the same way as ϕ_E, except that it couples to the velocity of the source. Hence for a moving point source one has

$$\vec{A} = \vec{\beta}\phi_E = \vec{\beta}\gamma\phi_H \tag{24}$$

resulting in electric field

$$
\begin{aligned}
\vec{E} &= -\vec{\nabla}\phi_E - \partial_t \vec{A} \\
&= -\vec{\nabla}\phi_E + \vec{\beta}(\vec{\beta}\cdot\vec{\nabla})\phi_E. \\
&= -\gamma\vec{\nabla}\phi_H + \gamma\vec{\beta}(\vec{\beta}\cdot\vec{\nabla})\phi_H. \tag{25}
\end{aligned}
$$

The gradient of the Higgs field, $\vec{\nabla}\phi_H$, does not remain radial about the moving center, but the electric field given by Eq.(25) is radial, as can be seen by substituting Eq.(22). The shape of the moving electric field, first calculated by Heaviside in 1888, is shown in Fig.3 below.
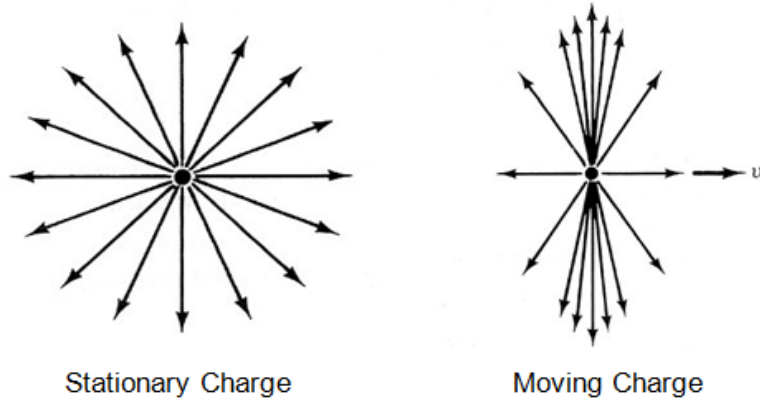

Stationary Charge   Moving Charge

FIG. 3. Electric field about a uniformly-moving source.

This famous and striking image shows the moving electric field "compressed" along the direction of motion, which certainly suggests that orbits and orbitals within the potential should experience similar contraction; at a minimum, they should undergo *some* shape

15

change. It is not hard to see the contraction approximately, for slowly-orbiting particles (Larmor and Lorentz did this during the decade prior to Einstein's work), but it is even easier to verify full Lorentz invariance and apply that. We do not attempt a more detailed treatment of orbits (or quantum orbitals) about a moving center, but we nevertheless feel confident in asserting, with Harvey Brown, that "shape deformation produced by motion is far from the proverbial riddle wrapped in a mystery inside an enigma".[2]

A precise construction of length contraction may not be essential for understanding the phenomenon, but students can still benefit from attempting an orbit calculation, if only to get a sense of just how complicated even this simple problem becomes in the wave-based setting. The first-order calculation is a simple exercise in perturbation theory, and one can also simulate the accelerating system on a computer, as suggested by Bell in his original manifesto.[1] A creative scenario due to Miller also allows the contraction to be seen without massive complication.[3]

VIII.   TIME DILATION OF DECAYS

Although usually treated within quantum field theory, the relativistic aspects of particle decays can be largely understood semiclassically. Indeed the essential factor can be seen already in the driven harmonic oscillator,

$$\ddot{x} + \omega^2 x = F. \tag{26}$$

Taking $F = \theta(t)\cos\omega t$ gives the solution $x = \frac{t}{2\omega}\theta(t)\sin\omega t$, showing that the amplitude increase is damped by a factor $\frac{1}{\omega}$. The same factor is seen more generally in the Green function

$$G(t - t') = \frac{1}{\omega}\theta(t - t')\sin\omega(t - t'). \tag{27}$$

The $\frac{1}{\omega}$ suppression carries over directly to particle decays since, as noted above, the particle fields are just collections of harmonic oscillators having $\vec{k}$-dependent frequency. Decays occur when one field drives one or more other fields at their resonant frequencies, and the factor of $\frac{1}{\omega}$ is essentially the relativistic dilation factor for moving wavepackets, as noted at the end of Sec.V. (By "resonant frequencies" in this context, we mean that both k and ω should satisfy Eq.4 for the field being driven.)

The simplest example comes not from true decay but rather oscillation between two particles having the same mass. In terms of the one-dimensional string model this corresponds to two parallel, identical strings attached to each other by additional springs running between them.

Letting X, Y be the displacements of the two strings, the springs running between them create a Hooke's law interaction energy $\frac{1}{2}g(X - Y)^2$. Expanding this, the X^2 and Y^2 terms just shift the mass m^2 of each string (recalling Eq.3), leaving the effective interaction term gXY. The coupled equations of motion then take the form

$$\ddot{X} - X'' + m^2 X = gY$$
$$\ddot{Y} - Y'' + m^2 Y = gX. \tag{28}$$

A complete basis of solutions is easily found by taking X, Y exactly in or out of phase, $X = \pm Y$. The out-of-phase modes stretch the springs connecting the two strings, hence have higher oscillation frequencies than the in-phase modes, and one finds

$$\Omega_\pm = \sqrt{k^2 + m^2 \mp g}$$
$$\approx \omega \mp \frac{g}{2\omega}, \tag{29}$$

where the last line is the small-g approximation.

To build a wave packet which initially exists only on the X string, one can combine two identical packets built using the in-phase and out-of-phase modes. The packets initially cancel each other on the Y string, but because they have slightly different frequencies $\Omega_\pm$, the initial "particle" will oscillate between the two strings at the beat frequency

$$\omega_{osc} = \frac{1}{2}(\Omega_- - \Omega_+)$$
$$= \frac{g}{2\omega}$$
$$= \frac{g}{2m\gamma}. \tag{30}$$

The last line shows the relativistic time dilation of the particle oscillation frequency (and used Eq.16). We note that the two packets will also separate over time, due to their differing group velocities, however this effect is of order $O(\frac{1}{\omega^2})$.

True multiparticle decays are more involved, but a semiclassical treatment can still be given showing the relativistic factors; this is done in Sec.XIV.

IX. DYNAMICAL EVOLUTION OF STATES

One aspect of relativity which is not always clear in the axiomatic viewpoint is which objects actually attain their Lorentz-transformed form after a dynamical acceleration process. Assume a system begins at rest in a state S and is adiabatically accelerated to velocity v, resulting in a state S'. Is this state identical with the state $L(S)$ obtained by Lorentz-transforming S to velocity v?

To make any statements about S', its evolution must be constrained by some quantity which is at least adiabatically invariant. To connect this to a Lorentz transformation, the same quantity must be Lorentz invariant. Such quantities can be found by looking at the phase and nodes of a standing wave; indeed, by counting cycles or nodes along well-defined lines or closed curves, one obtains integers which should be adiabatically invariant. Moreover, they will also be Lorentz invariant, since any proper Lorentz transformation can be built up "adiabatically" by applying numerous small transformations in succession. If these suffice to define the state S, then one must have $S' = L(S)$. Bound states in particular are constrained this way, since they are essentially three-dimensional standing waves, and since bound state solutions define the shape of objects, it follows that most objects will dynamically contract.

An example of distances which don't contract are the "disconnected" distances discussed by Miller, e.g. the distance between two separate spaceships.[3] If the ships begin at rest, with original distance D, and then accelerate to speed v, the final distance D' will not, in general, be equal to the contracted value $D\gamma_v^{-1}$; rather, it depends on the details of the acceleration (e.g., the rear ship could accelerate first and crash into the one in front!) If, however, the *observer* accelerates, then D' will contract as expected, because the real effect is occurring within the observer's own measuring devices, and hence applies uniformly to any distance which that observer subsequently measures (cf. Sec.XII).

Another example of objects which don't dynamically contract are the wavepackets themselves. To see this, consider the one-dimensional massive wave equation with a weak linear potential

$$\ddot{\xi} - \xi'' + m^2\xi = -fx\xi \tag{31}$$

and fourier-transform in the spatial variable to obtain

$$\partial_t^2\hat{\xi} + k^2\hat{\xi} + m^2\hat{\xi} = -if\partial_k\hat{\xi}. \tag{32}$$

For small f one has approximately the free-field behavior $\partial_t \hat{\xi}(k,t) \approx -i\omega_k \hat{\xi}(k,t)$, hence we can write a wave packet solution in the form $\hat{\xi}(k,t) = \psi(k,t)e^{i\omega_k t}$, where the envelope ψ is slowly-varying in time compared to $e^{i\omega_k t}$. Substituting into Eq.(32), and approximating $|\partial_t^2 \psi| \ll |\omega_k \partial_t \psi|$, gives

$$
\begin{aligned}
0 &= 2i\omega_k \partial_t \psi - \omega_k^2 \psi + k^2 \psi + m^2 \psi + if\partial_k \psi - ft\psi \frac{\partial \omega_k}{\partial k} \\
&= 2i\omega_k(\partial_t + \frac{f}{2\omega_k}\partial_k + \frac{if v_k}{2\omega_k}t)\psi
\end{aligned}
\tag{33}
$$

where $v_k \equiv \partial_k \omega_k = \frac{k}{\omega_k}$. This is solved by a k-space envelope which moves with time, plus a phase factor (the latter can be calculated as a power series in t; the lowest order is shown):

$$
\psi = \chi(k - \frac{f}{2\omega_k}t)e^{-i\frac{f v_k}{4\omega_k}t^2}.
\tag{34}
$$

As time progresses, the wave packet momentum therefore evolves approximately by

$$
\dot{k} = \frac{f}{2\omega_k},
\tag{35}
$$

where we focus on the case $f > 0$, $k > 0$, i.e. where the force is acting to increase the momentum. Since ω_k increases with k, this means that the packet is narrowing in k-space, hence broadening as a function of x; the packet expands rather than contracting (this is in addition to the natural spreading which every wave packet experiences over time).

The behavior changes if the potential is coupled to time derivatives of the field, so that f depends on ω. If $f \propto \omega$, the packet neither expands nor contracts moving in the potential field, but if $f = 2\omega^2$ (choosing the proportionality constant for convenience), the packet does contract relativistically, as can be seen by analyzing the change in time of its width in k-space (starting from Eq.35, applied to the momentum spread Δk of the packet):

$$
\begin{aligned}
\dot{k} &= \omega_k \\
\dot{\Delta k} &= \dot{\Delta \omega_k} = \frac{\partial \omega}{\partial k}\Delta k = v\Delta k
\end{aligned}
$$

$$
\tag{36}
$$

where v is the packet group velocity. Also (again using Eq.35)

$$
\dot{\omega} = \frac{\partial \omega}{\partial k}\dot{k} = \omega v
\tag{37}
$$

19

which is simply the Newtonian expression $\dot{E} = Fv$ for energy change in a force field. Combining the equations gives

$$\frac{\dot{\Delta k}}{\Delta k} = \frac{\dot{\omega}}{\omega} \tag{38}$$

which integrates to yield

$$\Delta k \propto \omega = m\gamma \tag{39}$$

showing that the packet's width in k-space expands by the relativistic factor, implying relativistic contraction of the width in position space.

Since $f \propto \omega^2$ corresponds to a force which couples to energy (recalling the $\dot{\xi}^2$ term in the expression for the energy of the vibrating string, Eq.8), i.e., to a gravitational force, the contraction might have been expected in this case, in view of the equivalence principle.

X. MASS AND ENERGY

In a world described by field theories, everything is "energy", hence understanding the mass/energy relationship mainly means understanding where mass comes from in the field theory context. According to Einstein, mass should be just another word for the energy content of the field configuration, so the question becomes how this actually translates into inertia, i.e., resistance to acceleration.

One of the primary mechanisms was already seen in Sec.V, namely the intrinsic mass of a field. This can be visualized as arising from "springs" having force constant m^2, which both impede the motion of wave packets, i.e. increase their inertia, and increase their internal frequency of oscillation, i.e. their energy. This is exactly in accordance with Einstein's formula (recall Eq.13; for $k = 0$ one has just $\omega = m$).

Closely related to this is the relativistic mass effect, by which kinetic energy of motion results in an increase of effective mass. This was also seen in the waves-on-a-string model (cf. Eqs.13,16), and is implicit in the fact that wave packets in the presence of "springs" (i.e, having $m > 0$) can never exceed the speed of spring-free packets ($m = 0$). Hence the packets must become harder to accelerate with speed, i.e., their inertia must grow. The energy which one applies to a wave packet ends up going, not into speed, but rather into internal oscillations of the field; the particle resembles an old, rattletrap car which vibrates badly at high speeds and can't be further accelerated.

The other principle mode of mass generation in field theories was actually the first piece of the relativity puzzle to be glimpsed, by J.J. Thomson, in the year of Einstein's second birthday.[13] Thomson realized that the EM field of a charged particle will act back on the particle itself, resisting its acceleration and adding to its inertia; part of this effect is just the familiar phenomenon of self-induction. This "back-reaction" mechanism, operating in the strong nuclear force, in fact is the source of nearly all inertia in the everyday world, only a small fraction of which arises from intrinsic masses (or Higgs mechanism).[14]

Back-reaction, plus relativistic mass, account for most examples of mass/energy connection. To take a concrete example, the energy of an excited state of hydrogen is greater than that of the ground state, hence the excited state mass must also be greater. The additional energy is EM potential energy, but how can that physically act as inertia, i.e., how can it impede the acceleration of the atom? The only possible mechanism is through backreaction of the internal EM field onto the charged particles. The EM portion of the mass difference is then accounted for by differing backreaction from the differing internal fields of the two states.[15] (We note also that the additional EM energy in the excited state is partially offset by a decrease in the electron kinetic energy, which translates into a lower contribution to the atom's overall inertia, by the relativistic mass effect.)

Eventually the excited state decays back to the ground state, emitting a photon and lowering its mass. Since the extra mass of the excited state was due originally to EM potential energy, this "conversion" of mass to energy is actually nothing but conversion of electromagnetic field energy to a different form.

XI. MAXIMUM SPEED

One of the most common questions asked about relativity is "why can't anything move faster than light?" To such a simple question one would expect a simple answer, but the most common answer is anything but. Typically one is told that superluminal signal propagation would allow backwards time travel, resulting in paradoxes, hence impossibility. In our view this answer, although not exactly incorrect, is not very informative since it provides little insight into the physical mechanisms at work.

Another standard answer, providing somewhat greater insight, is to note that the relativistic mass of an object diverges as it approaches light speed, making it physically impos-

sible to exceed this speed. However, this answer only applies to objects with mass, and it also begs further explanation of the relativistic mass effect itself.

In our view, a more comprehensible and arguably more accurate answer is to appeal directly to the underlying wave nature of the universe, and to the questioner's intuitive understanding of wave phenomena. From everyday experience with water waves, people understand that waves cannot be made to move faster than their natural speed. One can't speed up an ocean wave by pushing on it; such an effort merely creates more waves, not faster waves. The sound from a jet airplane or rifle shot travels the same speed as that from a turtle. The reason nothing in our universe moves faster than light is then simply that everything in the universe is built out of waves that have that speed. The massless waves move at exactly c, exactly analogous to familiar waves, while the massive ones are hindered by the added "springs" (recalling Sec.V) and hence move at lower speeds.

Implicit in this explanation, of course, is the idea that *everything* is made of waves, so that there is no form of motion in the universe other than wave motion. There is no analog of "supersonic" travel or Mach numbers for the fundamental waves making up the universe, because these refer to the speed of objects *other* than sound waves; an object which was somehow made from sound waves would be inherently unable to travel faster than the speed of sound, and likewise the objects in our universe cannot travel faster than the waves from which they are made.[16]

The question "why can't anything move faster than light" is then seen to be equivalent to asking why sound waves don't move faster than the speed of sound. This answer, although tautological at first glance, in fact reflects the true, underlying origin of the cosmic speed limit. It is deeper than the second answer, because the relativistic mass effect itself comes from underlying waves having bounded group velocity, as was described in Sec.V. It is also more informative than the first answer, because it places the phenomenon of bounded speeds within the wider physical context of wave motion, rather than the very specific context of Lorentz invariance, which entails specific combinations of fields all having the same speed constant.

XII. CONNECTING TO THE PRINCIPLE APPROACH

If everything is "made out of waves", then motion-dependent changes to objects and processes are generic and inevitable, as we have tried to illustrate in the previous sections. Students should not have trouble understanding these generic effects, since they connect to the most elementary types of waves studied in every curriculum.

Students should then be ready to take the next step, realizing that measurement itself makes use of various objects and processes, hence measurements made by differently-moving observers must also differ. This conclusion sounds quite dramatic when derived abstractly from axioms, but the pedagogical goal should be for students to think of it as just an obvious consequence of basic physical mechanisms that they already understand.

Realizing that all objects, including measuring devices, must undergo motion-dependent changes, the next logical question is whether some order can be brought to what seems like a potentially very complex or chaotic situation. How is it that we humans can consistently measure things, or even exist at all, given that motion affects all things, and that our planet is certainly moving? At this point students should be ready to more clearly understand the content and importance of the Lorentz symmetry and the principles it embodies. These may, indeed, make a greater impression on students when introduced through this longer route, as they will already have an idea of the vast range of complexities which can arise from wave-based underlying theories, and will be better able to appreciate both how surprising and how essential the Lorentz symmetry really is.

XIII. APPENDIX: RELATIVISTIC PARTICLE ACTION AND WAVE PHASE

To cement the connection between the particle-oriented approach to relativity and the wave packet derivations provided in Sec.V, it is worthwhile to show how the customary relativistic particle action[17]

$$L = - \int \frac{m}{\gamma} \tag{40}$$

arises from the fundamental wave theory. For simplicity we take the wave field ξ to be complex, which permits exponential solutions rather than sines and cosines. In terms of the string picture this just means allowing the string to vibrate in both transverse directions, although the "springs" creating the mass term become hard to visualize.

23

The standard action for a vibrating string, with the added Hooke's law mass term, is

$$S = \int dt dx \left\{ |\dot{\xi}|^2 - |\xi'|^2 - m^2 |\xi|^2 \right\}. \tag{41}$$

The Euler-Lagrange equation of motion following from this action is just Eq.(3).

With constant m there is no applied force, which results in trivial constant-speed motion for the wave packets. As mentioned in Sec.V, one can include a force from an external Higgs-like potential simply by allowing m to vary with x and/or t. We will continue to work in one dimension, although the results carry easily to three dimensions by converting k to a vector $\vec{k}$.

We seek an action which will govern wave packets for which k and ω vary slowly in time, i.e. for which $\dot{k}, \dot{\omega}$ are treated as small parameters and retained to first order. The wave packet can be written as a plane wave times a slowly-varying (complex) envelope function, leading to the ansatz

$$\xi = \exp i(kx - \omega t) F(x,t)$$
$$\dot{\xi} = [(-i\omega - it\dot{\omega} + ix\dot{k})F + \partial_t F] \exp i(kx - \omega t),$$
$$\xi' = (ikF + \partial_x F) \exp i(kx - \omega t), \tag{42}$$

and the action becomes (discarding terms quadratic in derivatives)

$$S \approx \int dt dx \{ (\omega^2 + 2t\omega\dot{\omega} - 2x\omega\dot{k} - k^2 - m^2)|F|^2 - 2Im(F^*(\omega\partial_t F + k\partial_x F)) \} \tag{43}$$

To reduce this to an action governing the "slow" changes $\dot{k}, \dot{\omega}$, we apply as constraints the "fast" consequences of the wave equation, $\omega^2 = k^2 + m^2$ and $N^2 = \frac{1}{2\omega}$, where the latter is Eq.(11) with an extra factor appropriate to the complex field.[18] An additional constraint is $F(x,t) = F(x - vt)$, expressing the fact that a wave packet built from modes satisfying the first constraint will move at the group velocity $v = \frac{k}{\omega}$; this eliminates the term involving derivatives of F. Then inserting the first constraint gives

$$S \approx \int dt dx (2t\omega\dot{\omega} - 2x\omega\dot{k})|\xi|^2. \tag{44}$$

Assuming the packet is relatively narrow, we can approximate x by the packet center

location and perform the dx integral, leading to

$$
\begin{aligned}
S &\approx \int dt (2t\omega\dot{\omega} - 2x\omega\dot{k})N^2 \\
&= \int dt (t\dot{\omega} - x\dot{k}) \\
&= \int dt (-\omega + \dot{x}k) \\
&= -\int dt \frac{m}{\gamma},
\end{aligned} \tag{45}
$$

which is the customary relativistic particle action (the last step used the group velocity $\dot{x} = \frac{k}{\omega}$; the third step integrated by parts, discarding the boundary terms, which don't affect the equations of motion arising from the action).

The derivation highlights the connections between the phase of the underlying wave, the particle's proper time, and Lorentz invariance. The third line can be written in terms of the phase θ as

$$
\begin{aligned}
S &= \int dt (-\omega + \dot{x}k) = \int dt \left(\frac{\partial\theta}{\partial t} + \dot{x}\frac{\partial\theta}{\partial x}\right) \\
&= \int dt \frac{d\theta}{dt} = \int d\theta.
\end{aligned} \tag{46}
$$

Hence the variational principle governing the particle's motion is to minimize (or extremize) the number of cycles of its underlying wave. Also, Lorentz invariance of S is equivalent to invariance of the phase, which can be seen in customary 4-vector form by noting

$$
d\theta = -\omega dt + \vec{k} \cdot d\vec{x} = k^\mu x_\mu. \tag{47}
$$

Lastly, from the last two steps of Eq.(45) the particle's proper time and phase are related by

$$
md\tau = d\theta. \tag{48}
$$

XIV. APPENDIX: MULTIPARTICLE DECAYS

Multiparticle decays require quantum mechanics for a correct treatment, but a heuristic sketch shows the important relativistic factors. We extend the previous two-field case of Sec.VIII to let a field X of mass m interact with N others $X_1, ..., X_N$, having masses m_i, with interaction $\mathcal{L}_I = gXX_1X_2 \cdots X_N$. We consider an initial wave packet in X, which we

will assume to be narrow enough in momentum space to be treated as a pure mode with fourier components k, ω.

The packet on X will also have amplitude proportional to $\frac{1}{\sqrt{\omega}}$, in accordance with Eq.(11); indeed the main concern will be to show how the various ω and ω_i factors combine to create a time-dilated decay rate. Details not connected to this, e.g. whether the fields are real or complex, will be ignored.

Classically, a solution with $X_i = 0$ for all i is stable and no decay occurs. In quantum mechanics, however, a field is never truly zero, but contains irreducible zero-point fluctuations, which we will label $\bar{X}_i$. No energy can be drawn from these fluctuations, but they can combine with true excitations to create resonances which do transfer energy. Focusing on X_1, for example, its equation of motion is

$$\ddot{X}_1 - X_1'' + m_1^2 = gXX_2 \cdots X_N \tag{49}$$

and one sees that the zero-point fluctuations $\bar{X}_2, ..., \bar{X}_N$ can modulate the X wavepacket to create a resonant driving term for X_1, transferring energy into that field.

The "vacuum" of zero-point fluctuations must be stable under adiabatic changes in external potentials, hence the amplitudes of these fluctuations must be governed by Eq.(11), so each mode of frequency ω_i comes with an amplitude factor $\frac{1}{\sqrt{\omega_i}}$, and we can write schematically

$$\bar{X}_i = \int \frac{dk_i}{\sqrt{\omega_i}} e^{i(k_i x - \omega_i t)}. \tag{50}$$

As in the driven harmonic oscillator, an exact resonance with an X_1 mode k_1, ω_1 leads to a linearly increasing amplitude, damped by $\frac{1}{\omega_1}$ (recall Eqns. 26,27). Hence, assuming that the X wavepacket comes into existence at $t = 0$ (or the interaction "turns on" at that time), as a first approximation one expects that each X_1 mode will grow as

$$X_1^{k_1, \omega_1} \propto \frac{gt}{\omega_1 \sqrt{\omega}} \int \frac{dk_2 \cdots dk_N}{\sqrt{\omega_2 \cdots \omega_N}} \delta(k - k_1 - k_2 - \cdots - k_n) \delta(\omega - \omega_1 - \omega_2 - \cdots - \omega_n). \tag{51}$$

Here the delta functions express the conservation of energy and momentum.

Next we note that the time dilation of the decay rate of X takes a simpler form when expressed in terms of energy. The single-particle X wavepacket has energy $\omega = m\gamma$, so a decay rate proportional to γ^{-1} is equivalent to a rate of energy loss which is independent of ω. To put this another way, if the rate of energy loss is independent of ω, then the time

taken to lose half of the starting energy is proportional to ω, i.e., to γ. Accordingly we will study the rate of energy emission into the fields X_i, rather than looking at the amplitudes directly, as would be done in a quantum mechanical treatment.

Looking again at X_1, by Eq(9) its energy is given in terms of the mode amplitudes as

$$E_1 = \int dk_1 \, \omega_1^2 \left| X_1^{k_1, \omega_1} \right|^2. \tag{52}$$

In view of Eq.(51) it is tempting to say the energy should grow as t^2, and this is correct for a problem with exactly matching modes, such as the X, Y case of Sec.VIII; indeed t^2 is the first term in the expansion of $\sin^2(t)$ for the energy transferred to the Y string in the oscillating solution.

Multiparticle decays don't have exactly matching modes, but rather each X mode is smeared by interaction with the $\bar{X}_i$, and the X_1 equation of motion then amplifies that vanishingly small subset of modes satisfying $\omega_1^2 = k_1^2 + m_1^2$. This would imply a rate of zero, were it not for an additional subtlety (here we are giving a heuristic explanation of "Fermi's Golden Rule" of quantum mechanics). Because the interaction begins at a particular time $t = 0$, the frequency ω_1 is not well-defined at the outset, but rather is subject to the "energy-time" uncertainty principle

$$\Delta\omega_1 \propto \frac{1}{t}, \tag{53}$$

which arises because it takes time for destructive interference to weed out the non-resonant modes (as seen explicitly writing the solution in terms of the Green function). The response is therefore increased at early times due to contributions from a wider range of driving modes, but for late times the contributions of modes not satisfying $\omega_1^2 = k_1^2 + m_1^2$ increasingly cancel out, so the response comes from an ever-narrower subset of the whole $k_2, k_3, ..., k_N$ space. The net energy emitted into X_i is then proportional to t, not t^2 (or zero).

The full result result for E_1 then looks as shown below, where again the main goal is to track the various ω factors which create the relativistic dilation effect. We note that the factors of ω_1 cancel, and that each factor of $\frac{1}{\sqrt{\omega_i}}$ for $i > 1$ is squared, as is the factor of $\frac{1}{\sqrt{\omega}}$:

$$E_1 \propto \frac{g^2 t}{\omega} \int dk_1 \int \frac{dk_2 \cdots dk_N}{\omega_2 \cdots \omega_N} \delta(k - k_1 - k_2 - \cdots - k_N)\delta(\omega - \omega_1 - \omega_2 - \cdots - \omega_N). \tag{54}$$

Writing $dk_1 = \omega_1 \frac{dk_1}{\omega_1}$ and adding together all of the E_i gives the total energy emission

$$\Delta E \propto g^2 t \int \frac{dk_1 \cdots dk_N}{\omega_1 \cdots \omega_N} \left(\frac{\omega_1 + \cdots + \omega_N}{\omega} \right) \delta(k - k_1 - k_2 + \cdots - k_N) \delta(\omega - \omega_1 - \omega_2 - \cdots - \omega_N)$$

$$= g^2 t \int \frac{dk_1 \cdots dk_N}{\omega_1 \cdots \omega_N} \delta(k - k_1 - k_2 + \cdots - k_N) \delta(\omega - \omega_1 - \omega_2 - \cdots - \omega_N). \tag{55}$$

In the final expression, the integral simply calculates the volume of $k-$space for the fields X_i for which the modes multiply to a resonant mode of X, but with the unusual volume element $(\omega_1 \cdots \omega_N)^{-1}$. In fact, as is well known, this volume element is exactly right to make the integral independent of k, ω! This can be seen by writing each dk_i integral in manifestly Lorentz-invariant form as

$$\int \frac{dk_i}{\omega_i} = 2 \int dk_i d\omega_i \, \delta(\omega_i^2 - k_i^2 - m_i^2). \tag{56}$$

A change to the initial k, ω (i.e., starting with a faster or slower moving packet) must preserve $\omega^2 = k^2 + m^2$, hence corresponds to a Lorentz transformation in k, ω. This may be counteracted by applying the opposite Lorentz transformation to each ω_i, k_i throughout the integral, leading to a velocity-independent rate of energy emission from the X wave packet into the fields X_i.

XV. ACKNOWLEDGEMENTS

XVI. REFERENCES

* wmn@cox.net

[1] J.S. Bell, reprinted in *Speakable and Unspeakable in Quantum Mechanics*, 2nd ed. (Cambridge University Press, Cambridge, 2004), pp. 67-80.

[2] H.R. Brown, *Physical Relativity*, 1st ed. (Oxford University Press, Oxford, 2005).

[3] D.J. Miller, "A Constructive Approach to the Special Theory of Relativity," Am. J. Phys. **78**, 633-638.

[4] W.M. Nelson, *Relativity Made Real*, 2nd ed. (Amazon/CreateSpace Publishing, 2013).

[5] A. Kostelecky, "The Search for Relativity Violations", Scientific American, September 2004

[6] A. Kostelecky, "Topics in Lorentz and CPT Violation", Arxiv:1010.4559

[7] This "slow-transport" method of clock synchronization was apparently first considered by Eddington; see Eddington, A., *The Mathematical Theory of Relativity* (Cambridge University Press, 1923).

[8] Wave packets and group velocity may not be familiar to students as they are not always discussed in introductory treatments of waves. See e.g. Hirose, A., and Lonngren, K, *Fundamentals of Wave Phenomena*, SciTech Publishing, 2010; Whitham, G.B., *Linear and Nonlinear Waves*, Wiley, 1974.

[9] N. Mott, "The Wave Mechanics of Alpha Ray Tracks", Proc. Roy. Soc. A **126**, p. 79. Also discussed by Bell, op. cit., ch. 15.

[10] For a recent treatment see C. Wells, S. Siklos, "The Adiabatic Invariance of the Action Variable in Classical Mechanics", Eur. J. Phys. **28**, 105-112.

[11] A recent discussion of this interesting episode can be found in H.R. Brown, "Michelson, FitzGerald and Lorentz: the origins of special relativity revisited", e-print PITT-PHIL-SCI 00000987; http://philsci-archive.pitt.edu/987/1/Michelson.pdf.

[12] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (John Wiley & Sons, New York, 1975)

[13] J.J. Thompson, "On the Electric and Magnetic Effects Produced by the Motion of Electrified Bodies," Philosophical Magazine **11** (1881), 229-249.

[14] F. Wilczek, "Happy Birthday, Electron," Sci. Am. June 2012, 24.

[15] W.M. Nelson, "On Back-reaction in Special Relativity", Arxiv:1303.6188; Am. J. Phys., to appear.

[16] A similar conception of a "water-world" is dicussed in Shupe, M., "The Lorentz-Invariant Vacuum Medium", American Journal of Physics 53, p. 122, 1985.

[17] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison Wesley, 1980), ch. 7.

[18] Note Eq.(41) transforms into two copies of the real scalar action using ansatz $\xi = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2)$.