Equally aged twins under different accelerations

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Abstract

In a very recent book on the philosophy of space and time, Maudlin discusses Langevin's famous twin paradox, emphasizing the incorrectness of attributing the different agings of the two twins to the different accelerations they suffered. Here we extend the argument a little by considering to and fro rectilinear motions under finite accelerations (and continuous velocities).

1 Introduction

Langevin's twin paradox has generated a large literature on its own, and by now its dissipation should be clear enough to everyone. However, even today one can read examples of mistaken explanations here and there, some of them attributing the difference in ages between the two twin brothers to the fact that one of them felt an acceleration and thus the symmetry between their reference frames would have been destroyed, since the traveling twin is the only one to feel the corresponding effects. There are multiple critiques to this interpretation, but here our aim is

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to show, in a straightforward manner, that different accelerations do not lead necessarily to different agings. So the assumed cause, i.e., different accelerations, does not in fact entail the pretended effect of different agings.

2 The case of two twins

Following the discussion in Maudlin [1], we will first use only the invariance of the relativistic interval between any two events in our discussion, without recourse to the Lorentz transformations, by considering the extreme case of instantaneous changes in the direction of motion along a straight line. In this manner, the geometrical characteristics of the Minkowski metric of spacetime are all that is needed to put forward the argument in a purely algebraic way.

In Minkowskian special relativity one considers occurrences or events as the building elements of all physical phenomena. Examples would be collisions between bodies, reflections of light rays on surfaces, and so on. These events take place in space and time and, once one adopts some appropriate inertial reference frame R, one can attach definite coordinates to each one of them. For the present purposes, this common approach should be enough, but a deeper discussion can be followed in Maudlin's book. Then, following Minkowski, one defines the relativistic interval Δs between any two events E_1 and E_2, with coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2), by the positive solution of the equality

$$\Delta s = \sqrt{c^2(t_2 - t_1)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2}, \quad (1)$$

a result expressed in meters, naturally ($c = 3 \times 10^8$ m/s being the speed of light in vacuum). The geometry of Minkowski spacetime is such that the number obtained for the interval between any two events is independent of the reference frame in which their respective coordinates are being measured. That is to say, spacetime intervals are relativistic invariants, if $\Delta s'$ is associated with the prime coordinates from another frame R', then $\Delta s = \Delta s'$. In particular, one can choose the reference frame attached to some physical object, its proper frame (such as one of the twin brothers), and then, not only is the difference in spatial coordinates between any two events that happen to that body zero (we are considering twins reduced to single mathematical points !), but also the time lapse between those same events has now a special meaning, being called the "proper time" between those events, traditionally symbolized by the greek letter τ. Consequently, one measures the interval $c\Delta\tau$ in the proper frame (with $\Delta x = \Delta y = \Delta z = 0$), and the same value in any other inertial frame even though the spatial distances are no longer zero, that is to say, $c\Delta\tau = \Delta s$.

Let us now consider the Minkowskian spacetime diagram in the figure below. For simplicity, only the x space coordinate is depicted, along the horizontal axis, and the time, also measured in meters through the product ct, runs along the vertical axis. This is the frame attached to twin A, its proper frame. As time goes by, twin A occupies successive positions along the positive vertical axis, always at $x = 0$ (the green line). Twin B, on his part, starts moving along the positive x-axis of twin A's frame, and his successive positions are depicted on the Minkowski diagram as occupying the red line from the origin $(0, 0)$ to spacetime point $(5, 4)$, i.e., reaching the $x = 4$ m coordinate of this frame at the instant $ct = 5$ m as measured in twin A's frame, and then from this point in spacetime to point $(ct = 10, x = 0)$, that is to say, back to the origin, meeting twin A again (here we are using exactly the same numerical example as in Maudlin's).

Now, given the metric properties of Minkowski spacetime, the interval between the two events, of twin A seeing his twin B leave and having him back again will be

$$\Delta s\,|^{A}_{(0,0)\to(10,0)}= \sqrt{(10-0)^2-(0-0)^2} = 10 = c\Delta\tau \tag{2}$$

as measured by twin A. So, A aged $\Delta\tau = 10/c$ seconds between these two events.

Between these same two events, labeled as $(0, 0)$ and $(10, 0)$ in this frame, the total interval measured with respect to twin B, which occupies coordinate $x = 4$ at the intermediate time $ct = 5$, is given by the sum of the intervals obtained for each leg:

$$\Delta s\,|^{B}_{total}= \Delta s\,|^{B}_{(0,0)\to(5,4)} + \Delta s\,|^{B}_{(5,4)\to(10,0)}\ . \tag{3}$$

Substituting for the concrete values in the example, we get

$$\Delta s\,|^{B}_{total}= \sqrt{(5-0)^2-(4-0)^2} + \sqrt{(10-5)^2-(0-4)^2} = 6, \tag{4}$$

there is to say, the time lapse between the two events, as measured by the B twin in his own frame (not used above) but using the data above from A's frame, will be $\Delta s\,|^{B}_{total}= 6/c = \Delta\tau$ seconds, less the one corresponding to twin A, $10/c$.

3 Mistaken explanations

Maudlin [1] considers three kinds of confusion surrounding the explanation of the twin paradox. The first one claims that the effect can not occur because it would represent a breach of the supposed total equivalence between reference frames ("all motion is relative"), which is in fact false. The second confusion, perhaps the most common, argues that what distinguishes the two twins A and B is the acceleration felt unequivocally by twin B; this different acceleration would by itself imply the different agings (he cites both Rindler [2] and Feynman [3], in this regard, but, in fact, the examples are numerous). As a matter of fact, what matters, as Maudlin points out, is the length of the trajectories in the spacetime diagrams, and not their particular shape (in this regard, see also reference [5], page 125 : "proper time is not integrable"). It is this second confusion that matters to us, presently. Finally, there is the confusion about the supposed effects of "speed" at which some clock moves on the time it gives, whereas, in fact, speed is irrelevant.

As just mentioned, what matters here is the common argument according to which it is the fact that twin B suffered an acceleration at the intermediate position, here $(ct = 5, x = 4)$, that justifies him aging less, establishing a causal connection of the kind *different accelerations* $\Rightarrow$ *different agings*.

4 Unequal accelerations, equal agings. I.

We consider, then, the common explanation that invokes the fact that the two twins suffer different accelerations and that it is this circumstance that ultimately justifies their different agings. Now, if it were true that different accelerations imply a brake up of symmetry and thus different agings, then, by *modus tollens*, equal agings should necessarily imply having suffered equal accelerations. That this is not so will be clear from an analysis of the diagram. We take again the two twins, A, B and, as before, A stays put at $(ct, x = 0)$ starting at $(ct = 0, x = 0)$, whereas B moves from the origin $(ct = 0, x = 0)$ to spacetime point $(ct = 5, x = 4)$ on the first leg, and from this to spacetime point $(ct = 10, x = 0)$ on the second leg, i.e., back to the origin.

But we now consider a third twin brother, C, who performs a more crooked trajectory in spacetime, leaving the vicinity of twin A at the same time and place as twin B (i.e., at the spacetime origin $(0, 0)$), but moving in the negative direction along the x axis at a constant speed until he reaches the spacetime point $(ct = 2.5, x = -2)$ where he inverts his unidimensional trajectory and returns to the space origin $x = 0$, meeting twin A at spacetime point $(ct = 5, x = 0)$. Here he rebounds once more, moves along the negative x axis to point $x = -2$ again, reaching it at time $ct = 7.5$, and then back to the position of twin A, at $x = 0$, which he reaches at time $ct = 10$ (the blue line in the diagram).

Let us now compute the total relativistic interval for twin C. We have

$$\Delta s \,|^{C}_{total} = \Delta s \,|^{C}_{(0,0)\to(2.5,-2)} + \Delta s \,|^{C}_{(2.5,-2)\to(5,0)} + \Delta s \,|^{C}_{(5,0)\to(7.5,-2)} + \Delta s \,|^{C}_{(7.5,-2)\to(10,0)} \tag{5}$$

$$= \sqrt{(2.5-0)^2 - (-2-0)^2} + \sqrt{(5-2.5)^2 - (0+2)^2} + \sqrt{(7.5-5)^2 - (-2-0)^2} + \sqrt{(10-7.5)^2 - (0+2)^2} = 6$$

in meters. One can see that this is exactly the same value obtained for the relativistic interval in the case of twin B.

5 Unequal accelerations, equal agings. II.

The calculations above are very common in introductory texts and extend a little the exposition in Maudlin's book. But one may be interested in verifying that the same conclusion can be drawn in the realistic case of finite accelerations. To that effect, we draw a second diagram, Fig. 2 below, in which twins B and C suffer one–g accelerations in particular time intervals, as observed in their own frames. Specifically, twin B suffers a constant acceleration in the positive x-axis direction, not with respect to the A-twin's frame ([4], Prob. 51), but rather with respect to the instantaneous co-moving inertial frame(s), for a proper time of 10 years, say, until he reaches position B_1 along his world-line on the diagram. He then reverses the thrust of the engines and begins to de-accelerate, again feeling a one–g force but in the opposite direction inside the spaceship, for another period of 10 years of his own time. At point B_2 on the diagram he reaches speed zero and a maximum distance x_R from twin A. Here he begins his journey back to meet twin A, always at rest at the origin, by first accelerating, again with a proper acceleration of one–g, in the negative x-axis direction for 10 years more of his proper time, reaching point B_3, where he again reverses the thrust and slows down, with a positive proper acceleration of one–g, to meet his brother A, at point B_4, 10 years later than point $B3$, in B's proper time. Hence, C's total journey took him 40 years of his lifetime.

In the mean time, twin C starts his trip as well, also feeling accelerations of one–g in opposite directions inside his own spaceship, as he moves back and fro along the negative x-axis, as depicted in Fig.2. But, in this case, the proper time periods for each acceleration only last 5 years of C's proper time. Consequently, since his journey has 8 legs, he will age a total 40 years, the same as his brother B, in spite of their different acceleration histories.

In the present situation we appeal to the Lorentz transformations in order to compute the positions of the points along the world-lines of twins B and C on the Minkowski diagram. But, as explained in [4], it is convenient to re-formulate these transformations using the so-called velocity-parameter, θ, due to the non-additivity of velocities in special relativity. So, from the inverse Lorentz transformations expressed for infinitesimal changes in space and time values,

$$\begin{aligned} dx &= (1-\beta_r^2)^{-1/2}dx' + \beta_r(1-\beta_r^2)^{-1/2}cdt' \\ cdt &= (1-\beta_r^2)^{-1/2}cdt' + \beta_r(1-\beta_r^2)^{-1/2}dx', \end{aligned} \tag{6}$$

where, as usual, $\beta_r = V/c$ represents the speed of some inertial frame R' relative to some other inertial frame R. From the same Lorentz transformations one can easily deduce the expression for the addition of velocities,

$$v = \frac{v' + V}{1 + v'V} \tag{7}$$

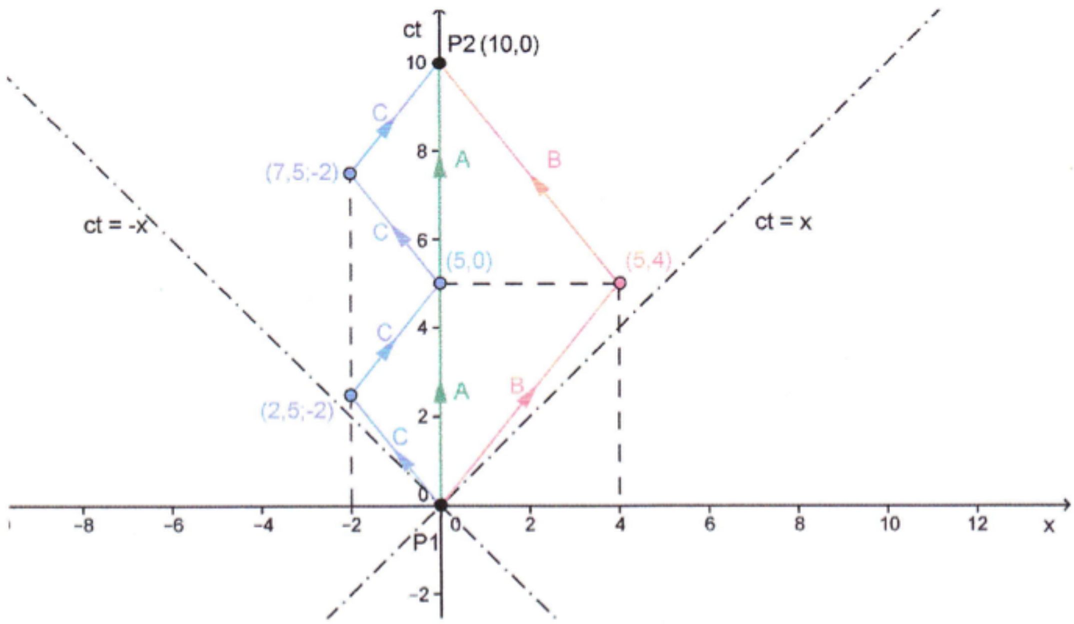

Figure 1: Instantaneous changes in speed

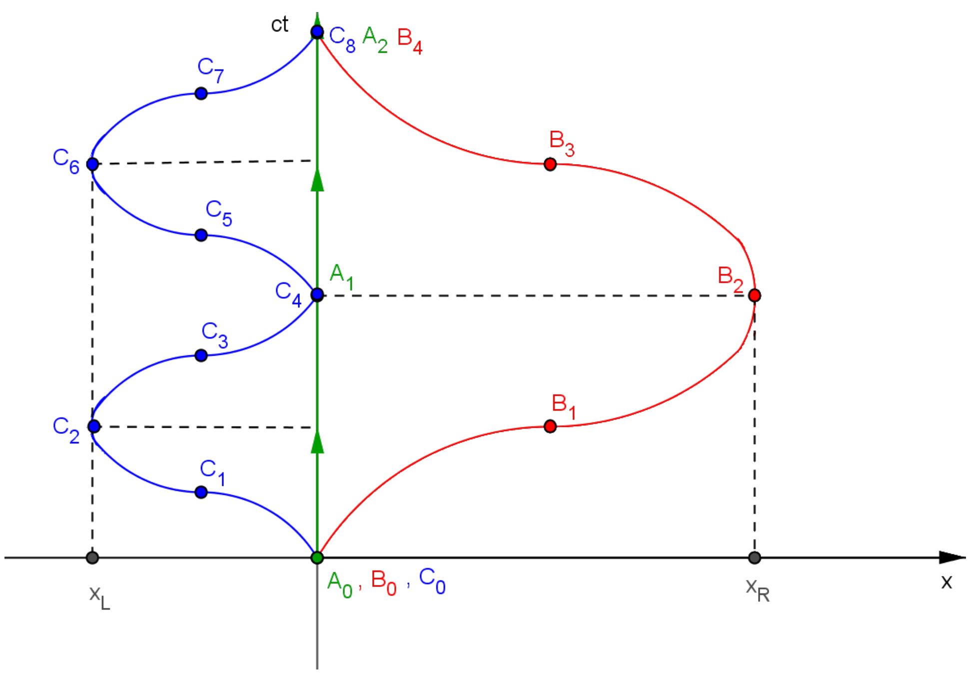

Figure 2: Smooth changes in speed

or

$$\beta = \frac{\beta' + \beta_r}{1 + \beta'\beta_r} \tag{8}$$

in which v is the velocity of some object with respect to frame R, whereas v' will be the velocity of the same object but now with respect to frame R'. Now, instead of using the velocities themselves, one can use the velocity parameter θ defined in general by the equality $\beta = \tanh(\theta)$ with the convenience that velocity parameters are simply additive, $\theta = \theta' + \theta_r$, basically because $\tanh(x+y) = [\tanh(x) + \tanh(y)]/[1 + \tanh(x)\tanh(y)]$.

In terms of the velocity parameter θ_r, the (inverse) Lorentz transformations become

$$\begin{aligned} dx &= \cosh(\theta_r)dx' + \sinh(\theta_r)cdt' \\ cdt &= \cosh(\theta_r)cdt' + \sinh(\theta_r)dx', \end{aligned} \tag{9}$$

In describing the case of the accelerating twins, the fundamental relations are then $\theta = (g/c)\tau$, where τ is the proper time of a moving twin subject to a proper acceleration of value g, $dt = \cosh(g\tau/c)d\tau$, and $dx = \sinh(g\tau/c)d\tau$, since obviously $dx' = 0$ for a twin in his own (instantaneous) moving frame [4]. From this last expression we can compute the distance travelled by twins B and C in the first leg of their respective journeys. We have then, for twin B,

$$\begin{aligned} \Delta x(B_0 \to B_1) &= \int_{\tau_0=0}^{\tau_1=10yrs} \cosh\left(\frac{g_B\tau}{c}\right) d\tau \\ &= \frac{c^2}{g_B}\cosh\left(\frac{g_B\tau}{c}\right)\Big|_{\tau_0}^{\tau_1} \\ &= 14434 \;\; ly \end{aligned} \tag{10}$$

where ly = one light-year = 9.46×10^{15}m. Here we are taking the acceleration of twin B to be $g_B = 9.8$ m/s^2. Also, the time between events B_0 and B_1 as measured in the rest frame of twin A is given by

$$\begin{aligned} \Delta t(B_0 \to B_1) &= \int_{\tau_0=0}^{\tau_1=10yrs} \sinh\left(\frac{g\tau}{c}\right) d\tau \\ &= \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right)\Big|_{\tau_0}^{\tau_1} \\ &= 14480 \;\; years \end{aligned} \tag{11}$$

Taking into due account the velocity of twin B at point B_1, we can similarly compute the the distance travelled for the next 10 years of proper time, between points B_1 and B_2 and conclude that one obtains the same value, so that $x_R =$

$2 \times 14434 = 28868$ ly. So, the total 40 year journey for twin B will correspond to $4 \times 14480 = 57920$ years of Earth time, when twin A is long dead !

Now, C's acceleration on each leg of his world-line as given can not be equal to 9.8 m/s^2 also because we want to make sure that, in A's frame, the time coordinate of event C_2 equals the time coordinate of event B_1 and that, ultimately, twins B and C arrive back on Earth simultaneously (i.e, that events B_4 and C_8 occur at the same time and place in all frames), both having aged a total of 40 years. Imposing this condition then, and taking advantage of the relation above $dt = \cosh(g\tau/c)d\tau$ in its integral form, we obtain $g_C = 18.3$ m/s^2 (!).

6 Conclusion

This pedagogical exercise aims to show the well but not universally known fact that it is not the acceleration history of the traveling twins that ultimately justifies their different agings. In fact, what we tried to show in this example was that even though twins B and C suffer very different fates along their respective journey, they end up by aging exactly the same number of years. Of course, being such an old paradox, it has been properly solved many times by different authors in more or less complicated expositions. The above, we believe, can be followed by anyone familiar with an introductory text such as Taylor and Wheeler's.

References

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