Solution approaching Riemann Hypothesis

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ABSTRACT By following Riemann'a method, this paper extended discussion on $J(x)$ prime formula. We showed relationship between this formula with $J_q(x)$. Further more, we presented a new expression for Euler-Mascheroni constant γ, and showed a way for approaching proving Riemann Hypothesis.

Contents

1. INTRODUCTION

Nonation
x - a large real number
$p_i, i = 1, 2, 3, ...$ - a prime number where $2=p_1 < p_2 < ... < p_i < p_{i+1}$
$\pi(x)$ - number of prime numbers less than a given x
$J(x)$ - Riemann prime formula
$\zeta(s)$ - Riemann zeta function
p - a prime number
q -an integer number
$\mu(n)$ - the Mobius function
$\zeta_q(s)$ - zeta function exclude prime factor(s) of q's
$J_q(x)$ - Riemann prime formula exclude prime factor(s) of q's

For definiteness we recall the statement of Riemann Hypothesis. For $\Re(s) > 1$, the zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}(\frac{1}{n^s}) = \Pi_p(1 - p^{-s})^{-1} \qquad (1)$$

the product being over the prime numbers. Riemann found an explicit formula for the number of primes $\pi(x)$ less than a given number x. His formula was given in terms of the related function,

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \frac{1}{4}\pi(x^{1/4}) + \frac{1}{5}\pi(x^{1/5}) + ... \qquad (2)$$

which counts primes and a prime power up to x, counting p^n as $\frac{1}{n}$ of a prime. The number of primes can be recovered from this fucntion by

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{\frac{1}{n}})$$

where μ is the Mobius function.

Riemann found following equation,

$$\frac{1}{s} log(\zeta(s)) = \int_1^{\infty} J(x) x^{-s-1} dx, \qquad (3)$$

After applying Fourier's theorem, finally Riemann's prime formula is

$$J(x) = \mathrm{Li}(x) - \sum_{\rho} (\mathrm{Li}(x^{\rho})) - log(2) + \int_x^{\infty} \frac{1}{t(t^2-1)log(t)} dt \qquad (4)$$

where

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log(x)} dx$$

The sum is over the non-trivial zeros of the zeta function and where J_0 is a slightly modified version $J(x)$ that replaces its value at its points of discontinuity by the average of its upper and lower limits,

$$J_0(x) = \lim_{\epsilon -> 0} \frac{J(x-\epsilon) + J(x+\epsilon)}{2}$$

In the strip $0 < \Re(s) < 1$, the zeta function satisfies the function equation,

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) \qquad (5)$$

The Riemann Hypothesis (RH) is the assertion that all of the non-trivial zeros of the Riemann zeta function have real part equal to $\frac{1}{2}$.

It is also known that the Riemann Hypothesis is equivalent to

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x} \log x) \qquad (6)$$

2. EXTENDED DISCUSSION WITH RIEMANN PRIME FORMULA

Following Riemann's method, For $\Re(s) > 1$, if defines

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, (q, n) = 1 \qquad (7)$$

we can see (7) has similar product as (1), which satisfies,

$$\zeta_q(s) = \sum_{n=1}^{\infty} (\frac{1}{n^s}) = \Pi_p (1 - p^{-s})^{-1} \qquad (8)$$

the product being over the prime numbers excepts all q's prime factor(s). If defines

$$J_q(x)$$

which counts primes and a prime power up to x, counting p^n as $\frac{1}{n}$ of a prime, but here it needs $(p, q) = 1$.
It is easy to see

$$\frac{1}{s} log(\zeta_q(s)) = \int_1^{\infty} J_q(x) x^{-s-1} dx, \qquad (9)$$

and further more, we get an extended of Riemann prime function, which relates with non-trivial zeros of ζ_q,

$$J_q(x) = \mathrm{Li}(x) - \sum_{\rho} (\mathrm{Li}(x^{\rho})) + R(q, x) \qquad (10)$$

The sum is over the non-trivial zeros of ζ_q function, $R(q, x)$ is the function relates with q.

It will be not difficult to get what $R(q, x)$ is, and currently it is not quite certain that the main factors are

$$\mathrm{Li}(x) - \sum_{\rho} (\mathrm{Li}(x^{\rho}))$$

although it is most possible. I put here as opened as I really don't know. ;)

Lemma - 1
In $0 < \Re(s) < 1$, $\zeta(s)$ and $\zeta_q(s)$ has exact the same non-trivial zeros.

Seems to me there should be at least one different zero (Siegel Zero), but some mathematician told that it was obvious that Lemma-1 is correct without question. For Dirichlet L-functions the only interesting case for non-existence of Siegel zeros is for real (thus quadratic) nontrivial character.

After analysis extending to $\Re(s) < 1$, q's prime factor(s) contribute zeros only locating on $\Re(s) = 0$.

3. FURTHER DISCUSSION ON $J_q(x)$

Following discussion will be based on Lemman-1 and extended Riemann prime formula (10). For formula (10), we are still uncertain what $R(q, x)$ is, but it should be a simple one which relating with q's prime factor(s) only, which is easy to control the boundary.

Let's assume x is a sufficient large real number,
(I) If q contains only 1 prime factor p, $2 \leq p < x^{\frac{1}{2}}$,

$$J(x) - J_q(x) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... \qquad (11)$$

as zeros for $\zeta(s)$ and ζ_q are exact the same, they will not show up in (11). We have following product for EulerMascheroni constant γ,

$$\gamma = \lim_{x \to \infty} (J(x) - J_q(x) - log(log_p(x))) \qquad (12)$$

(II)

$$J(x) > J_2(x) > ...J_{p_i}(x) > J_{p_{i+1}} > ... \qquad (13)$$

(III) If q contains only 1 prime factor p, $x^{\frac{1}{2}} < p < x$,

$$J(x) - J_q(x) = 1 \qquad (14)$$

(IX) If q contains all prime factors within $(x^{\frac{1}{2}}, x)$,

$$J(x) - J_q(x) < O(\sqrt{x} \log x) \qquad (15)$$

(X) If q contains all prime factors within $(1, x^{\frac{1}{2}})$,

$$J(x) - J_q(x) = \pi(x) - \pi(x^{\frac{1}{2}}) \qquad (16)$$

(XI)

$$J(x) > J_2(x) > J_{2*3}(x) > .... > J_{2*3*5*...*p_i*p_{i+1}...}(x) > ... \qquad (17)$$

We can see if i goes to sufficient large in

$$J_{2*3*5*...*p_i*p_{i+1}...}(x)$$

there will be only one or less prime factor can be counted, hence,

$$0 < J_{2*3*5*...*p_i*p_{i+1}...}(x) \leq 1.$$

If recall formula (10), we have following boundary,

$$0 < J_q(x) = \mathrm{Li}(x) - \sum_{\rho}(\mathrm{Li}(x^{\rho})) + R(q,x) \leq 1 \qquad (18)$$

therefore,

$$\mathrm{Li}(x) - \sum_{\rho}(\mathrm{Li}(x^{\rho}))$$

can be evaluated by $R(q, x)$ only, which becomes essential part of this paper.

It is reasonable to get more positive result for approaching proving RH if one can,
(a) evaluate the RH equivalent product for formula $J(x)$.
(b) calculate $R(q, x)$.

By clarifying relationship on (a), (b) and (18), hopefully, we could answer the question from Hilbert when he awakes after 100 years.

4. CONCLUSION

Riemann Hypothesis plays very important role not only in number theory but also in other area, even in physics area. There are several different versions on different fields, like GRH, ERH, etc. Searching documents, we can not find someone has deep research on Riemann prime formula, as which is replaced by another "convenient" function $\varphi(x)$. Riemann's prime formula is the primary one which connects ζ's non-trivial zeros with primes. $\pi(x)$ shows relation with ζ's zeros only after Mobius Funtion, however, nowadays, we paid much efforts on it. We may get relationship for different Dirichlet L-functions if we define a good formula (like Riemann prime formula), when investigating them with nontrivial character.

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