Methods of finding infinitely many integer and rational solutions of wide class of Diophantine equations

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Abstract // Streszczenie

This article contains my Diophantine equations solutions. **I am presenting this mathematical work mainly to attract attention to my proof that special relativity is false** that you can find on vixra.org under title "***Proof that special relativity is false***".

I have worked on diophantine solutions for more than two years. I can prove that my work is completely independent from the work of others and that two years ago I had solution to (as I call it) *general case* for solutions without little Fermat theorem and *simple case* with little Fermat theorem, which is much more than others achieved, but I didn't want to publish it until it would be complete. I sent it to the Polish profesors of mathematics and to myself so I really can prove and document that I had it two years ago. I sent it for example on 10/26/2011 to polish full professor PhD. Edmund Puczylowski (http://www.mimuw.edu.pl/wydzial/organizacja/pracownicy/edmund.puczylowski.xml) from Univeristy of Warsaw and I can prove it with my correspondence with him (I gave full content of this document that I sent to him in A*ppendix 1*). I sent also some diophantine solutions (the simplest case with use of little Fermat theorem) to full professor PhD. Jerzy Tiuryn from Univeristy of Warsaw (http://www.mimuw.edu.pl/wydzial/organizacja/pracownicy/jerzy.tiuryn.xml) on 02/23/2011 and I can prove it too.

I've searched the Internet and found very little work on this matter:

1.) Wolfram – nothing.
2.) Wikipedia: Fermat Last Theorem/Diophantine equations – single special case;
3.) http://cp4space.files.wordpress.com/2012/10/moda-ch12.pdf – that does not define all solutions

But what I've seen is that:

1.) There is given really very little solutions in comparison to my solutions,
2.) There are not all solutions of (as I call it) "*general*" or at least "*simple*" case of presented equations for the cases like for example: $ua^x + wb^y = vc^z$
3.) There is not proof that presented solutions are all such (wich I call "*complex not derived*") solutions for any case, like for example: $ua^x + wb^y = vc^z$,
4.) There is not proof when there exist such (*complex not derived*) solutions,
5.) There are not solutions for simultaneous equations,
6.) There are not solutions for rational exponents,
7.) As I know work of others contains only case of solution when

$$\sum_{i=1}^{n} \frac{c_i}{d} a_i^{x_i} = b^z = \left(\sum_{i=1}^{n} \frac{c_i}{d} l_i^{x_i} \right)^{t*lcm(x)+1}$$

or even only $\sum_{i=1}^{n} a_i^{x_i} = b^z = \left(\sum_{i=1}^{n} l_i^{x_i}\right)^{t*lcm(x)+1}$ which is very little. And does not show how to solve equation without solving $qz = t * lcm(x) + 1$, so this algorithm to solve equation has not complexity $O(1)$ while my has $O(1)$.
8.) There is no solution given for any case (especially for general case) to equations that has coefficient not equal to 1 on the right side.

Which all and much more I've done in this article.

If my Diophantine equation solutions are not enough I also give a inverse function to $Li(n)$ function. I think it should be enough.

I named this kind of Diophantine equation that I've described in this article after my surname, because I need to refere to them in this article.

Finally I can present part of my work. Thanks for reading. I have more and I will publish it in my book that should come out next year.

Please, give me an endorsement on arxiv (on physics, math), If you can. My username on arxiv: Zbigniew_Plotnicki

(and let me know at my e-mail address: Zbigniew.Plotnicki.proofs@hotmail.com)

If you find any error, let me know too.

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Important note

Where there is not stated otherwise, there variables with the same name but different indexes are different variables. Often for example a is a set of variables a_i for every i or set of variables $a_{i,j}$ for every i, j, but only in these cases when it is stated so. Sometimes there is used a variable with name $x_{i,}$ where there is comma after i, which means that it is set of $x_{i,j}$ for every j. The same is for case $x_{,i}$ where comma is before i, which means that it is set of $x_{j,i}$ for every j.

Whenever there is a talk about integers or rationals there are on mind only positive intergers or rationals unless it is told otherwise, eg. : $x(-)$ is integer.

All variables are variable, unless there is told othwerwise, eg.: $(\rightarrow)x$.

And that is all – there is no other rules in variable names reading and identification. You will see that it is very clear notation when it will comes to more complicated cases.

A - Plotnicki's equations – part I

Method 1 – Płotnicki’s equation with use of little Fermat theorem – the simplest case

Method 1: Method of finding infinitely many complex solutions for equation like this:

$$\sum_{i=1}^{n} \boldsymbol{c_i a_i^{x_i}} = \boldsymbol{db^z}$$

where $\mathbf{gcd}\left(\prod_{i=1}^{n} \boldsymbol{x_i}, \boldsymbol{z}\right) = \mathbf{1}$ **and** $\boldsymbol{z}$ **is prime.**

where for every $\boldsymbol{i}$**:** $(\rightarrow)\boldsymbol{c_i}, \boldsymbol{a_i}, \boldsymbol{b}, (\rightarrow)\boldsymbol{d}$ **are rationals and** $(\rightarrow)\boldsymbol{n}, (\rightarrow)\boldsymbol{x_i}, (\rightarrow)\boldsymbol{z}$ **are integers.**

Proof

First of all we can use little Fermat’s theorem:

When z is prime and $gcd\left(\prod_{i=1}^{n} x_i, z\right) = 1$ then we can use little Fermat’s theorem:

$\left(p\left(r_i * lcm(x_1, \dots, x_n)\right)^{z-1} mod\ z\right) = p, gcd(r_i * lcm(x_1, \dots, x_n), z) = 1$, then z divides $(qz - k)(r_i * lcm(x_1, \dots, x_n))^{z-1} + k$

So we have infinitely many solutions in form:

$$\sum_{i=1}^{n} \boldsymbol{c_i} \left(\left(\frac{\sum_{i=1}^{n} \boldsymbol{c_i l_i^{x_i}}}{\boldsymbol{d}} \right)^{\frac{(\boldsymbol{qz-k}) * \frac{(\boldsymbol{r * lcm(x_1, \dots, x_n)})^{\boldsymbol{z-1}}}{\boldsymbol{x_i}}}{\boldsymbol{k}}} * \boldsymbol{l_i} \right)^{\boldsymbol{x_i}} = \boldsymbol{d} \left(\left(\frac{\sum_{i=1}^{n} \boldsymbol{c_i l_i^{x_i}}}{\boldsymbol{d}} \right)^{\frac{\mathbf{1}}{\boldsymbol{k}}} \right)^{(\boldsymbol{qz-k})(\boldsymbol{r * lcm(x_1, \dots, x_n)})^{\boldsymbol{z-1}} + \boldsymbol{k}}$$

For any integer r such that $\gcd(r, z) = 1$.

For any rationals c_i, d, l_i.

And for any integer k, q such that $k < qz$ and k is prime or 1 and $\sum_{i=1}^{n} c_i l_i^{x_i} = dn^k$ then this equation could be solved the same way for $k > 1$ and could be any l_i for $k = 1$.

In general we have rational solutions above and when $\frac{\sum_{i=1}^{n} c_i l_i^{x_i}}{d}$ and for every i: l_i are integers, then we have integer solutions.

QED.

Example:

$$wa^x + vb^y = c^z$$

We have:

$$w\left((wl^x + vm^y)^{\frac{(qz-k)(xy)^{z-2}y}{k}} * l\right)^x + \; v\left((wl^x + vm^y)^{\frac{(qz-k)(xy)^{z-2}x}{k}} * m\right)^y$$

$$= (wl^x + vm^y)\left((wl^x + vm^y)^{\frac{1}{k}}\right)^{(qz-k)(xy)^{z-1}}$$

$$= \left((wl^x + vm^y)^{\frac{1}{k}}\right)^k \left((wl^x + vm^y)^{\frac{1}{k}}\right)^{(qz-k)(xy)^{z-1}}$$

$$= \left((wl^x + vm^y)^{\frac{1}{k}}\right)^{(qz-k)(xy)^{z-1}+k} = \left(\left((wl^x + vm^y)^{\frac{1}{k}}\right)^{\mathrm{p}}\right)^{\mathrm{z}}$$

For $w = v = 1$:

$$\left((l^x + m^y)^{\frac{(qz-k)(xy)^{z-2}y}{k}} * l\right)^x + \left((l^x + m^y)^{\frac{(qz-k)(xy)^{z-2}x}{k}} * m\right)^y$$

$$= \left((l^x + m^y)^{\frac{1}{k}}\right)^{(qz-k)(xy)^{z-1} + k} = \left(\left((l^x + m^y)^{\frac{1}{k}}\right)^{\mathrm{p}}\right)^{\mathrm{z}}$$

Example

$$\mathbf{2x^2 + 3x^3 = x^5}$$

$$l = 2, m = 1$$

$$2l^2 + 3m^3 = 11$$

$$2\left(11^{(5-1)(2*3)^{(5-2)}*3} * 2\right)^2 + 3\left(11^{(5-1)(2*3)^{(5-2)}*2} * 2\right)^3 = 11^{(5-1)(2*3)^{5-1}}(11) = 11^{4*6^4+1}$$
$$= 11^{5185} = (11^{1037})^5$$

Method 2 – Płotnicki's equation with use of little Fermat theorem – simple case

Method 2: Method of finding infinitely many complex solutions for equation like this:

$$\sum_{i=1}^{n} c_i a_i^{x_i} = d b^z$$

where $\gcd(\prod_{i=1}^{n} x_i, z) = 1$ **and** z **is prime.**

where for every i**:** $(\rightarrow) c_i, a_i, (\rightarrow) d, b$ **are rationals and** $(\rightarrow) n, (\rightarrow) x_i, (\rightarrow) z$ **are integers.**

for every i: for every rational l_i and for every j: for every rational p_j, t and every integer q_j, f that suffices equation:

$$\sum_{i=1}^{n} c_i l_i^{x_i} = d t^{f*z} \sum_{j=1}^{m} p_j^{q_j}$$

where f could be 0, for every j: $\gcd(q_j, z) = 1$, we have infinitely many solutions:

$$\sum_{i=1}^{n} c_i \left(\prod_{j=1}^{m} p_j^{(t_j z - q_j)*\frac{\left(r_j * lcm(x_1,\ldots,x_n)\right)^{z-1}}{x_i}} * y^{\frac{lcm(x_1,\ldots,x_n,z)}{x_i}} * l_i \right)^{x_i} =$$

$$= \sum_{i=1}^{n} c_i l_i^{x_i} \prod_{j=1}^{m} p_j^{(t_j z - q_j)*\left(r_j * lcm(x_1,\ldots,x_n)\right)^{z-1}} * y^{lcm(x,z)}$$

$$= d t^{f*z} * \prod_{j=1}^{m} p_j^{(t_j z - q_j)*\left(r_j * lcm(x_1,\ldots,x_n)\right)^{z-1} + q_j} * y^{lcm(x,z)} = d c^z$$

Where y is any rational.

Where for every j: $gcd(r_j, z) = 1$.

Where c_i, d, l_i are any rationals and for every j: $q_j < t_j z$, where q_j, t_j are any integers.

In general we have rational solutions above, and when $\frac{\sum_{i=1}^{n} c_i l_i^{x_i}}{d}$ and for every i: l_i, y, t, p_i are integers, then we have integer solutions.

Example

$$\mathbf{4x^5 + 2y^3 = x^2}$$

For $l_1 = 1, l_2 = 2$:

$$4 * 1^5 + 2 * 2^3 = 4 + 2 * 8 = 20 = 2^2 * 5$$

$$4\left((2^2)^{(2-1)*\frac{(15)^{2-1}}{5}}(5)^{(2*2-1)*\frac{(15)^{2-1}}{5}}*1\right)^5+2\left((2^2)^{(2-1)*\frac{(15)^{2-1}}{3}}(5)^{(4-1)*\frac{(15)^{2-1}}{3}}*2\right)^3$$

$$=(2^2)^{(2-1)*(15)^{2-1}}(5)^{(4-1)*(15)^{2-1}}(4*1^5+2*2^3)=(2^2)^{15+1}(5)^{3*15+1}=((2^2)^8(5)^{23})^2$$

Method 3 – Płotnicki's equation with use of little Fermat theorem – general case

Method 3: Method of finding infinitely many nontrivial complex solutions for equation like this:

$$\sum_{i=1}^{n} \boldsymbol{c_i} \prod_{j=1}^{m_i} \boldsymbol{a_{i,j}^{x_{i,j}}} = \boldsymbol{d} \prod_{j=1}^{m_0} \boldsymbol{b_j^{z_j}}$$

where gcd $\left(\prod_{i=1}^{n} \prod_{j=1}^{m_i} \boldsymbol{x_{i,j}}, \prod_{j=1}^{m_0} \boldsymbol{z_j}\right) = \mathbf{1}$ **and for every** $\boldsymbol{j}$: $\boldsymbol{z_j}$ **is prime,** $\boldsymbol{a_{i,j}} > 1$.

where for every $\boldsymbol{i,j}$: $(\to)\boldsymbol{c_i}, \boldsymbol{a_{i,j}}, (\to)\boldsymbol{d}, \boldsymbol{b_j}$ **are rationals and** $(\to)\boldsymbol{n}, (\to)\boldsymbol{m_i}$ $(\to)\boldsymbol{x_{i,j}}, (\to)\boldsymbol{z_j}$ **are integers.**

for every i,j: for every rational $l_{i,j}$ and every rational $p_{i,j}, t_i$ and every integer $q_{i,j}, f_i$ that suffices equation:

$$\sum_{i=1}^{n} \boldsymbol{c_i} \prod_{j=1}^{m_i} \boldsymbol{l_{i,j}^{x_{i,j}}} = \boldsymbol{d} \prod_{i=1}^{u} \left(\boldsymbol{t_i^{f_i * z_i}} \prod_{j=1}^{v_i} \boldsymbol{p_{i,j}^{q_{i,j}}} \right)$$

where for every i: f_i could be 0, for every i,j: $\gcd(q_{i,j}, z_i) = 1$, we have infinitely many solutions:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} \left(\prod_{s=1}^{u} \left(\prod_{k \in S_{i,j,s}} p_{s,k}^{(t_{s,k} z_s - q_{s,k}) * \frac{\left(r_{s,k} * lcm(x)\right)^{z_s - 1}}{x_{i,j}}} * \prod_{k \in T_{i,j,s}} y_{s,k}^{\frac{lcm(x, z_s)}{x_{i,j}}} \right) * l_{i,j} \right)^{x_{i,j}}$$

$$= d \prod_{i=1}^{u} \left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j} z_i - q_{i,j}) * \left(r_{i,j} * lcm(x)\right)^{z_i - 1} + q_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{lcm(x, z_i)} \right)$$

$$= d \prod_{i=1}^{u} \left(t_i^{f_i} \prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{\frac{lcm(x, z_i)}{z_i}} \right)^{z_i}$$

Where for every i,j: $y_{i,j}$ is any rational.

Where for every i,j: $\gcd(r_{i,j}, z_i) = 1$.

Where for every i,s: $\bigcup_{j=1}^{m_i} S_{i,j,s} = \{1, \dots, v_i\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \dots, w_i\}$,

for every i,j,k,s where $j \neq k$: $S_{i,j,s} \cap S_{i,k,s} = \emptyset, T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

x is a set of all $x_{i,j}$, z is a set of all z_i.

Where c_i, d, l_i are any rationals and for every s,k: $q_{s,k} < t_{s,k} z_s$, where $q_{s,k}, t_{s,k}$ are any integers.

In general we have rational solutions above and when $\frac{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}}{d}$ and for every i, j: $l_{i,j}, y_{i,j}, t_i, p_{i,j}$ are integers, then we have integer solutions.

More generally:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} \left(\prod_{s=1}^{u} \left(\prod_{k \in U_{i,j,s}} p_{s,k}^{u_{i,j,s,k} * \frac{lcm(x)}{x_{i,j}}} * \prod_{k \in T_{i,j,s}} y_{s,k}^{\frac{lcm(x, z_s)}{x_{i,j}}} \right) * l_{i,j} \right)^{x_{i,j}}$$
$$= d \prod_{i=1}^{u} \left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j} z_i - q_{i,j}) * \left(r_{i,j} * lcm(x) \right)^{z_i - 1} + q_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{lcm(x, z_i)} \right)$$
$$= d \prod_{i=1}^{u} \left(t_i^{f_i} \prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{\frac{lcm(x, z_i)}{z_i}} \right)^{z_i}$$

for every i, s: $\bigcup_{j=1}^{m_i} U_{i,j,s} = \{1, \dots, v_s\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \dots, w_s\}$,

for every i, j, k, s where $j \neq k$: $T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

for every i, j: $z_i \mid \left((t_{i,j} z_i - q_{i,j}) * lcm(x)^{z-1} + q_{i,j} \right)$ {little Fermat theorem},

x is a set of all $x_{i,j}$, z is a set of all z_i.

Where for every i, s, k: $\sum_{j=1}^{m_i} u_{i,j,s,k} = (t_{s,k} z_s - q_{s,k}) * (r_{s,k})^{z_s - 1} * lcm(x)^{z_s - 2}$

B – Plotnicki's equations – part II

Theorem 1 – useful theorem

Theorem: $ab = t\prod_{i=1}^{n} c_i + x$, has integer solution for every a for given c_i, and given x, where $gcd(a, \prod_{i=1}^{n} c_i) = 1$.

You can use Chinese remainder theorem to get proof of the this problem, so there is always infinitely many solutions for:

$$\begin{cases} w = x\left(mod\left(\prod_{i=1}^{n} c_i\right)\right) \\ w = 0(mod\ a) \end{cases}$$

Where $gcd(\prod_{i=1}^{n} c_i\,, a) = 1$.

So every solution have to be in form $w_{k,l} = w + k * \prod_{i=1}^{n} c_i = w + l * a \Leftrightarrow k * \prod_{i=1}^{n} c_i = l * a$

So as $gcd(\prod_{i=1}^{n} c_i\,, a) = 1$ then:

$$w_k = w + k * a * \prod_{i=1}^{n} c_i = \left(\frac{w - x}{\prod_{i=1}^{n} c_i} + k * a\right) * \prod_{i=1}^{n} c_i + x$$

And that will be used in almost every Plotnicki's equation without use of a little Fermat theorem.

The simplest Diophantine equation and how to deal with d (part I)

$$\boldsymbol{wa^x = vb^y}$$

Where $\gcd(x, y) = 1$.

First of all we can divide equation by $\gcd(w, v)$, so we can assume $\gcd(w, v) = 1$

$$w\left(v^p * w^k * u^{\frac{lcm(x,y)}{x}}\right)^x = v\left(w^q * v^l * u^{\frac{lcm(x,y)}{y}}\right)^y$$

Now we can solve $qy = xk + 1, px = yl + 1$ {see *Theorem 1*}

$$\boldsymbol{w\left(v^p * w^{\frac{qy-1}{x}} * u^{\frac{lcm(x,y)}{x}}\right)^x = v\left(w^q * v^{\frac{px-1}{y}} * u^{\frac{lcm(x,y)}{y}}\right)^y}$$

And these are all integer solutions when w and v are primes, x, y, k, l, q, p are positive integers.

All solutions for:

$$w = \prod_{i=1}^{m} w_i^{q_i}$$

$$v = \prod_{i=1}^{m} v_i^{p_i}$$

$$\boldsymbol{w\left(\prod_{i=1}^{m} v_i^{\frac{lcm(p_i,p,x)}{x}} * \prod_{i=1}^{m} w_i^{\frac{lcm(q_i,q,y)-q_i}{x}} * u^{\frac{lcm(x,y)}{x}}\right)^x = v\left(\prod_{i=1}^{m} w_i^{\frac{lcm(q_i,q,y)}{y}} * \prod_{i=1}^{m} v_i^{\frac{lcm(p_i,p,x)-p_i}{y}} * u^{\frac{lcm(x,y)}{y}}\right)^y}$$

Where of course every $\boldsymbol{\frac{lcm(q_i,q,y)-q_i}{x}, \frac{lcm(p_i,p,x)-p_i}{y}}$ is solved to be positive integer {see *Theorem 1*}.

For three (where $\gcd(x, y) = \gcd(x, z) = \gcd(y, z) = 1$):

$$w\left(v^p * f^{r_1} * w^{\frac{qy-1}{x}} * u^{\frac{lcm(x,y,z)}{x}}\right)^x = v\left(w^q * f^{r_2} * v^{\frac{px-1}{y}} * u^{\frac{lcm(x,y,z)}{y}}\right)^y = f\left(w^{\frac{qy}{z}} * v^{\frac{px}{z}} f^{\frac{r_1 x-1}{z}} * u^{\frac{lcm(x,y,z)}{z}}\right)^z$$

$$r_2 y = r_1 x \Rightarrow r_1 = h\frac{lcm(x, y)}{x}, r_2 = h\frac{lcm(x, y)}{y}$$

$$w\left(v^{p\frac{lcm(x,z)}{x}} * f^{h\frac{lcm(x,y)}{x}} * w^{\frac{q*lcm(y,z)-1}{x}} * u^{\frac{lcm(x,y,z)}{x}}\right)^x = v\left(w^{q\frac{lcm(y,z)}{y}} * f^{h\frac{lcm(x,y)}{y}} * v^{\frac{p*lcm(x,z)-1}{y}} * u^{\frac{lcm(x,y,z)}{y}}\right)^y = f\left(w^{q\frac{lcm(y,z)}{z}} * v^{p\frac{lcm(x,z)}{z}} f^{\frac{h*lcm(x,y)-1}{z}} * u^{\frac{lcm(x,y,z)}{z}}\right)^z$$

And there is solution for general case (where for every different i, j: $\gcd(x_i, x_j) = 1$):

$$\boldsymbol{c_1 a_1^{x_1} = \cdots = c_n a_n^{x_n}}$$

$$\boldsymbol{c_k\left(\prod_{i=1}^{k-1} c_i^{p_i \frac{lcm(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}{x_k}} \prod_{i=k+1}^{n} c_i^{p_i \frac{lcm(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}{x_k}} * c_k^{\frac{p_k * lcm(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)-1}{x_k}} * u^{\frac{lcm(x_1,\ldots,x_n)}{x_k}}\right)^{x_k}}$$

Where of course every $\boldsymbol{\frac{p_k * lcm(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)-1}{x_k}}$ is solved to be positive integer {see *Theorem 1*}.

And all solutions for:

$$\boldsymbol{\left\{ for\ every\ i: c_i = \prod_{j=1}^{m_i} c_{i,j}^{p_{i,j}} \right.}$$

$$\boldsymbol{c_k\left(\prod_{i=1}^{k-1}\prod_{j=1}^{m_i} c_{i,j}^{\frac{lcm(p_i,p_{i,j},x_k)}{x_k} * \frac{lcm(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}{x_k}} \prod_{i=k+1}^{n}\prod_{j=1}^{m_i} c_{i,j}^{\frac{lcm(p_i,p_{i,j},x_k)}{x_k} * \frac{lcm(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}{x_k}} * \prod_{j=1}^{m_k} c_{k,j}^{\frac{\frac{lcm(p_k,p_{k,j},x_k)}{x_k} * lcm(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)-p_{k,j}}{x_k}} * u^{\frac{lcm(x_1,\ldots,x_n)}{x_k}}\right)^{x_k}}$$

Where of course every $\boldsymbol{\frac{\frac{lcm(p_k,p_{k,j},x_k)}{x_k} * lcm(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)-p_{k,j}}{x_k}}$ is solved to be positive integer {see *Theorem 1*}.

So first of all when x or y is odd we can solve:

$$wr_1^x + vr_2^y = 0$$

So we can solve for every f and $\gcd(k, l) = 1$:

$$\boldsymbol{w(gk)^x + v(gl)^y = fc^z}$$

Using analogous method we can solve for every d and $\gcd\left(\prod_{j=1}^{m_1} a_{1,j}^{x_{1,j}}, \prod_{j=1}^{m_2} a_{2,j}^{x_{2,j}}\right) = 1$

$$\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d \prod_{j=1}^{m_0} b_j^{z_j}$$

And we can easily find infinitely many solutions for:

$$\sum_{i=1}^{n} \boldsymbol{c_i a_i^{x_i}} = \boldsymbol{ndb^z}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} \boldsymbol{c_j a_j^{x_j}} = \sum_{i=1}^{n} \boldsymbol{m_i db^{z_i}}$$

And for example:

$$\sum_{i=1}^{n} \boldsymbol{c_i a_i^{x_i}} = \sum_{i=n+1}^{2n} \boldsymbol{c_i a_i^{x_i}}$$

So we can find infinitely many solutions if at least half of factors of sum of equation has odd power or negative coefficient.

As you can see the simplest Diopantine equations allow to solve not only equations when $\gcd(x, z) = 1$, where x is a set of exponents of variables $a_{i,j}$ on the left side of the equation and z is a set of exponents of variavbles b_i on the right side of the equation. To solve equation it is enough for example to pair factors of the equation such a way that for every factor on the left side $c_i a_i^{x_i}$ there is corresponding factor on the right side $c_j a_j^{x_j}$ for which $gcd(x_i, x_j) = 1$, so thanks to this we can solve every such pair $c_i a_i^{x_i} = c_j a_j^{x_j}$, so we can solve whole equation.

This method is really good also for more complex examples where you for example firstly solve equations like this $\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d \prod_{j=1}^{m_0} b_j^{z_j}$, to solve whole equation portion by portion. Here is very simple example:

$$a^3 + b^3 + c^3 = x^5 + y^5 + z^5$$

$$a^3 + b^3 = z^5$$

$$x^5 + y^5 = c^3$$

Then:

$$a^3 + b^3 - z^5 = x^5 + y^5 - c^3 = 0 \Rightarrow a^3 + b^3 + c^3 = x^5 + y^5 + z^5$$

Whats more in such a case we can solve:

$$\sum_{i=1}^{2n} c_i r_i^{x_i} = 0$$

So we can solve for any d and $gcd(l_1, \dots, l_{2n}) = 1$ {where l_i are defined as in method 4 furtther in this document}:

$$\sum_{i=1}^{n} c_i a_i^{x_i} = db^z$$

So equation:

$$\sum_{i=1}^{n} c_i a_i^{x_i} = db^z$$

that has at least half of factors with odd power or negative coefficient and where $a_i \neq 0, b \neq 0$, can be solved:

a.) When n is odd and $gcd\left(x_{p(i)}, x_{p\left(\frac{n+1}{2}+i\right)}\right) = 1$, then it can be solved with:

$$\sum_{i=1}^{\frac{n+1}{2}} c_{p(i)} a_{p(i)}^{x_{p(i)}} = \sum_{i=\frac{n+1}{2}+1}^{n+1} -c_{p(i)} a_{p(i)}^{x_{p(i)}}$$

where p is some permutation of $1, \ldots, 2n$.

b.) When n is even and $gcd\left(x_{p(i)}, x_{p\left(\frac{n+1}{2}+i\right)}\right) = 1$, then it can be solved firstly with:

$$\sum_{i=1}^{n} c_i r_i^{x_i} = 0$$

and then:

$$\sum_{i=1}^{\frac{n+1}{2}} c_{p(i)} r_{p(i)}^{x_{p(i)}} = \sum_{i=\frac{n+1}{2}+1}^{n+1} -c_{p(i)} r_{p(i)}^{x_{p(i)}}$$

where p is some permutation of $1, \ldots, 2n$.

This way we can find such a rests for which $\sum_{i=1}^{n} c_i l_i^{x_i}$ is divisible by d. Then there are solution of equation like is showed in Method 4.

There is of course also a generalization:

$$\prod_{i=1}^{n_1} w_{1,i} a_{1,i}^{x_{1,i}} = \cdots = \prod_{i=1}^{n_k} v_{k,i} b_{k,i}^{k,y}$$

So all that is said above applies also for general case of Płotnicki's equation.

So for example it can be used to solve for every d and $gcd(l) = 1$:

$$\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d \prod_{j=1}^{m_0} b_j^{z_j}$$

but it is not for this article. I will probably write about it in my book that will come out next year.

And that all is not all. The same easy we can find solutions for:

$$\sum_{i=1}^{m_1} d_{1,i} b_{1,i}^{x_{1,i}} = \cdots = \sum_{i=1}^{m_k} d_{k,i} b_{k,i}^{x_{k,i}} = c_1 a_1^{x_1} = \cdots = c_n a_n^{x_n}$$

Where:

for every i, l where $i \neq l$: $\gcd\left(\prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_l} x_{l,j}\right) = 1$.

for every i, l: $\gcd\left(\prod_{j=1}^{m_i} x_{i,j}, x_l\right) = 1$.

for every i, j where $i \neq j$: $\gcd(x_i, x_j) = 1$.

To find solutions it is enough to treat value of every $\sum_{i=1}^{m_j} d_{j,i} l_{j,i}^{x_{j,i}}$ for every $l_{j,i}$ for any i, j as a coefficient in equation. Then we have from equation above simply the same kind of equation for any $l_{i,j}$ for any i, j:

$$a_{-1}^{lcm(x_{1,})} \sum_{i=1}^{m_1} d_{1,i} l_{1,i}^{x_{1,i}} = \cdots = a_{-k}^{lcm(x_{k,})} \sum_{i=1}^{m_k} d_{k,i} l_{k,i}^{x_{k,i}} = c_1 a_1^{x_1} = \cdots = c_n a_n^{x_n}$$

The same is possible for general case of Plotnicki's equations.

Method 4 – **Płotnicki's equation – simple case**

Method 4: Method of finding infinitely many complex solutions for equation like this:

$$\sum_{i=1}^{n} \boldsymbol{c_i a_i^{x_i}} = \boldsymbol{db^z}$$

where $\mathbf{gcd}\left(\prod_{i=1}^{n} \boldsymbol{x_i}, \boldsymbol{z}\right) = \mathbf{1}$

where for every $\boldsymbol{i}$**:** $(\rightarrow)\boldsymbol{c_i}, \boldsymbol{a_i}, (\rightarrow)\boldsymbol{d}, \boldsymbol{b}$ **are rationals and** $(\rightarrow)\boldsymbol{n}, (\rightarrow)\boldsymbol{x_i}, (\rightarrow)\boldsymbol{z}$ **are integers.**

for every i: for every rational l_i and for every j: for every rational p_j, t and every integer q_j, f that suffices equation:

$$\sum_{i=1}^{n} \boldsymbol{c_i l_i^{x_i}} = \boldsymbol{dt^{f*z}} \prod_{j=1}^{m} \boldsymbol{p_j^{q_j}}$$

where f could be 0, for every j: $\gcd(q_j, z) = 1$, we have infinitely many solutions:

$$\sum_{i=1}^{n} c_i \left(\prod_{j=1}^{m} p_j^{(t_j+f_j*z)*\frac{r_j*lcm(x_1,\dots,x_n)}{x_i}} * y^{\frac{lcm(x_1,\dots,x_n,z)}{x_i}} * l_i \right)^{x_i} =$$

$$= \sum_{i=1}^{n} c_i l_i^{x_i} \prod_{j=1}^{m} p_j^{(t_j+f_j*z)*r_j*lcm(x_1,\dots,x_n)} * y^{lcm(x_1,\dots,x_n,z)}$$

$$= dt^{f*z} * \prod_{j=1}^{m} p_j^{(t_j+f_j*z)*r_j*lcm(x_1,\dots,x_n)+q_j} * y^{lcm(x_1,\dots,x_n,z)} = dc^z$$

Where y is any rational.

Where for every i: r_i is any integer such that $gcd(r_i, z) = 1$

Where for every j: t_j is any integer such that $z | \left(t_j * r_j * lcm(x_1, \dots, x_n) + q_j\right)$ {for details see: *Theorem 1*}

Where for every j: f_j is any integer.

In general we have rational solutions above and when $\frac{\sum_{i=1}^{n} c_i l_i^{x_i}}{d}$, and for every j: l_j, y, p_j, t are integers we have integer solutions.

And these are the only solutions for $gcd(a) > 1$ for most cases (of course only for l_i that is not a solution, BECAUSE THERE IS NOT GIVEN METHOD OF FINDING SUCH SOLUTIONS without using l_i that is not solution - that is obvious), where a is set of variables, which is proved later in this docoument for example for case $c_1 a_1^{x_1} \pm c_2 a_2^{x_2} = db^z$.

So for every l_i we have as much subclasses of solutions as much “images of divisibility” of given $\sum_{i=1}^{n} \frac{c_i}{d} l_i^{x_i}$ exists in form:

$$t^{f*z} \prod_{j=1}^{m} p_j^{q_j}$$

So for given $gcd(a_1, \ldots, a_n) = t^{f*z} \prod_{j=1}^{m} p_j^{t_j lcm(x_1,\ldots,x_n)}$

there is only one image of divisibility $t^{f*z} \prod_{j=1}^{m} p_j^{q_j}$ for which are constant numbers of l_i such that $\sum_{i=1}^{n} \frac{c_i}{d} l_i^{x_i} = t^{f*z} \prod_{j=1}^{m} p_j^{q_j}$, which has only above solutions.

And if equation has one solution : $\sum_{i=1}^{n} \frac{c_i}{d} l_i^{x_i} = b^z$, then it has infinitely many solutions:

$$\sum_{i=1}^{n} \frac{c_i}{d}\left(g^{t*\frac{lcm(x_1,\ldots,x_n,z)}{x_i}} * l_i\right)^{x_i} = \sum_{i=1}^{n} \frac{c_i}{d} l_i^{x_i} \; g^{t*lcm(x_1,\ldots,x_n,z)} = \left(b * g^{t*\frac{lcm(x_1,\ldots,x_n,z)}{z}}\right)^{z}$$

for every g,t

And those are all solutions that can be derived from $\sum_{i=1}^{n} \frac{c_i}{d} l_i^{x_i} = b^z$.

Derivation also works when $gcd(z, \prod_{i=1}^{n} x_i) > 1$.

Definitons:

When $gcd(a_1, \ldots, a_n) = 1$ then it is not complex solution.

When $gcd(a_1, \ldots, a_n) > 1$ then it is complex solution.

Where a is variables set.

And those are all solutions (derived from all not complex solutions) when there are not complex not derived solutions (when gcd($\prod_{i=1}^{n} x_i$,z) > 1).

So putting both together, when we know all not complex solutions (that the amount of is constant number or zero and such a is small), we know all solutions of Diophantine equation.

Method 5 – **Płotnicki's equation – general case**

Method 5: Method of finding infinitely many nontrivial complex solutions for equation like this:

$$\boldsymbol{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d \prod_{j=1}^{m_0} b_j^{z_j}}$$

where gcd $\left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} \boldsymbol{x_{i,j}}, \prod_{j=1}^{m_0} \boldsymbol{z_j}\right) = \mathbf{1}, \boldsymbol{a_{i,j}} > 1$

where for every $\boldsymbol{i,j}$ **:** $(\rightarrow)\boldsymbol{c_i}, \boldsymbol{a_{i,j}}, (\rightarrow)\boldsymbol{d}, \boldsymbol{b_j}$ **are rationals and** $(\rightarrow)\boldsymbol{n}, (\rightarrow)\boldsymbol{m_i}, (\rightarrow)\boldsymbol{x_{i,j}}, (\rightarrow)\boldsymbol{z_j}$ **are integers.**

for every i, j: for every rational $l_{i,j}$ and every rational $p_{i,j}, t_i$ and every integer $q_{i,j}, f_i$ that suffices equation:

$$\boldsymbol{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}} = d \prod_{i=1}^{u}\left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{q_{i,j}}\right)}$$

where for every i: f_i could be 0, for every i, j: $\gcd(q_{i,j}, z_i) = 1$, we have infinitely many solutions:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i}\left(\prod_{s=1}^{u}\left(\prod_{k\in S_{i,j,s}} p_{s,k}^{(t_{s,k}+f_{s,k}*z_s)*\frac{r_{s,k}*lcm(x)}{x_{i,j}}} * \prod_{k\in T_{i,j,s}} y_{s,k}^{\frac{lcm(x,z_s)}{x_{i,j}}}\right) * l_{i,j}\right)^{x_{i,j}}$$
$$= d\prod_{i=1}^{u}\left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j}+f_{i,j}*z_i)*r_{i,j}*lcm(x)+q_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{lcm(x,z_i)}\right)$$
$$= d\prod_{i=1}^{u}\left(t_i^{f_i} \prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{\frac{lcm(x,z_i)}{z_i}}\right)^{z_i}$$

Where

for every i, j: $y_{i,j}$ is any rational,

for every i, s: $\bigcup_{j=1}^{m_i} S_{i,j,s} = \{1, \dots, v_s\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \dots, w_s\}$,

for every i, j, k, s where $j \neq k$: $S_{i,j,s} \cap S_{i,k,s} = \emptyset$, $T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

for every i, j: $t_{i,j}$ is any integer such that: $z_i | \left((t_{i,j}) * r_{i,j} * lcm(x) + q_{i,j}\right)$ {for details see: *Theorem 1*},

for every i, j: $f_{i,j}$ is any integer,

for every i, j: $r_{i,j}$ is any integer such that $\gcd(r_{i,j}, z_i) = 1$,

x is a set of all $x_{i,j}$, z is a set of all z_i.

In general we have rational solutions above and when $\frac{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}}{d}$, and for every i,j: $l_{i,j}$, $y_{i,j}, p_{i,j}$, t_i are integers we have integer solutions.

More generally:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} \left(\prod_{s=1}^{u} \left(\prod_{k \in U_{i,j,s}} p_{s,k}^{u_{i,j,s,k} * \frac{lcm(x)}{x_{i,j}}} * \prod_{k \in T_{i,j,s}} y_{s,k}^{\frac{lcm(x,z_i)}{x_{i,j}}} \right) * l_{i,j} \right)^{x_{i,j}}$$
$$= d \prod_{i=1}^{u} \left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j}+f_{i,j}*z_i)*r_{i,j}*lcm(x)+q_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{lcm(x,z_i)} \right)$$
$$= d \prod_{i=1}^{u} \left(t_i^{f_i} \prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} y_{i,j}^{\frac{lcm(x,z_i)}{z_i}} \right)^{z_i}$$

for every i,j: $y_{i,j}$ is any rational integer,

for every i,s: $\bigcup_{j=1}^{m_i} U_{i,j,s} = \{1, \ldots, v_s\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \ldots, w_s\}$,

for every i,j,k,s where $j \neq k$: $T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

for every i,j: $z_i | \left((t_{i,j}) * r_{i,j} * lcm(x) + q_{i,j} \right)$ {for details see: *Theorem 1*},

for every i,j: $t_{i,j}$ is any integer such that: $z_i | \left((t_{i,j}) * r_{i,j} * lcm(x) + q_{i,j} \right)$ {for details see: *Theorem 1*},

for every i,j: $f_{i,j}$ is any integer (f_i is completely other integer with other meaning),

for every i,j: $r_{i,j}$ is any integer such that $\gcd(r_{i,j}, z_i) = 1$,

x is a set of all $x_{i,j}$, z is a set of all z_i.

Where for every i,s,k: $\sum_{j=1}^{m_i} u_{i,j,s,k} = (t_{s,k} + f_{s,k} * z_s) * r_{s,k}$

In general we have rational solutions above and when $\frac{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}}{d}$, and for every i,j: c_i, $l_{i,j}$, $y_{i,j}, p_{i,j}$, t_i are integers we have integer solutions.

Example 1

$$a^x + b^y c^z = d^w, where \gcd(xyz, w) = 1$$

$$k^x + l^y m^z = p_1^{q_1} * \ldots * p_m^{q_m} * t^{f*w}$$

Any divisor $p_i^{(t_i+f_i*z)*r_i}$ below can be divided between variables b and c like this: $p_i^{(t_i+f_i*z)*r_i} = \mathrm{p}_\mathrm{i}^{\mathrm{u}_1+\mathrm{u}_2}$, where $p_i^{u_1}$ is for b and $p_i^{u_2}$ is for c, where u_1 or u_2 can be 0. For example:

$$\left(\prod_{i=1}^{m} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}} * \prod_{i=1}^{k} y_i^{\frac{lcm(x,y,z,w)}{x}} * k\right)^x$$
$$+ \left(\prod_{i\in P_1} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}} * \prod_{i\in Q_1} y_i^{\frac{lcm(x,y,z,w)}{x}} * l\right)^y$$
$$* \left(\prod_{i\in P_2} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}} * \prod_{i\in Q_2} y_i^{\frac{lcm(x,y,z,w)}{x}} * m\right)^z$$
$$= \prod_{i\in(P_1+P_2)} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}+q_i} * t^{f*w} * \prod_{i\in(Q_1+Q_2)} y_i^{lcm(x,y,z,w)} = d^w$$

And simplier:

$$\left(\prod_{i=1}^{m} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}} * (y_1y_2)^{\frac{lcm(x,y,z,w)}{x}} * k\right)^x$$
$$+ \left(\prod_{i\in P_1} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}} * (y_1)^{\frac{lcm(x,y,z,w)}{y}} * l\right)^y$$
$$* \left(\prod_{i\in P_2} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}} * (y_2)^{\frac{lcm(x,y,z,w)}{z}} * m\right)^z$$
$$= \prod_{i\in(P_1+P_2)} p_i^{(t_i+f_i*z)*\frac{r_i*lcm(x,y,z)}{x}+q_i} * t^{f*w} * (y_1y_2)^{lcm(x,y,z,w)} = d^w$$

Where $P_1 + P_2 = \{1,..,m\}, Q_1 + Q_2 = \{1,..,k\}, P_1 \cap P_2 = \emptyset, Q_1 \cap Q_2 = \emptyset$

For example:

$$a^2 + b^3c^5 = d^7$$

$$2^2 + 2^3 * 2^5 = 260 = 26 * 10$$

$$t1 * (2 * 3 * 5) + 1 = 7q1$$

$$t2 * (2 * 3 * 5) + 1 = 7q2$$

$$t1 = 3, t2 = 3 + 7 = 10$$

so:

$$(26^{3*3*5} * 10^{10*3*5} * 2)^2 + (26^{3*2*3} * 2)^3 * (10^{10*2*3} * 2)^5 = (26)^{3*30+1} * 10^{10*30+1}$$
$$= (26^{13} * 10^{43})^7$$

For d^w it will give all complex solutions.

Example 2

$b^y c^z$ can be calculated as f^{y+z}, but it will not give all possible solutions, but there still is a way to calculate them:

$$d^w - a^x = b^y c^z, where \gcd(wx, yz) = 1$$

$$k^w - l^x = p_1^{q_1} * \dots * p_m^{q_m} * t_b^{f*y} * t_c^{g*z}$$

So p_i have to be selected such a way to contruct $b^y c^z$.

For example :

$$d^7 - a^2 = b^3 c^5$$

$$2^7 - 2^2 = 124 = 2^2 * 31$$

$$2 * 7 * t1 + 2 = 3q1$$

$$2 * 7 * t2 + 1 = 5q2$$

$$t1 = 2, t2 = 1$$

$$\left((61^{1*2} * 2^{2*2}) * 2\right)^7 - \left((61^{1*7} * 2^{2*7}) * 2\right)^2 = (2^7 - 2^2) * (2^{14} * 61^{14}) = (2^2 * 61) * (2^{28} * 61^{14}) = 2^{30} * 61^{15} = (2^{10})^3 * (61^3)^5$$

The same is for derivation:

$$\left(g^{t_1 * \frac{lcm(x,y,z,w)}{w}} * h^{t_2 * \frac{lcm(x,y,z,w)}{w}} k\right)^w - \left(g^{t_1 * \frac{lcm(x,y,z,w)}{x}} h^{t_2 * \frac{lcm(x,y,z,w)}{x}} * l\right)^x = (k^w - l^x)\left(g^{t_1 * \frac{lcm(x,y,z,w)}{y}}\right)^y * \left(h^{t_1 * \frac{lcm(x,y,z,w)}{z}}\right)^z$$

And the same is for combinations when there exist partial solved solution:

$$d^7 - a^3 = b^3 c^5$$

$$2^7 - 2^3 = 120 = 2^3 * (3 * 5)$$

There is always infinitely many complex not derived solutions only when $gcd(x, z) = 1$, where x is multiplication of all powers except those that are at some position (z); or there exists combination (there exist partially solved solution, eg.: $d^7 - a^3 = b^3 c^5, 2^7 - 2^3 = 120 = 2^3 * (3 * 5)$, where the condition should be sufficed only for those $x_{i,j}$ that are not solved; of course for example for $d^{11} - a^2 = b^3 c^5$ even for partially solved solution $(2^3) * (14^1)$ divisibilities could be exchanged $3 \to 5, 1 \to 3$); and there exist always infinitely many complex derived solutions if there exist at least one solution – proved.

So in general this is the way to calculate all rational complex solutiuons of Diophantine equations where there exist such j that $gcd(x, z) = 1$, where z is a multiplication of powers at some position in equation, eg.: $2x^3 + 3y^5 v^3 = 5z^7 w^2$, etc.

How to deal with d – part II – the most important part

When we have solution for:

$$\sum_{i=1}^{n} c_i a_i^{x_i} = b^z$$

Where for every i: $gcd(x_i, z) = 1$.

Then we can multiply both sides for example by $d^{pz+1} = d^{q*lcm(x)}$:

$$d^{q*lcm(x)} \sum_{i=1}^{n} c_i a_i^{x_i} = \sum_{i=1}^{n} c_i \left(d^{q\frac{lcm(x)}{x_i}} a_i \right)^{x_i} = d^{pz+1} b^z = d(d^p b)^z$$

Where x is a set of x_i for every i.

For every i: for every rational l_i and for every j: for every rational p_j, d_j, t and every integer q_j, v_j, u_j, f that suffices equation:

$$\sum_{i=1}^{n} c_i l_i^{x_i} = \prod_{j=1}^{o} d_j^{v_j} \; t^{f*z} \prod_{j=1}^{m} p_j^{q_j} \; , where\ d = \prod_{j=1}^{o} d_j^{u_j}$$

where f could be 0, for every j: $\gcd(q_j, z) = 1$, we have infinitely many solutions:

$$\sum_{i=1}^{n} c_i \left(\prod_{j=1}^{m} p_j^{(t_j+f_j*z)*\frac{r_j*lcm(x_1,\dots,x_n)}{x_i}} * \prod_{j=1}^{o} d_j^{s_j\frac{lcm(x)}{x_i}} * y^{\frac{lcm(x_1,\dots,x_n,z)}{x_i}} * l_i \right)^{x_i} =$$

$$= \sum_{i=1}^{n} c_i l_i^{x_i} \prod_{j=1}^{m} p_j^{(t_j+f_j*z)*r_j*lcm(x_1,\dots,x_n)} * \prod_{j=1}^{o} d_j^{s_j*lcm(x)} * y^{lcm(x_1,\dots,x_n,z)}$$

$$= \prod_{j=1}^{o} d_j^{v_j} \; t^{f*z} * \prod_{j=1}^{m} p_j^{(t_j+f_j*z)*r_j*lcm(x_1,\dots,x_n)+q_j} * \prod_{j=1}^{o} d_j^{w_j*z+(u_j-v_j)}$$

$$* \, y^{lcm(x_1,\dots,x_n,z)} = dc^z$$

Where y is any rational.

Where for every i: r_i is any integer such that $gcd(r_i, z) = 1$

Where for every j: t_j is any integer such that $z | (t_j * r_j * lcm(x_1, \dots, x_n) + q_j)$ {for details see: *Theorem 1*}

Where for every j: w_j is any integer such that $s_j * lcm(x) = w_j * z + (u_j - v_j)$ {for details see: *Theorem 1*}

Where for every j: f_j is any integer.

In general we have rational solutions above and when for every j: l_j, p_j, t_j, d_j, y are integers we have integer solutions.

And for $n = 2$ that are all complex not derived solutions.

The same is for Płotnicki's equation with use of little Fermat theorem.

The same is for:

$$\sum_{i=1}^{n} \boldsymbol{c_i} \prod_{j=1}^{m_i} \boldsymbol{a_{i,j}^{x_{i,j}}} = \boldsymbol{d} \prod_{j=1}^{m_0} \boldsymbol{b_j^{z_j}}$$

Where we have simply just more possible places to place $\boldsymbol{d^{q*lcm(x)}}$

Using Chinese remainder theorem we could also find solutions for:

$$\sum_{i=1}^{n} \boldsymbol{c_i a_i^{x_i}} = \boldsymbol{d_1 b_1^{z_1}} = \cdots = \boldsymbol{d_k b_k^{z_k}}$$

Where for every i, j: $\gcd(x_i, z_j) = 1$.

For every i: for every rational l_i and for every j: for every rational p_j, d_j and every integer q_j, v_j, u_j, f that suffices equation:

$$\sum_{i=1}^{n} \boldsymbol{c_i l_i^{x_i}} = \prod_{j=1}^{o} \boldsymbol{d_j^{v_j}} \prod_{j=1}^{m} \boldsymbol{p_j^{q_j}} \text{ , } \boldsymbol{where\ d} = \prod_{j=1}^{o} \boldsymbol{d_j^{u_j}}$$

we have infinitely many solutions:

We have to find solution of:

for every $i = 1, \dots, k, j = 1, \dots, m$:

$$Q_j = -q_j (mod\ z_i)$$

$$Q_j = 0\big(mod\ lcm(x)\big)$$

for every $i = 1, \dots, k, j = 1, \dots, o$:

$$V_j = u_j - v_j (mod\ z_i)$$

$$V_j = 0\big(mod\ lcm(x)\big)$$

Then we have solutions in form:

$$\sum_{i=1}^{n} c_i \left(\prod_{j=1}^{m} p_j^{\frac{Q_j}{x_i}} * \prod_{j=1}^{o} d^{\frac{V_j}{x_i}} * y^{\frac{lcm(x,z)}{x_i}} * l_i \right)^{x_i} == \sum_{i=1}^{n} c_i l_i^{x_i} \prod_{j=1}^{m} p_j^{Q_j} * \prod_{j=1}^{o} d_j^{V_j} * y^{lcm(x,z)}$$

$$= \prod_{j=1}^{m} p_j^{Q_j+q_j} * \prod_{j=1}^{o} d_j^{V_j+v_j+(u_j-u_j)} * y^{lcm(x,z)}$$

$$= \prod_{j=1}^{m} p_j^{Q_j+q_j} * \prod_{j=1}^{o} d_j^{V_j-(u_j-v_j)+u_j} * y^{lcm(x,z)} = d_1 b_1^{z_1} = \cdots = d_k b_k^{z_k}$$

Where x is a set of x_j for every j.

Where z is a set of z_j for every j.

Analogous solutions exist of course also for general case of Plotnicki's equations.

There is also a way to find solutions for:

$$\sum_{i=1}^{m_1} \boldsymbol{c_{1,i} a_{1,i}^{x_{1,i}}} = \cdots = \sum_{i=1}^{m_k} \boldsymbol{c_{k,i} a_{k,i}^{x_{k,i}}} = \boldsymbol{d_1 b_1^{z_1}} = \cdots = \boldsymbol{d_m b_m^{z_m}}$$

when we have:

$$\sum_{i=1}^{m_1} \boldsymbol{c_{1,i} a_{1,i}^{x_{1,i}}} = \cdots = \sum_{i=1}^{m_k} \boldsymbol{c_{k,i} a_{k,i}^{x_{k,i}}} = \boldsymbol{b_1^{z_1}} = \cdots = \boldsymbol{b_m^{z_m}}$$

That I will probably describe in details in my coming next year book.

Here is simplified example for simple case of Plotnicki's equation:

$$\boldsymbol{d_1^{p_1 z_1 + 1} = d_1^{q_1 * \frac{lcm(x,z)}{z_1}}}$$

…

$$\boldsymbol{d_m^{p_m z_m + 1} = d_m^{q_m * \frac{lcm(x,z)}{z_m}}}$$

Where x is a set of $x_{i,j}$ for every i, j.

Where z is a set of z_i for every i.

$$\prod_{i=1}^{m} \boldsymbol{d_i^{q_i * \frac{lcm(x,z)}{z_i}}} \sum_{i=1}^{m_1} \boldsymbol{c_{1,i} a_{1,i}^{x_{1,i}}} = \sum_{i=1}^{m_1} \boldsymbol{c_{1,i}} \left(\prod_{j=1}^{m} \boldsymbol{d_j^{q_j * \frac{lcm(x,z)}{x_{1,j} z_j}}} \boldsymbol{a_{1,i}} \right)^{\boldsymbol{x_{1,i}}} = \cdots =$$

$$= \prod_{i=1}^{m} d_i^{q_i * \frac{lcm(x,z)}{z_i}} \sum_{i=1}^{m_k} c_{k,i} a_{k,i}^{x_{k,i}} = \sum_{i=1}^{m_k} c_{k,i} \left(\prod_{j=1}^{m} d_j^{q_j * \frac{lcm(x,z)}{x_{k,j} z_j}} a_{k,i} \right)^{x_{k,i}} =$$

$$= d_1^{p_1 z_1 + 1} \prod_{i=2}^{m} d_i^{q_i * \frac{lcm(x,z)}{z_i}} b_1^{z_1} = d_1 \left(d_1^{p_1} \prod_{i=2}^{m} d_i^{q_i * \frac{lcm(x,z)}{z_1 z_i}} b_1 \right)^{z_1} = \cdots =$$

$$= d_1^{p_1 z_1 + 1} \prod_{i=1}^{m} d_i^{q_i * \frac{lcm(x,z)}{z_i}} b_1^{z_1} = d_m \left(d_m^{p_m} \prod_{i=1}^{m-1} d_i^{q_i * \frac{lcm(x,z)}{z_m z_i}} b_m \right)^{z_m}$$

How to deal with d – part III

...in:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d \prod_{j=1}^{m_0} b_j^{z_j}$$

where $\gcd\left(\prod_{i=1}^{n} \prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_0} z_j\right) = 1$

For this example

$$\mathbf{2x^3 + 3y^5v^3 = 5z^7w^2}$$

it is enough to find such $2k^3 + 3l^5m^3$ that is divisible by 5, which in this example is really very simple (eg: $k = l = m = 1$) or solve in rational numbers without such a requirement. When $c_1 = \cdots = c_{n=2k} = c$ and at least half of x_i are odd it is simple to find such l_i that d divides $\sum_{i=1}^{2k=n} c\left(du_i + (-1)^{g(x_i)}\right)^{x_i}$.

In general infinitely many complex not derived solution exist when $\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} r_{i,j}^{x_{i,j}} = 0$ has a solution (which can be solved often with the same method and so on). Because then for any $k_{i,j}$ for every i and j: d divides $\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i}\left(dk_{i,j} + r_{i,j}\right)^{x_{i,j}}$

Imagine that we have for example equation $\sum_{i=1}^{n} prime_i a_i^{prime_i} = prime_{n+1} b^{prime_{n+1}}$

Then we need to solve

$$\sum_{i=1}^{n} \boldsymbol{prime_i a_i^{prime_i} = 0}$$

So for:

$$\sum_{i=1}^{n-1} \boldsymbol{prime_i a_i^{prime_i} = prime_i(-a_n)^{prime_n}}$$

We use the same method and so on...

Then we go to the equation:

$$\mathbf{2x^2 + 3v^3 + 5z^5 = 7(-w)^7}$$

Where we need to solve

$$\mathbf{2x^2 + 3v^3 = 5(-z)^5}$$

And here we need to solve (see *The simplest Diophantine equation*):

$$2l_1^2 + 3l_2^3 = 0 \Leftrightarrow 2l_1^2 = 3(-l_2)^3 \Leftrightarrow 2(2*3^2*k^3)^2 = 3(2*3*k^2)^3 \Leftrightarrow$$

$$l_1 = (5 * l_1' + 2 * 3^2 * k^3), l_2 = (5 * l_2' - 2 * 3 * k^2)$$

For $k = 1, l_1' = 1, l_2' = 2$:

$$2(18+5)^2 + 3(10-6)^3 = 1058 + 192 = 1250 = \mathbf{5 * 250}$$

As we have l_1, l_2 we can solve:

$$\boldsymbol{2x^2 + 3v^3 = 5(-z)^5}$$

When we solve this, we can solve:

$$\boldsymbol{2x^2 + 3v^3 + 5z^5 = 7w^7}$$

And so on… to the equation:

$$\sum_{i=1}^{n} prime_i a_i^{prime_i} = prime_{n+1} b^{prime_{n+1}}$$

That we can solve now.

The last method is to select all $l_{i,j}$ divisible by d or select some subset of $l_{i,j}$ to be divisible by d and calculate rest with this method that is showed above, for example for:

$$\boldsymbol{2x^2 + 3v^2 + 5z^3 = 7w^7}$$

you could put $7k$ to l_x and find solution to

$$\boldsymbol{3r_v^2 + 5r_z^3 = 0}$$

Of course it is very simple (see *The simplest Diophantine equation*).

To find all solutions use a computer. Complexity of such an algorithm is $O(d^n)$.

Theorem 4 – how equations can be simplified

Theorem: every equation that can be simplified using:

$$\boldsymbol{Q(x) * R(x) = q * R(x)}$$

$$\boldsymbol{\frac{R(x)}{Q(x)} = \frac{R(x)}{q}}$$

$$\boldsymbol{R(x)^{Q(x)} = R(x)^q}$$

where $\boldsymbol{R(X)}$ is acceptable polynomial and $\boldsymbol{Q(x)}$ is every function that could give rational (in first and second rule) or integer (in third rule) result,

to the form of acceptable polynomial:

$$\boldsymbol{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d \prod_{j=1}^{m_0} b_j^{z_j}}$$

where gcd $\boldsymbol{\left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_0} z_j\right) = 1}$, has infinitely many complex not derived solutions.

So there are two simple rules in a formulation of Plotnicki equation:

1. Use every variable always in the same power or in expression where it could be simplified to the constance.
2. Reduce, if you want, everything that does not introduce alone standing constance to expression.

So acceptable equation suffices mainly three conditions:

a.) does not evaluate to expression that have some variable two times with different expontents or this variable can have set the same value in all places
b.) does not evaluate to expression that have alone standing constance.
c.) gcd $\left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_0} z_j\right) = 1$

So for example you may think that there is no such solution to the:

$$a^3 + b^5c^7 = d^7$$

But you would be wrong, because you can put any number to c and get for example:

$$a^3 + 128b^5 = d^7$$

Other example is:

$$\frac{x+1}{x-1}\left(a^y + (b^z)^{c^2-d^3}\right) = e^u$$

Which can be simplified for example to $(x = 3, c = 3, d = 2)$:

$$2a^y + 2b^z = e^u$$

You can also solve equation like this:

$$a^x + b^y(c^z + d^w) = e^v, where\, \gcd(xyzw, v) = 1$$

So any x^y can occur any number of time but under condition that all those occurances can be simplified to $x^y Q$, where Q is every acceptable polynomial that haven't got x and any variable from outer expression or this variable can be set to the same value.

And by the way there is a simple rule that every QR can be always simplified, when Q or R is acceptable polynomial, by putting any number to every variable that R use (when Q is acceptable polynomial) or Q use (when R is acceptable polynomial). Then simply $x^y Q = qx^y$, where q is a constant or $x^y Q = pQ$, where p is a constant. So for example:

$$a^x(c^z - d^w) + b^y(c^z + d^w) = e^v$$

Could be very easily solved:

$$p(c^z - d^w) + q(c^z + d^w) = e^v$$

Or:

$$pa^x + qb^y = e^v$$

The same is for $\frac{Q}{R}$ where Q is acceptable polynomial:

$$\frac{a^x + b^y}{(e^g - f^h)} + \frac{c^z}{(e^g + f^h)} - \frac{d^w}{(e^g - f^h)(e^g + f^h)} = e^v$$

Could be easily solved:

$$qa^x + qb^y + pc^z - pqe^v = d^w$$

There is of course a possiblity to solve using the same metod equation like this:

$$\left(x^a + y^b z^c\right)\left(w^d - v^e\right) = p^r q^s$$

or:

$$\left(x^a + y^b z^c\right)\left(w^d - v^e\right) = (p^r)(q^s)$$

or:

$$\frac{\left(x^a + y^b z^c\right)}{(w^d - v^e)} = \frac{p^r q^s}{k^m l^n}$$

And that is not all, because you can solve equations like this:

$$\left(x^a + y^b z^c - p^r q^s\right)\left(w^d - v^e + f^g\right) = 0$$

Etc.

In the end you could think that you can not solve equation like this:

$$x^{10} + y^9 + z^6 + w^5 + v^3 + h^2 = 0$$

Because there is not such power f that $gcd\left(\frac{10*9*6*5*3*2}{f}, f\right) = 1$, but you would be wrong, because you can solve it for example this way:

$$x^{10} + y^9 + z^6 + w^5 + v^3 + h^2 = 0$$

$$-w^5 = y^9 + z^6$$

$$-v^3 = x^{10} + h^2$$

$$(y^9 + z^6 + w^5) + (x^{10} + v^3 + h^2) = 0 + 0 = 0$$

The same easy you can solve:

$$7\sqrt{x^3 + y^5} = 2z^7$$

For example like this:

$$49(x^3 + y^5) = 4z^{14}$$

Proof that there are not other complex not derived solutions

Proof for the case:

// Polish: Dowód dla przypadku

$$\boldsymbol{wa^x + vb^y = fc^z}$$

$$w(g^p k)^x + v(g^q l)^y = f(g^r m)^z$$

Of course we can assume that $gcd(wk^x, vl^y, wk^x + vl^y) = s = 1$, because when we align power of divisors of s to z then equation can be divided by these divisors which does not applies for other divisors of $wk^x + vl^y$.

Secondly, when we assume that $gcd(f, wk^x + vl^y) = \gcd(w, fm^z - vl^y) = \gcd(v, fm^z - wk^x) = 1$, then coefficients w, v, f can be always choosed, because they do not depend on the k, l, m.

As you will notice, if $gcd(a, b, c) = g > 1$, $\gcd(wk, vl) = 1$, then at least two factors of sum must have g in the same power, so they must be aligned. In addition, you must ensure that all divisors of $wk^x + vl^y$ had the power divisible by z at the right side of the equation. If some prime factor of g is aligned for the sum of the two factors and will be in power z for the third, then another prime factor can not be aligned for another pair of factors of sum in the equation, because it will lost alignment of this firstly aligned prime factor. What leads to the template solution presented in this document.

If we align divisors for concrete two factors of sum in eqation then we assume some l and k, which implicates what we need to align on the right side, so before we align some of them, there is no (we don't know any) m for fc^z, and so the alignment of two other factors of the sum in equation is not possible. If we tried to define in some moment such m on the basis of aligned to z dividers of $wk^x + vl^y$, then if we wanted to keep the $gcd(m, k') = gcd\ (m, l') = 1$, then it means that:

1' when m has all prime divisors of $wk^x + vl^y$:

$$k' = g_k \frac{k}{t_k}, l' = g_l \frac{l}{t_l}$$

Where t_k is eventually divisor of k, g_k is eventually divisor of $g^{\frac{lcm(x,y)}{x}}$, and g_l is eventually divisor of $g^{\frac{lcm(x,y)}{y}}$. f' is a divisor of f.

Then $\frac{g^{lcm(x,y)}(wk^x+vl^y)}{f'm^z} = \frac{w\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{w(k')^x} = \frac{\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{g_k^x\left(\frac{k}{t_k}\right)^x} = t_k^x\left(\frac{g^{\frac{lcm(x,y)}{x}}}{g_k}\right)^x \Leftrightarrow g_k^x(wk^x + vl^y) =$ $f'm^z t_k^x$, but $\gcd(wk^x + vl^y, f't_k^x) = 1$, so $f't_k^x p = g_k^x$, but then $p(wk^x + vl^y) = m^z$, and that means that m has all divisors of c^z aligned, so m divides c, what is possible only at the end, when all divisors of c are aligned to z, so there is nothing to be aligned.

2' when m has not all prime divisors of $wk^x + vl^y$:

$$k' = g_k s_k \frac{k}{t_k}, l' = g_l s_l \frac{l}{t_l}, \gcd(s_k, t_k m) = \gcd(s_l, t_l m) = 1$$

Where t_k is eventually divisor of k, w' is eventually divisor of w, g_k is eventually divisor of $g^{\frac{lcm(x,y)}{x}}$, and g_l is eventually divisor of $g^{\frac{lcm(x,y)}{y}}$, s_k, s_l are divisble at most by these prime divisors of $wk^x + vl^y$ (in some powers), that does not divide m. f' is a divisor of f.

$\frac{g^{lcm(x,y)}(wk^x+vl^y)}{f'm^z} = \frac{w\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{w(k')^x} = \frac{\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{\left(g_k s_k \frac{k}{t_k}\right)^x} = \frac{t_k^x g^{lcm(x,y)}}{g_l^x s_k^x} \Leftrightarrow g_l^x s_k^x (wk^x + vl^y) = f' m^z t_k^x$, but $\gcd(wk^x + vl^y, f' t_k^x) = 1$, so $f' t_k^x p = g_k^x s_k^x$, then $p(wk^x + vl^y) = m^z$, so m has all divisors of $wk^x + vl^y$. Contradicition.

Proof for the case:

// Dowód dla:

Secondly, when we assume that:

$$gcd\left(d, \sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}\right) = \gcd\left(c_1, d\prod_{j=1}^{m_0} l_{b_j}^{z_j} - c_2 \prod_{j=1}^{m_2} l_{2,j}^{x_{2,j}}\right)$$

$$= \gcd\left(c_2, d\prod_{j=1}^{m_0} l_{b_j}^{z_j} - c_1 \prod_{j=1}^{m_1} l_{1,j}^{x_{1,j}}\right) = 1$$

then coefficients c_1, c_2, d can be always choosed, because they do not depend on the $l_{i,j}, l_{b\,j}$.

$$\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d\prod_{j=1}^{m_0} b_j^{z_j}$$

If we align divisors for concrete two factors of sum in equation then we assume some $l_{i,j}$, which implicates what we need to align on the right side, so before we align some of them, there is no (we don't know any) m_j for $d\prod_{j=1}^{m_0} b_j^{z_j}$, and so the alignment of two other factors of the sum in equation is not possible. If we tried to define in some moment such m_j on the basis of aligned to z_j dividers of $\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}$, then if we wanted to keep the $gcd(m_j, l'_{i,j}) = 1$, then it means that:

1' m_j has not all prime divisors $\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}$ .Then

$$l'_{i,j} = g_{i,j} s_{i,j} \frac{l_{i,j}}{t_{i,j}}$$

$$\gcd(s_{i,j}, t_{i,j} m) = 1$$

d' is a divisor of d.

$$\frac{g^{lcm(x)}\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}}{d'\prod_{j=1}^{m_0}m_j^{z_j}}=\frac{c_i g^{lcm(x)}\prod_{j=1}^{m_i}\left(l_{i,j}\right)^{x_{i,j}}}{c_i\prod_{j=1}^{m_i}\left(l'_{i,j}\right)^{x_{i,j}}}=\frac{g^{lcm(x)}\prod_{j=1}^{m_i}\left(l_{i,j}\right)^{x_{i,j}}}{\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\frac{l_{i,j}}{t_{i,j}}\right)^{x_{i,j}}}=\frac{\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}g^{lcm(x)}}{\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\right)^{x_{i,j}}}$$

$\Leftrightarrow\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\right)^{x_{i,j}}\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right)=\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}d'\prod_{j=1}^{m_0}m_j^{z_j}$, but $\gcd\left(\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right),d'\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}\right)=1$, so $d'\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}p=\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\right)^{x_{i,j}}$, then $p\left(\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right)\right)=\prod_{j=1}^{m_0}m_j^{z_j}$, so m_j has all divisors of $\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right)$. Contradiction.

So for every j: m_j has all prime divisors of $\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}$.

2' m_j has all prime divisors of $\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}$. Then

$$l'_{i,j}=g_{i,j}\frac{l_{i,j}}{t_{i,j}}$$

d' is a divisor of d.

and:

$$\frac{g^{lcm(x)}\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}}{d'\prod_{j=1}^{m_0}m_j^{z_j}}=\frac{c_i g^{lcm(x)}\prod_{j=1}^{m_i}\left(l_{i,j}\right)^{x_{i,j}}}{c_i\prod_{j=1}^{m_i}\left(l'_{i,j}\right)^{x_{i,j}}}=\frac{g^{lcm(x)}\prod_{j=1}^{m_i}\left(l_{i,j}\right)^{x_{i,j}}}{\prod_{j=1}^{m_i}\left(c'_{i,j}\frac{l_{i,j}}{t_{i,j}}\right)^{x_{i,j}}}=\frac{\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}g^{lcm(x)}}{\prod_{j=1}^{m_i}\left(g_{i,j}\right)^{x_{i,j}}}$$

$$\Leftrightarrow\prod_{j=1}^{m_i}\left(g_{i,j}\right)^{x_{i,j}}\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}=\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}d'\prod_{j=1}^{m_0}m_j^{z_j}$$

but $\gcd\left(\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}},d'\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}\right)=1$, so $d'\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}p=\prod_{j=1}^{m_i}\left(g_{i,j}\right)^{x_{i,j}}$, but then $p\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}=\prod_{j=1}^{m_0}m_j^{z_j}$. That means that for every j: m_j has all divisors of $b_j^{z_j}$ aligned, so m_j divides b_j, what is possible only at the end, when all divisors of b_j are aligned to z_j, so there is nothing to be aligned.

It can be probably proved also for more complex equations, but it is much more complicated. Probably for most, if not all, equations presented solutions are all solutions for $gcd\ (a)>1$.

// Polish:

Po drugie, kiedy założymy, że $gcd(f,wk^x+vl^y)=\gcd(w,fm^z-vl^y)=\gcd(v,fm^z-wk^x)=1$, wtedy współczynniki w,v,f mogą być zawsze dobrane, ponieważ nie zależą od k,l,m.

Jak łatwo zauważyć, jeśli $gcd(a,b,c)=g>1,\gcd(wk,vl)=1$, to przynajmniej dwa czynniki sumy muszą mieć g w tej samej potędze, czyli muszą być wyrównane. Dodatkowo trzeba zadbać o to, żeby wszystkie podzielniki wk^x+vl^y miały potęgę podzielną przez z po prawej stronie równania. Jeśli jakiś czynnik pierwszy g zostanie wyrównany dla danych dwóch czynników sumy i będzie w potędze z dla trzeciego czynnika, to inny czynnik pierwszy g nie może być wyrównany dla innej pary czynników sumy równania, bo zostanie utracone wyrównanie do z

tego pierwszego czynnika. Co już prowadzi wprost do szablonu rozwiązania przedstawionego w tym dokumencie.

Jeśli wyrównujemy podzielniki dla dwóch czynników dodawania w wyrażeniu to zakładamy jakieś l i k, z których wynika jakie podzielniki musimy wyrównać do z po prawej stronie, a więc zanim nie wyrównamy pewnych podzielników nie istnieje żadne (nie znamy żadnego) m dla fc^z, a więc wyrównanie dwóch innych czynników równania nie jest możliwe. Gdybyśmy próbowali określić w pewym momencie takie m na podstawie wyrównanych do z podzielników $wk^x + vl^y$, to gdybyśmy chcieli zachować $gcd(m, k') = \gcd(m, l') = 1$, to okazałoby się, że:

1' m ma wszystkie pierwsze podzielniki $wk^x + vl^y$

$$k' = g_k \frac{k}{t_k}, l' = g_l \frac{l}{t_l}$$

Gdzie t_k to ewentualny podzielnik k , a g_k i g_l to ewentualne podzielniki odpowiednio $g^{\frac{lcm(x,y)}{x}}, g^{\frac{lcm(x,y)}{y}}$. f' jest podzielnikiem f.

i że $\frac{g^{lcm(x,y)}(wk^x+vl^y)}{f'm^z} = \frac{w\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{wk'^x} = \frac{\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{\left(g_k\frac{k}{t_k}\right)^x} = \frac{t_k^x g^{lcm(x,y)}}{g_k^x} \Leftrightarrow g_k^x(wk^x + vl^y) = f'm^z t_k^x$, ale $\gcd(wk^x + vl^y, f't_k^x) = 1$, więc $f't_k^x p = g_k^x$, ale wtedy $p(wk^x + vl^y) = m^z$, co by oznaczało, że m ma wszystkie wyrównane podzielniki c^z, więc m dzieli c, co jest możliwe tylko na samym końcu, gdy wszystkie podzielniki c są już wyrównane do z, więc nie ma co wyrównywać.

2' m nie ma wszystkich podzielników $wk^x + vl^y$

$$k' = g_k s_k \frac{k}{t_k}, l' = g_l s_l \frac{l}{t_l}, \gcd(s_k, t_k m) = \gcd(s_l, t_l m) = 1$$

Gdzie t_k to ewentualny podzielnik k, a w' to ewentualny podzielnik w, g_k i g_l to ewentualne podzielniki odpowiednio $g^{\frac{lcm(x,y)}{x}}$, $g^{\frac{lcm(x,y)}{y}}$, a s_k jest podzielne tylko conajwyżej przez te podzielniki pierwsze $wk^x + vl^y$ (w pewnych potęgach), przez które nie jest podzielne m. f' jest podzielnikiem f.

$\frac{g^{lcm(x,y)}(wk^x+vl^y)}{f'm^z} = \frac{w\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{w(k')^x} = \frac{\left(g^{\frac{lcm(x,y)}{x}}k\right)^x}{\left(g_k s_k\frac{k}{t_k}\right)^x} = \frac{t_k^x g^{lcm(x,y)}}{g_k^x s_k^x} \Leftrightarrow g_k^x s_k^x(wk^x + vl^y) = f'm^z t_k^x$, ale $\gcd(wk^x + vl^y, f't_k^x) = 1$, więc $f't_k^x p = g_k^x s_x^x$, wtedy $p(wk^x + vl^y) = m^z$, więc m ma wszystkie podzielniki $wk^x + vl^y$. Sprzeczność.

Proof for the case:

// Dowód dla:

Po drugie, jeśli założymy, że:

$$gcd\left(d, \sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}\right) = \gcd\left(c_1, d\prod_{j=1}^{m_0} l_{b_j}^{z_j} - c_2 \prod_{j=1}^{m_2} l_{2,j}^{x_{2,j}}\right)$$

$$= \gcd\left(c_2, d\prod_{j=1}^{m_0} l_{b_j}^{z_j} - c_1\prod_{j=1}^{m_1} l_{1,j}^{x_{1,j}}\right) = 1$$

wtedy współczynniki c_1, c_2, d can moga być zawsze dobrane, ponieważ nie zależa od $l_{i,j}, l_{b\,j}$.

$$\boldsymbol{\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = d\prod_{j=1}^{m_0} b_j^{z_j}}$$

Jeśli wyrównujemy podzielniki dla dwóch czynników dodawania w wyrażeniu to zakładamy jakieś $l_{i,j}$, z których wynika jakie podzielniki musimy wyrównać do z_i po prawej stronie, a więc zanim nie wyrównamy pewnych podzielników nie istnieje żadne (nie znamy żadnego) m dla $d\prod_{j=1}^{m_0} b_j^{z_j}$, a więc wyrównanie dwóch innych czynników równania nie jest możliwe. Gdybyśmy próbowali określić w pewym momencie takie m na podstawie wyrównanych do z_i podzielników $\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}$, to gdybyśmy chcieli zachować $gcd(m, l'_{i,j}) = 1$, to okazałoby się, że:

1’ m_j ma wszystkie pierwsze podzielniki $\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}$

$$l'_{i,j} = g_{i,j}\frac{l_{i,j}}{t_{i,j}}$$

d' jest podzielnikiem d.

i że:

$$\frac{g^{lcm(x)}\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}}{d'\prod_{j=1}^{m_0} m_j^{z_j}} = \frac{c_i g^{lcm(x)}\prod_{j=1}^{m_i}(l_{i,j})^{x_{i,j}}}{c_i\prod_{j=1}^{m_i}(l'_{i,j})^{x_{i,j}}} = \frac{g^{lcm(x)}\prod_{j=1}^{m_i}(l_{i,j})^{x_{i,j}}}{\prod_{j=1}^{m_i}\left(g_{i,j}\frac{l_{i,j}}{t_{i,j}}\right)^{x_{i,j}}} = \frac{\prod_{j=1}^{m_i} t_{i,j}^{x_{i,j}}\, g^{lcm(x)}}{\prod_{j=1}^{m_i}(g_{i,j})^{x_{i,j}}}$$

$$\Leftrightarrow \prod_{j=1}^{m_i}(g_{i,j})^{x_{i,j}}\sum_{i=1}^{2} c_i\prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}} = d' c_i \prod_{j=1}^{m_i} t_{i,j}^{x_{i,j}}\prod_{j=1}^{m_0} m_j^{z_j}$$

ale $\gcd\left(\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}, d'\prod_{j=1}^{m_i} t_{i,j}^{x_{i,j}}\right) = 1$, więc $d'\prod_{j=1}^{m_i} t_{i,j}^{x_{i,j}}\, p = \prod_{j=1}^{m_i}(g_{i,j})^{x_{i,j}}$, ale wtedy $p\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}} = \prod_{j=1}^{m_0} m_j^{z_j}$, co by oznaczało, że m_j ma wszystkie wyrównane podzielniki $b_j^{z_j}$, więc m_j dzieli b_j, co jest możliwe tylko na samym końcu, gdy wszystkie podzielniki b_j są już wyrównane do z_j, więc nie ma co wyrównywać.

2’ m_j nie ma wszystkich podzielników $\sum_{i=1}^{2} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}$

$$l'_{i,j} = g_{i,j}s_{i,j}\frac{l_{i,j}}{t_{i,j}}$$

$$\gcd(s_{i,j}, t_{i,j}m) = 1$$

d' jest podzielnikiem d.

$$\frac{g^{lcm(x)}\sum_{i=1}^{2}c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}}{d'\prod_{j=1}^{m_0}m_j^{z_j}}=\frac{c_i g^{lcm(x)}\prod_{j=1}^{m_i}\left(l_{i,j}\right)^{x_{i,j}}}{c_i\prod_{j=1}^{m_i}\left(l'_{i,j}\right)^{x_{i,j}}}=\frac{g^{lcm(x)}\prod_{j=1}^{m_i}\left(l_{i,j}\right)^{x_{i,j}}}{\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\frac{l_{i,j}}{t_{i,j}}\right)^{x_{i,j}}}=\frac{\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}g^{lcm(x)}}{\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\right)^{x_{i,j}}}$$

$$\Leftrightarrow\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\right)^{x_{i,j}}\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right)=d'c_i\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}\prod_{j=1}^{m_0}m_j^{z_j}$$

ale $\gcd\left(\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right),d'\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}\right)=1$, więc $d'\prod_{j=1}^{m_i}t_{i,j}^{x_{i,j}}p=\prod_{j=1}^{m_i}\left(g_{i,j}s_{i,j}\right)^{x_{i,j}}$, wtedy $p\left(\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right)\right)=\prod_{j=1}^{m_0}m_j^{z_j}$, więc m_j ma wszystkie podzielniki $\sum_{i=1}^{2}\left(c_i\prod_{j=1}^{m_i}l_{i,j}^{x_{i,j}}\right)$. Sprzeczność.

Dowód da się prawdopodobnie przeprowadzić także dla bardziej złożonych równań, jednak jest to o wiele bardziej skomplikowane. Prawdodpobonie dla większości, jeśli nie wszystkich, równań przedstawione rozwiązania są wszystkimi rozwiązaniami dla $gcd(a)>1$.

Proof – when there are complex not derived solutions

There are complex not derived solutions only when

$$\mathbf{gcd}\left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_0} z_j\right) = \mathbf{1}$$

Proof for the case:

$$\boldsymbol{wa^x + vb^y = fc^z}$$

If for each $w = x \; or \; y \; or \; z$: $gcd\left(\frac{xyz}{w}, w\right) > 1$, it is impossible to align the powers by this method, so the only possible alignment is:

// Jeśli dla każdego $w = x \; lub \; y \; lub \; z$: $gcd\left(\frac{xyz}{w}, w\right) > 1$, to nie da się wyrównać potęg tą metodą, więc jedyne możliwe wyrównanie to:

$$w\left(g^{\frac{lcm(x,y,z)}{x}}k\right)^x + v\left(g^{\frac{lcm(x,y,z)}{y}}l\right)^y = f\left(g^{\frac{lcm(x,y,z)}{z}}m\right)^z$$

$$\frac{w}{f}\left(g^{\frac{lcm(x,y,z)}{x}}k\right)^x + \frac{v}{f}\left(g^{\frac{lcm(x,y,z)}{y}}l\right)^y = \left(g^{\frac{lcm(x,y,z)}{z}}\right)^z\left(\frac{w}{f}k^x + \frac{v}{f}l^y\right) = c^z$$

Then, as can be seen $\frac{w}{f}k^x + \frac{v}{f}l^y = m^z$, so we have a solution. Hence the equation has complex not derived solution then and only then when for some $w = x \; or \; y \; or \; z$: $gcd\left(\frac{xyz}{w}, w\right) = 1$, and has an infinite number of them.

// Wtedy jak widać $\frac{w}{f}k^x + \frac{v}{f}l^y = m^z$, czyli mamy rozwiązanie pochodne. Stąd równanie to ma złożone niepochodne rozwiązania wtedy i tylko wtedy gdy dla pewnego $w = x \; lub \; y \; lub \; z$: $gcd\left(\frac{xyz}{w}, w\right) = 1$, i ma ich nieskończenie wiele.

More general proof

$$\sum_{i=1}^{2}\frac{c_i}{d}\prod_{j=1}^{m_i}\left(\prod_{k=1}^{s} y_k^{u_{i,j,k}\frac{lcm(x,z)}{x_{i,j}}} * l_{i,j}\right)^{x_{i,j}} = \left(\sum_{i=1}^{2}\frac{c_i}{d}\prod_{j=1}^{m_i} l_{i,j}{}^{x_{i,j}}\right)\prod_{i=1}^{m_0}\left(\prod_{k=1}^{s} y_k^{p_f\frac{lcm(x,z)}{z_i}}\right)^{z_i} = \prod_{j=1}^{m_0} b_j^{z_j}$$

Where for every i, k: $\sum_{j=1}^{m_i} u_{i,j,k} = p_f$

Then: $\sum_{i=1}^{2}\frac{c_i}{d}\prod_{j=1}^{m_i} l_{i,j}{}^{x_{i,j}} = \prod_{j=1}^{m_0} B_j^{z_j}$, so we have derived solution. So equation has complex not derived solutions then and only then when for some $z = x_1 \; or \; \ldots . \; or \; x_2$:gcd$(x, z) = 1$.

Where $x_i = \prod_{j=1}^{m_i} x_{i,j}$, $x = \prod_{i=1}^{n} x_i$.

Simultanous Płotnicki's equations

And if there is a solution for: $\sum_{k=1}^{n}(c_{i,k}-c_{j,k})\prod_{j=1}^{m_i} l_{k,j}^{x_{k,j}} = 0$, and $gcd(a)>1$ then and only then (because otherwise $b_i \neq b_j$) simultaneous equation has infinitely many solutions:

$$\sum_{k=1}^{n} c_{i,k}\prod_{j=1}^{m_i}(P_{k,j} * l_{k,j})^{x_{k,j}} = b^z$$

where $c_{i,j}$ are rationals.

So for two equations there is always a solution when there is at least one such x_k that $\gcd(x\ without\ x_k, x_k) = 1$, because then it is Plotnicki's equation.

Example 1

$$\begin{cases} \boldsymbol{x^2+y^3=z^5} \\ \boldsymbol{2x^2-3y^3=z^5} \end{cases}$$

There is the smallest l_i such that $l_1^2+l_2^3 = 2l_1^2-3l_2^3 \Leftrightarrow \left(\frac{l_1}{2}\right)^2 = l_2^3 : l_1 = 2x^3 = 2, l_2 = x^2 = 1, l_1^2 + l_2^3 = 5$, so:

$$t_1 = 4 \Rightarrow 5|(2*3*t_1+1)$$

$$\begin{cases} (5^{3*4}*2)^2+(5^{2*4}*1)^3 = (2^2+1^3)*5^{24} = 5*5^{24} = 5^{25} = (5^5)^5 \\ 2(5^{3*4}*2)^2-3(5^{2*4}*1)^2 = (2*2^2-3*1^3)*5^{24} = 5*5^{24} = (5^5)^5 \end{cases}$$

So probably the smallest complex solution is:

$$(x,y,z) = (5^{3*4}*2, 5^{2*4}*1, 5^5) = (\mathbf{244140625}, \mathbf{390625}, \mathbf{3125})$$

Example 2

$$\begin{cases} \boldsymbol{x^2+y^3=2z^5} \\ \boldsymbol{x^2-y^3=z^5} \end{cases}$$

$$\begin{cases} \boldsymbol{\frac{x^2}{2}+\frac{y^3}{2}=z^5} \\ \boldsymbol{x^2-y^3=z^5} \end{cases}$$

There is the smallest l_i such that $\frac{l_1^2}{2}+\frac{l_2^3}{2} = l_1^2-l_2^3 \Leftrightarrow l_1^2 = 3l_2^3 : l_1 = 9x^3 = 9, l_2 = 3x^2 = 3, \frac{l_1^2}{2} + \frac{l_2^3}{2} = \frac{81+27}{2} = 54$, so:

$$54 = 2*3^3$$

$$t_1 = 2 \Rightarrow 5|2t_1+1$$

$$t_2 = 4 \Rightarrow 5|1t_2+1$$

$$\begin{cases} \frac{(3^{3*2}*2^{3*4}*9)^2}{2}+\frac{(3^{2*2}*2^{2*4}*3)^3}{2} = \frac{(9^2+3^3)}{2}*2^{24}*3^{18} = 2*3^3*2^{24}*3^{12} = 2^{25}*3^{15} = (2^5*3^3)^5 \\ (3^{3*2}*2^{3*4}*9)^2-(3^{2*2}*2^{2*4}*3)^3 = 2*3^3*2^{24}*3^{12} = (2^5*3^3)^5 \end{cases}$$

So probably the smallest complex solution is:

$$(x,y,z) = (\ 3^{3*2} * 2^{3*4} * 9, 3^{2*2} * 2^{2*4} * 3, (2^5 * 3^3)^5)$$

Example 3

$$\begin{cases} \boldsymbol{2x^2 - y^2 = w^7} \\ \boldsymbol{x^2 + z^2 = w^7} \end{cases}$$

$l_x^2 - l_y^2 - l_z^2 = 0$ then we have pitagorean triple $(l_z, l_y, l_x) = (p^2 - q^2, 2pq, p^2 + q^2)$

The smallest pitagorean triple is (3,4,5):

$$2l_x^2 - l_y^2 = l_x^2 + l_z^2 = 25 + 9 = 34 = 2 * 17$$

$$t_1 = 3 \Rightarrow 7|(2 * t_1 + 1)$$

$$\begin{cases} 2(2^3 * 17^3 * 5)^2 - (2^3 * 17^3 * 4)^2 = 2^6 * 17^6 * (50 - 16) = 2^6 * 17^6 * (2 * 17) = (2 * 17)^7 \\ (2^3 * 17^3 * 5)^2 + (2^3 * 17^3 * 3)^2 = 2^6 * 17^6 * (25 + 9) = 2^6 * 17^6 * (2 * 17) = (2 * 17)^7 \end{cases}$$

So probably the smallest complex solutuion is:

$$(x,y,z,w) = (34^3 * 5, 34^3 * 4, 34^3 * 3, 34) = (\mathbf{39304 * 5, 39304 * 4, 39304 * 3, 34})$$

Example 4

$$\begin{cases} \boldsymbol{2x^4 - y^2 = w^7} \\ \boldsymbol{x^4 + z^2 = w^7} \end{cases}$$

$l_x^4 - l_y^2 - l_z^2 = 0$ then we have pitagorean triple $(l_z, l_y, l_x^2) = (p^2 - q^2, 2pq, p^2 + q^2)$

The smallest l_x will be from pitagorean triple (3,4,5): $l_x^2 = 3^2 + 4^2 = 25$

$$2l_x^4 - l_y^2 = l_x^4 + l_z^2 = 5^4 + (3 * 5)^2 = 850 = 2 * 5^2 * 17$$

$$t_1 = 5 \Rightarrow 7|(4 * t_1 + 1)$$

$$t_1 = 3 \Rightarrow 7|(4 * t_1 + 2)$$

$$\begin{cases} 2(2^5 * 5^3 * 17^5 * 5)^4 - (2^{5*2} * 5^{3*2} * 17^{5*2} * 5 * 4)^2 = 2^{20} * 5^{12} * 17^{20} * (850) = (2^3 * 5^2 * 17^3)^7 \\ (2^5 * 5^3 * 17^5 * 5)^4 + (2^{5*2} * 5^{3*2} * 17^{5*2} * 5 * 3)^2 = 2^{20} * 5^{12} * 17^{20} * (850) = (2^3 * 5^2 * 17^3)^7 \end{cases}$$

So probably the smallest complex solutuion is:

$$(x,y,z,w)$$

$$= (2^5 * 5^3 * 17^5 * 5, 2^{5*2} * 5^{3*2} * 17^{5*2} * 5 * 4, 2^{5*2} * 5^{3*2} * 17^{5*2} * 5 * 3, 2^3 * 5^2 * 17^3)$$

$$= (\mathbf{28397140000, 645118048143680000000, 483838536107760000000, 982600})$$

Theorem 5 – complex solutions with alone standing constance

There is complex solution for equation like this:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} + C = d \prod_{j=1}^{m_0} b_j^{z_j}$$

where $\gcd\left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_0} z_j\right) = 1$**, when there is sufficed condition**

$$C = \left(\prod_{i=1}^{u}\left(t_i^{f_i * z_i}\prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j}+f_{i,j}*z_i)*r_{i,j}*lcm(x)}\prod_{j=1}^{w_i} y_{i,j}^{lcm(x,z_i)}\right)\right) * l_c$$

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}} + l_c = \prod_{i=1}^{u}\left(t_i^{f_i * z_i}\prod_{j=1}^{v_i} p_{i,j}^{q_{i,j}}\right)$$

where l_c **could be 1.**

So alone standing constance have to be treated like new variable with exponent 1. And that is all. And then and only then when there is such solution that new variable is equal to C, there is complex not derived solution for the equation.

Example 1

$$x^2 + 25 = y^3$$

For $l_c = 1, l_x = 2$:

$$t = 1 \Rightarrow 3|2 * t + 1$$

$$(5^{1*1} * 2)^2 + 5^2 = (2^2 + 1) * (5^2) = 5^{2+1} = 5^3$$

So solution is $(x, y) = (\mathbf{10}, \mathbf{5})$.

Example 2

$$x^2 + 123 = y^3$$

$$123 = 3 * 41$$

So: l_c could be 1 or 3 or 41

1’ l_c =1

Then $l_x^2 + 1 = 3 * 41 = 123 \Rightarrow l_x = \sqrt{122}$, so there is no solutions.

2’ l_c =3

Then $l_x^2 + 3 = 41 \Rightarrow l_x = 2\sqrt{7}$, so there is no solutions.

3' $l_c = 41$

Then $l_x^2 + 41 > 41 = 3$, so there is no solutions.

Conclusion: There is not solutions of this equation for $\mathbf{gcd}(\boldsymbol{x}, \boldsymbol{y}) > 1$.

C – Plotnicki's equations – part III

Theorem 1 – useful theorem II

Theorem: $\frac{a}{b}q = t\prod_{i=1}^{n} c_i + x$**, has integer solution for every** a **for given** c_i**, and** x **(1.), where** $gcd(a,b) = 1, gcd(a, \prod_{i=1}^{n} c_i) = 1.$

It is enough to see that

$$\frac{a}{b}q = t\prod_{i=1}^{n} c_i + x \Leftrightarrow a|(t\prod_{i=1}^{n} c_i + x)$$

So this is classical example of Chinese remainder theorem:

$$\begin{cases} w = x\left(mod \prod_{i=1}^{n} c_i\right) \\ w = 0(mod\ a) \end{cases}$$

Example

$$\frac{3}{5}q = 2t + 1 \Leftrightarrow 3r = 2t + 1$$

$$t = 1 + 3k, r = 1 + 2k, q = (1 + 2k) * 5$$

Method 6 – **Płotnicki's equation with use of little Fermat theorem – general case – rational exponents**

Method 6: Method of finding infinitely many complex nontrivial solutions for equation like this:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{\frac{x_{i,j}}{y_{i,j}}} = d \prod_{j=1}^{m_0} b_j^{\frac{z_j}{z'_j}}$$

where $\gcd\left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_0} z_j\right) = 1$

where for every i,j **:** $(\rightarrow)c_i, a_{i,j}, (\rightarrow)d, b_j$ **are rationals and** $(\rightarrow)n, (\rightarrow)m_i, (\rightarrow)x_{i,j}, (\rightarrow)y_{i,j}, (\rightarrow)z_j, (\rightarrow)z'_j$ **are integers.**

for every i,j: for every rational $l_{i,j}$ and every rational $p_{i,j}, t_i$ and every integer $q_{i,j}, f_i$ that suffices equation:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{\frac{x_{i,j}}{y_{i,j}}} = d \prod_{i=1}^{u} \left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{q_{i,j}} \right)$$

where for every i: f_i could be 0, for every i,j: $\gcd(q_{i,j}, z_i) = 1$, we have infinitely many solutions:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} \left(\prod_{s=1}^{u} \left(\prod_{k \in S_{i,j,s}} p_{s,k}^{(t_{s,k} z_s - q_{s,k}) * y_{i,j} * \frac{(r_{s,k} * lcm(x))^{z_s - 1}}{x_{i,j}}} * \prod_{k \in T_{i,j,s}} g_{s,k}^{y_{i,j} \frac{lcm(x,z_s)}{x_{i,j}}} \right) * l_{i,j} \right)^{\frac{x_{i,j}}{y_{i,j}}}$$

$$= d \prod_{i=1}^{u} \left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j} z_i - q_{i,j}) * (r_{i,j} * lcm(x))^{z_i - 1} + q_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{lcm(x,z_i)} \right)$$

$$= d \prod_{i=1}^{u} \left(\left(t_i^{f_i} \prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{\frac{lcm(x,z_i)}{z_i}} \right)^{z'_i} \right)^{\frac{z_i}{z'_i}}$$

Where for every i,j: $\gcd(r_{i,j}, z_i) = 1$.

Where for every i,s: $\bigcup_{j=1}^{m_i} S_{i,j,s} = \{1, \dots, v_i\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \dots, w_i\}$,

for every i,j,k,s where $j \neq k$: $S_{i,j,s} \cap S_{i,k,s} = \emptyset, T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

x is a set of all $x_{i,j}$, z is a set of all z_i.

Where c_i, d, l_i are any rationals and for every s,k: $q_{s,k} < t_{s,k} z_s$, where $q_{s,k}, t_{s,k}$ are any integers.

For every i,j: $g_{i,j}$ is any rational.

In general we have rational solutions above and when $\frac{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{\frac{x_{i,j}}{y_{i,j}}}}{d}$, and for every i, j: $l_{i,j}, g_{i,j}, t_i, p_{i,j}$ are intergers, we have integer solutions.

More generally:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} \left(\prod_{s=1}^{u} \left(\prod_{k \in U_{i,j,s}} p_{s,k}^{u_{i,j,s,k} * y_{i,j} * \frac{lcm(x)}{x_{i,j}}} * \prod_{k \in T_{i,j,s}} g_{s,k}^{y_{i,j} \frac{lcm(x,z_s)}{x_{i,j}}} \right) * l_{i,j} \right)^{\frac{x_{i,j}}{y_{i,j}}}$$

$$= d \prod_{i=1}^{u} \left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j} z_i - q_{i,j}) * \left(r_{i,j} * lcm(x)\right)^{z_i - 1} + q_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{lcm(x,z_i)} \right)$$

$$= d \prod_{i=1}^{u} \left(\left(t_i^{f_i} \prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{\frac{lcm(x,z_i)}{z_i}} \right)^{z_i'} \right)^{\frac{z_i}{z_i'}}$$

for every i, s: $\bigcup_{j=1}^{m_i} U_{i,j,s} = \{1, \dots, v_s\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \dots, w_s\}$,

for every i, j, k, s where $j \neq k$: $T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

for every i, j: $z_i | \left(\left(t_{i,j} z_i - q_{i,j}\right) * lcm(x)^{z-1} + q_{i,j} \right)$ {little Fermat theorem},

x is a set of all $x_{i,j}$, z is a set of all z_i.

Where for every i, s, k: $\sum_{j=1}^{m_i} u_{i,j,s,k} = \left(t_{s,k} z_s - q_{s,k}\right) * \left(r_{s,k}\right)^{z_s - 1} * lcm(x)^{z_s - 2}$

For every i, j: $g_{i,j}$ is any integer.

Method 7 – Płotnicki's equation – general case – rational exponents

Method 7: Method of finding infinitely many nontrivial complex solutions for equation like this:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{\frac{x_{i,j}}{y_{i,j}}} = d \prod_{j=1}^{m_0} b_j^{\frac{z_j}{z'_j}}$$

where gcd $\left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} x_{i,j}, \prod_{j=1}^{m_0} z_j\right) = 1, a_{i,j} > 1$

where for every i, j **:** $(\rightarrow)c_i, a_{i,j}, (\rightarrow)d, b_j$ **are rationals and** $(\rightarrow)n, (\rightarrow)m_i, (\rightarrow)x_{i,j}, (\rightarrow)y_{i,j}, (\rightarrow)z_j, (\rightarrow)z'_j$ **are integers.**

for every i, j: for every rational $l_{i,j}$ and every rational $p_{i,j}, t_i$ and every integer $q_{i,j}, f_i$ that suffices equation:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{\frac{x_{i,j}}{y_{i,j}}} = d \prod_{i=1}^{u}\left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{q_{i,j}}\right)$$

where for every i: f_i could be 0, for every i, j: $\gcd(q_{i,j}, z_i) = 1$, we have infinitely many solutions:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i}\left(\prod_{s=1}^{u}\left(\prod_{k\in S_{i,j,s}} p_{s,k}^{(t_{s,k}+f_{s,k}*z_s)*\frac{r_{s,k}*y_{i,j}*lcm(x)}{x_{i,j}}} * \prod_{k\in T_{i,j,s}} g_{s,k}^{y_{i,j}*\frac{lcm(x,z_s)}{x_{i,j}}}\right) * l_{i,j}\right)^{\frac{x_{i,j}}{y_{i,j}}}$$
$$= d\prod_{i=1}^{u}\left(t_i^{f_i*z_i}\prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j}+f_{i,j}*z_i)*r_{i,j}*lcm(x)+q_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{lcm(x,z_i)}\right)$$
$$= d\prod_{i=1}^{u}\left(\left(t_i^{f_i}\prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{\frac{lcm(x,z_i)}{z_i}}\right)^{z'_i}\right)^{\frac{z_i}{z'_i}}$$

Where

for every i, s: $\bigcup_{j=1}^{m_i} S_{i,j,s} = \{1, \dots, v_s\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \dots, w_s\}$,

for every i, j, k, s where $j \neq k$: $S_{i,j,s} \cap S_{i,k,s} = \emptyset, T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

for every i, j: $t_{i,j}$ is such integer that: $z_i | (t_{i,j} * r_{i,j} * lcm(x) + q_{i,j})$, {for details see: *Theorem 1*},

for every i, j: $f_{i,j}$ is any integer,

for every i, j: $r_{i,j}$ is any integer such that $\gcd(r_{i,j}, z_i) = 1$,

x is a set of all $x_{i,j}$, z is a set of all z_i.

For every i,j: $g_{i,j}$ is any integer.

In general we have rational solutions above and when $\frac{\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} l_{i,j}^{x_{i,j}}}{d}$, and for every i,j: $l_{i,j}$, $g_{i,j}$, $p_{i,j}$, t_i are integers we have integer solutions.

More generally:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} \left(\prod_{s=1}^{u} \left(\prod_{k \in U_{i,j,s}} p_{s,k}^{u_{i,j,s,k} * y_{i,j} * \frac{lcm(x)}{x_{i,j}}} * \prod_{k \in T_{i,j,s}} g_{s,k}^{y_{i,j} * \frac{lcm(x,z_i)}{x_{i,j}}} \right) * l_{i,j} \right)^{\frac{x_{i,j}}{y_{i,j}}}$$
$$= d \prod_{i=1}^{u} \left(t_i^{f_i * z_i} \prod_{j=1}^{v_i} p_{i,j}^{(t_{i,j} + f_{i,j} * z_s) * r_{i,j} * lcm(x) + q_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{lcm(x,z_i)} \right)$$
$$= d \prod_{i=1}^{u} \left(\left(t_i^{f_i} \prod_{j=1}^{v_i} p_{i,j}^{h_{i,j}} \prod_{j=1}^{w_i} g_{i,j}^{\frac{lcm(x,z_i)}{z_i}} \right)^{z_i'} \right)^{\frac{z_i}{z_i'}}$$

for every i,s: $\bigcup_{j=1}^{m_i} U_{i,j,s} = \{1, \dots, v_s\}$, $\bigcup_{j=1}^{m_i} T_{i,j,s} = \{1, \dots, w_s\}$,

for every i,j,k,s where $j \neq k$: $T_{i,j,s} \cap T_{i,k,s} = \emptyset$,

for every i,j: $t_{i,j}$ is such integer that: $z_i | \left(t_{i,j} * r_{i,j} * lcm(x) + q_{i,j}\right)$, {for details see: *Theorem 1*},

for every i,j: $f_{i,j}$ is any integer (f_i is completely other integer with other meaning),

for every i,j: $r_{i,j}$ is any integer such that $\gcd(r_{i,j}, z_i) = 1$,

x is a set of all $x_{i,j}$, z is a set of all z_i.

Where for every i,s,k: $\sum_{j=1}^{m_i} u_{i,j,s,k} = \left(t_{s,k} + f_{s,k} * z_s\right) * r_{s,k}$

For every i,j: $g_{i,j}$ is rational.

Example

$$\boldsymbol{5x^{\frac{2}{3}} + 3y^{\frac{3}{5}} = z^{\frac{5}{3}}}$$

$$5 * 1^{\frac{2}{3}} + 3 * 1^{\frac{3}{5}} = 8 = 2^3$$

$$5q = 2 * 3 * t + 3 \Leftrightarrow 5q = 3 * (2t + 1)$$

$$\boldsymbol{t = 2 + 5k} \Rightarrow 5 | 3 * (2t + 1)$$

$$\boldsymbol{5 * \left(2^{2*3*3} * 1\right)^{\frac{2}{3}} + 3 * \left(2^{2*2*5} * 1\right)^{\frac{3}{5}}} = 2^{12} * (5 + 3) = 2^{12} * 2^3 = 2^{15} = (2^3)^5 = \boldsymbol{\left(2^9\right)^{\frac{5}{3}}}$$

Proof that there are not other complex not derived solutions – rational exponents

Is almost the same as for integer exponents, so proof is not worth to be rewritten.

D – Plotnicki's equations – part IV – What is next? – the unlimited field of Płotnicki's equations.

Simple case

When we have to calculate solution for:

$$\sum_{i=1}^{k} c_i v_i^{x_i} = \sum_{i=1}^{l} d_i w_i^{y_i}$$

where $gcd(x,y) = 1$, x is a set of x_i, y is a set of y_i.

First we have to solve simultaneous equation in form:

$$\begin{cases} \sum_{i=1}^{k} c_i a_i^{x_i} = p^{lcm(y)} \\ \sum_{i=1}^{l} d_i b_i^{y_i} = q^{lcm(x)} \end{cases}$$

Then we can solve it from the equation:

$$\sum_{i=1}^{k} c_i a_i^{x_i} * \sum_{i=1}^{l} d_i b_i^{y_i} = \sum_{i=1}^{l} d_i b_i^{y_i} * \sum_{i=1}^{k} c_i a_i^{x_i}$$

We have solution in form:

$$\sum_{i=1}^{k} c_i \left(g^{\frac{lcm(x,y)}{x_i}} q^{\frac{lcm(x)}{x_i}} a_i \right)^{x_i} = \sum_{i=1}^{l} d_i \left(g^{\frac{lcm(x,y)}{y_i}} p^{\frac{lcm(y)}{y_i}} b_i \right)^{y_i}$$

For example:

$$wa^x + vb^y = uc^z$$

$$\begin{cases} wf^x + vg^y = p^z \\ ud^z = q^{lcm(x,y)} \end{cases}$$

There are given rules to solve both equations, so:

$$(wf^x + vg^y)ud^z = p^z ud^z = q^{lcm(x,y)}(wf^x + vg^y)$$

So here we have complex derived solution:

$$u(pd)^z = w\left(q^{\frac{lcm(x,y)}{x}} f\right)^x + v\left(q^{\frac{lcm(x,y)}{y}} g\right)^y$$

So this is the next method how to deal with “coefficient on the right side” and works always.

As we know how to solve:

$$\sum_{i=1}^{k} c_i v_i^{x_i} = \sum_{i=1}^{l} d_i w_i^{y_i}$$

We could solve for example:

$$\begin{cases} \sum_{i=1}^{k} c_i a_i^{x_i} = \sum_{i=1}^{l} g_i e_i^{y_i} \\ \sum_{i=1}^{m} d_i b_i^{z_i} = \sum_{i=1}^{n} h_i f_i^{w_i} \end{cases}$$

Where $\gcd(x, y) = \gcd(z, w) = 1$.

x is a set of x_i, y is a set of y_i, z is a set of z_i, w is a set of w_i.

And from there we have:

$$\sum_{i=1}^{k} c_i a_i^{x_i} \sum_{i=1}^{n} h_i f_i^{w_i} = \sum_{i=1}^{m} d_i b_i^{z_i} \sum_{i=1}^{l} g_i e_i^{y_i}$$

$$\sum_{i=1}^{k} c_i a_i^{x_i} \sum_{i=1}^{m} d_i b_i^{z_i} = \sum_{i=1}^{l} g_i e_i^{y_i} \sum_{i=1}^{n} h_i f_i^{w_i}$$

$$\frac{\sum_{i=1}^{k} c_i a_i^{x_i}}{\sum_{i=1}^{n} h_i f_i^{w_i}} = \frac{\sum_{i=1}^{m} d_i b_i^{z_i}}{\sum_{i=1}^{l} g_i e_i^{y_i}}$$

$$\frac{\sum_{i=1}^{k} c_i a_i^{x_i}}{\sum_{i=1}^{m} d_i b_i^{z_i}} = \frac{\sum_{i=1}^{l} g_i e_i^{y_i}}{\sum_{i=1}^{n} h_i f_i^{w_i}}$$

$$\sum_{i=1}^{k} c_i a_i^{x_i} \pm \sum_{i=1}^{n} h_i f_i^{w_i} = \sum_{i=1}^{m} d_i b_i^{z_i} \pm \sum_{i=1}^{l} g_i e_i^{y_i}$$

$$\sum_{i=1}^{k} c_i a_i^{x_i} \pm \sum_{i=1}^{m} d_i b_i^{z_i} = \sum_{i=1}^{l} g_i e_i^{y_i} \pm \sum_{i=1}^{n} h_i f_i^{w_i}$$

Etc.

And much more, eg.:

$$\boldsymbol{\prod_{j=1}^{l} \sum_{i=1}^{k_j} c_{j,i} a_{j,i}^{x_{j,i}} = \prod_{j=1}^{n} \sum_{i=1}^{m_j} d_{j,i} b_{j,i}^{y_{j,i}}}$$

$$\boldsymbol{\prod_{j=1}^{l} \sum_{i=1}^{k_j} c_{j,i} a_{j,i}^{x_{j,i}} + \sum_{i=1}^{k_j} e_{j,i} g_{j,i}^{x_{j,i}} = \prod_{j=1}^{n} \sum_{i=1}^{m_j} d_{j,i} b_{j,i}^{y_{j,i}} + \sum_{i=1}^{k_j} f_{j,i} h_{j,i}^{x_{j,i}}}$$

And so on…

General simple case

$$\sum_{i=1}^{k_1} c_{1,i} a_{1,i}{}^{x_{1,i}} = \cdots = \sum_{i=1}^{k_n} c_{n,i} a_{n,i}{}^{x_{n,i}}$$

where for every j: $gcd(x\ without\ x_j, x_j) = 1$, x is a set of x_i, x_i is a set of $x_{i,j}$.

We have for every j:

$$\left\{ \sum_{i=1}^{k_j} c_{j,i} a_{j,i}^{x_{j,i}} = p_j^{lcm(x\ without\ x_j)} \right.$$

And then solution in form:

$$\sum_{i=1}^{k_1} c_{1,i} \left(g^{\frac{lcm(x)}{x_{1,i}}} \prod_{j=2}^{n} p_j^{\frac{lcm(x\ without\ x_j)}{x_{1,i}}} a_{1,i} \right)^{x_{1,i}} = \cdots$$

$$= \sum_{i=1}^{k_l} c_{l,i} \left(g^{\frac{lcm(x)}{x_{l,i}}} \prod_{j=1}^{l-1} p_j^{\frac{lcm(x\ without\ x_j)}{x_{l,i}}} \prod_{j=l+1}^{n} p_j^{\frac{lcm(x\ without\ x_j)}{x_{l,i}}} a_{l,i} \right)^{x_{l,i}} = \cdots$$

$$= \sum_{i=1}^{k_n} c_{n,i} \left(g^{\frac{lcm(x,y)}{x_{n,i}}} \prod_{j=1}^{n-1} p_j^{\frac{lcm(x\ without\ x_j)}{x_{n,i}}} a_{n,i} \right)^{x_{n,i}}$$

The most general case

The same is in the most general case:

$$\sum_{i=1}^{n_1} c_{1,i} \prod_{j=1}^{m_{1,i}} a_{1,i,j}^{x_{1,i,j}} = \cdots = \sum_{i=1}^{n_k} c_{k,i} \prod_{j=1}^{m_{k,i}} a_{k,i,j}^{x_{k,i,j}}$$

where $gcd(x\ without\ x_j, x_j) = 1$, x is a set of x_i, x_i is a set of $x_{i,j}$.

For example:

$$\sum_{i=1}^{l} d_i \prod_{j=1}^{o_i} b_{i,j}^{y_{i,j}} * \sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = \sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} * \sum_{i=1}^{l} d_i \prod_{j=1}^{o_i} b_{i,j}^{y_{i,j}}$$

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = p^{lcm(y_{,j})}$$

$$\sum_{i=1}^{l} d_i \prod_{j=1}^{o_i} b_{i,j}^{y_{i,j}} = q^{lcm(x_{,j})}$$

Or:

$$\sum_{i=1}^{n} c_i \prod_{j=1}^{m_i} a_{i,j}^{x_{i,j}} = \prod_{j=1}^{\max(o_i)} p_j^{lcm(y_{,j})}$$

$$\sum_{i=1}^{l} d_i \prod_{j=1}^{o_i} b_{i,j}^{y_{i,j}} = \prod_{j=1}^{\max(m_i)} q_j^{lcm(x_{,j})}$$

This method is not so difficult, but is to difficult to be elegantly showed. You could easily see it. The only diffference here is that you have simply more possibilities to place p_j and q_j for every i.

So in general you can easily solve equations like this:

$$\prod_{j=1}^{l} \sum_{i=1}^{k_j} c_{j,i} \prod_{e=1}^{g_{j,i}} a_{j,i,e}^{x_{j,i,e}} = \prod_{j=1}^{n} \sum_{i=1}^{m_j} d_{j,i} \prod_{f=1}^{h_{j,i}} b_{j,i,f}^{y_{j,i,f}}$$

And from this point we can solve:

$$\prod_{j=1}^{l_1} \sum_{i=1}^{k_{1,j}} c_{1,j,i} \prod_{e=1}^{g_{1,j,i}} a_{1,j,i,e}^{x_{1,j,i,e}} = \cdots = \prod_{j=1}^{l_n} \sum_{i=1}^{k_{n,j}} c_{n,j,i} \prod_{e=1}^{g_{n,j,i}} a_{n,j,i,e}^{x_{n,j,i,e}}$$

And so on… More about it you will find in my book.

Rational exponents

The same is for rational exponents.

For example:

First we have to solve simultaneous equation in form:

$$\begin{cases} \sum_{i=1}^{k} c_i a_i^{\frac{x_i}{x'_i}} = p^{lcm(y)} \\ \sum_{i=1}^{l} d_i b_i^{\frac{y_i}{y'_i}} = q^{lcm(x)} \end{cases}$$

Then we can solve it from the equation:

$$\sum_{i=1}^{k} c_i a_i^{\frac{x_i}{x'_i}} * \sum_{i=1}^{l} d_i b_i^{\frac{y_i}{y'_i}} = \sum_{i=1}^{l} d_i b_i^{\frac{y_i}{y'_i}} * \sum_{i=1}^{k} c_i a_i^{\frac{x_i}{x'_i}}$$

We have solution in form:

$$\sum_{i=1}^{k} c_i \left(g^{\frac{x'_i * lcm(x,y)}{x_i}} q^{\frac{x'_i * lcm(x)}{x_i}} a_i \right)^{\frac{x_i}{x'_i}} = \sum_{i=1}^{l} d_i \left(g^{\frac{y'_i * lcm(x,y)}{y_i}} p^{\frac{y'_i * lcm(y)}{y_i}} b_i \right)^{\frac{y_i}{y'_i}}$$

And so on…

Appendix 1 – Inverse function of Li(n)

```
int prime(int n)
{
	typedef double real_type;
	const int ilogsum_limit = 3;
	real_type* ilogsumt = new real_type[ilogsum_limit];
	for (int i = 0; i < ilogsum_limit; ++i) ilogsumt[i] = 0.0;
	for (int i = 2; i <= n; ++i)
	{
		ilogsumt[0] += log(real_type(i)*log(real_type(i)));
		for (int j = 1; j < ilogsum_limit; ++j)
			if (ilogsumt[j - 1] > 1.0) ilogsumt[j] += log(ilogsumt[j - 1]);
	}
	const int result = ilogsumt[ilogsum_limit-1];
	delete [] ilogsumt;
	return result;
}
```

$$\boldsymbol{prime(n) = f_\infty(n)}$$

$$\boldsymbol{f_0(n) = \sum_{i=1}^{n} \ln(i * \ln i)}$$

$$\boldsymbol{f_k(0) = 0, f_k(n) = f_k(n-1) + max\big(\ln\big(f_{k-1}(n)\big), 0\big)}$$

or:

$$\boldsymbol{f_k(n) = \sum_{i=1}^{n} max\big(\ln\big(f_{k-1}(i)\big), 0\big)}$$

Function $prime(n)$ runs in time $O(n) = O\left(\frac{p_n}{\ln p_n}\right)$ and tends very quickly to $\log(p_1 * \dots * p_n)$ and p_n, where p_i is i-th prime number. The best performance can be obtained calculating $prime(n)$ for all numbers in the range 1 ... n, or for a set of complexity $O(n)$, then the complexity of calculating each $prime(i)$ is $O(1)$.

For `ilogsum_limit = 4` with double precision (a higher value for this type causes already deterioration of result due to errors in floating point operations) it gets average percentage difference less than 1% for p_{1073} (p_{1096} for $\ln(p_1 * \ldots * p_n)$) 8623 (8803), and one promile for p_{18415} (p_{18491} for $\ln(p_1 * \ldots * p_n)$), which is the prime number 205417 (206273). Probably there is no better known approximation for p_i that does not use primes and it is very possible that in general it does not exist.

// Polish: Funkcja $prime(n)$ działa w czasie $O(n) = O\left(\frac{p_n}{\ln p_n}\right)$ i dąży bardzo szybko do $\ln(p_1 * \dots * p_n)$ i p_n, gdzie p_i to i-ta liczba pierwsza. Najlepszą wydajność można uzyskać licząc $prime(n)$ dla wszystkich liczb z przedziału $1 \dots n$, lub dla zbioru o złożoności $O(n)$, wtedy złożoność obliczenia każdego $prime(i)$ jest $O(1)$.

Na marginesie: oczywiście definicja liczby pierwszej powinna brzmieć: „liczba podzielna tylko przez samą siebie i 1", czyli powinna być nią również jedynka.

Już dla `ilogsum_limit==4` przy precyzji *double* (większa wartość dla tego typu powoduje już pogorszenie wyniku ze względu na błędy operacji zmiennoprzecinkowych) uzyskuje średnią różnicę procentową mniejszą od 1% już przy p_{1073} (p_{1096} dla $\ln(p_1 * \dots * p_n)$), czyli 8623 (8803), a jednopromilową różnicę przy p_{18415} (p_{18491} dla $\ln(p_1 * \dots * p_n)$), czyli liczbie pierwszej 205417 (206273). Przy kilkumilionowej liczbie pierwszej schodzi do około jednomilionowej. Prawdopodobnie nie istnieje żadne lepsze znane przybliżenie p_i nie wykorzystujące liczb pierwszych i bardzo możliwe, że w ogóle nie istnieje. Algorytmowi można również bardzo łatwo podać największą znaną liczbę pierwszą $p_i < p_n$ (wystarczy do i liczyć wszystkie poziomy poza ostatnim, dla i podać p_i na ostatnim poziomie i kontynuować obliczenia już dla wszystkich poziomów), zwiększając znacznie precyzję obliczeń. Na podstawie dwóch kolejnych liczb $prime(i)$ można osiągnąć precyzję taką, jakby zaczęło się obliczenia od nich i można rozpocząć obliczanie od nich, bo można na ich podstawie obliczyć całą tablicę `ilogsumt`. Tak więc przeznaczając na wcześniej obliczone pary sąsiednich liczb $prime(i)$ pamięć $O\left(p_n^{\frac{1}{3}}\right)$ tak jak to ma miejsce w najlepszym algorytmie Lagarias-Miller-Odlyzko dla $\pi(n)$, można uzyskać złożoność $O\left(\frac{\frac{p_n}{\ln p_n}}{p_n^{\frac{1}{3}}}\right) = O\left(\frac{p_n^{\frac{2}{3}}}{\ln p_n}\right)$. Ponadto mając zapamiętane `ilogsum_limit` liczb z tablicy `ilogsumt` dla $prime(n)$ można odtworzyć $prime(n-c)$ i $prime(n+c)$ w czasie $O(c)$.

W czasie $O\left(\frac{n}{\ln n}\right)$ da się zatem oszacować bardzo dokładnie $\pi(i)$ dla wszystkich liczb z przedziału $1 \dots n$, nie znając żadnej liczby pierwszej. Jest to zatem algorytm niemal tak szybki jak Lehmera ($O\left(\frac{n}{\ln^4 n}\right)$, 1994r.) i Meissela ($O\left(\frac{n}{\ln^3 n}\right)$, 1985-1994r.), przy czym zużywa tylko $O(1)$ pamięci, a nie $O\left(\frac{n^{\frac{1}{3}}}{\ln n}\right)$ lub odpowiednio $O\left(\frac{n^{\frac{1}{2}}}{\ln n}\right)$, a więc nie ma ograniczenia pamięciowego na obliczenie wielkich wartości n, oraz dla zbioru liczb $\pi(i)$ o złożoności $O(n)$ złożoność obliczenia pojedynczej wartości to $O(1)$.

Oto algorytm:

```
int pi(int n)
{
	typedef double real_type;
	const int ilogsum_limit = 3;
	real_type* ilogsumt = new real_type[ilogsum_limit];
	for (int i = 0; i < ilogsum_limit; ++i) ilogsumt[i] = 0.0;
	for (int i = 2; i <= n; ++i)
	{
		ilogsumt[0] += log(real_type(i)*log(real_type(i)));
		for (int j = 1; j < ilogsum_limit; ++j)
			if (ilogsumt[j - 1] > 1.0) ilogsumt[j] += log(ilogsumt[j - 1]);
		if (ilogsumt[ilogsum_limit - 1] > n)
		{
			delete [] ilogsumt;
			return i - 2;
		}
	}
}
```

The accuracy of the algorithm $pi(n)$ for `ilogsum_limit = 3` and double precision numbers is basically the same as $Li(n)$ from the table from Wikipedia: $pi(10^7) = 664919$, $Li(10^7) = 664918$, $\pi(10^7) = 664579$; $pi(10^8) = 5762211$, $Li(10^8) = 5762209$, $\pi(10^8) = 5761455$. For numbers of greater precision you can probably get exactly the same result as $Li(n)$. Therefore it seems that the $prime(n)$ is the inverse of the $Li(n)$, which gives a much smaller errors from the formula proposed in www.mathworld.wolfram.com/PrimeFormulas.html (15). It also maintains the relation $prime(n) < p_n$. $prime(n)$ algorithm also has much simpler form than this proposed there.

// Dokładność algorytmu $pi(n)$ dla `ilogsum_limit = 3` i liczb dokładności *double* jest w zasadzie identyczna jak $Li(n)$ z tabeli z wikipedii: $pi(10^7) = 664919$, $Li(10^7) = 664918$, $\pi(10^7) = 664579$; $pi(10^8) = 5762211$, $Li(10^8) = 5762209$, $\pi(10^8) = 5761455$. Dla liczb większej precyzji prawdopodobnie można uzyskać wynik identyczny albo nawet lepszy niż $Li(n)$.**Wydaje się więc, że $prime(n)$ jest funkcją odwrotną do $Li(n)$**, przy czym daje o wiele mniejsze błędy od wzoru zaproponowanego w www.mathworld.wolfram.com/PrimeFormulas.html (15) – zachowuje także relację $prime(n) < p_n$. Algorytm $prime(n)$ ma też o wiele prostszą postać od zaproponowanego tam rozwinięcia. Dla n około 50 milionów dokładność $pi(n)$ jest rzędu czterech pierwszych wiodących liczb. Precyzję tego algorytmu również można łatwo i znacznie podnieść podając największą znaną liczbę $p_i < p_n$.

Appendix 2

Solutions of equation:

$$\boldsymbol{a^2 \pm b^2 = c^z}$$

If z is not divisible by 2 then we have infinitely many non coprime solutions.

If z is divisible by 2 then we have the same problem

$$a = p_1^2 \mp q_1^2$$

$$b = 2p_1q_1$$

$$c^{\frac{z}{2}} = p_1^2 \pm q_1^2$$

So we always come to equation:

$$c^{\frac{z}{2^j}} = p_j^2 \pm q_j^2$$

Where $\frac{z}{2^j}$ is odd. So we always has infinitely many solutions. And that's are all non coprime solutions.

Additionally there is always coprime solution for:

$$c^{\frac{z}{2^j}} = p_j^2 - q_j^2 = (p_j - q_j)(p_j + q_j)$$

$$p_j - q_j = e^{z_j}$$

$$p_j + q_j = f^{z_j}$$

$$p_j = \frac{e^{z_j} + f^{z_j}}{2}$$

$$q_j = \frac{e^{z_j} - f^{z_j}}{2}$$

And that are not all coprime solutions to the equation:

$$\boldsymbol{a^2 - b^2 = c^z}$$

Because c^{z_j} can be $2p_jq_j$ where $gcd(p_j, q_j) = 1$.

For:

$$\boldsymbol{a^2 \pm b^2 = c^2}$$

As $gcd(2,2,2) > 1$, so there is no possibility to align common divisors to $2z$ other than:

$$(gk)^2 \pm (gl)^2 = (gm)^2$$

which can be divided by g^2, to get

$$k^2 \pm l^2 = m^2$$

so all non coprime solutions are derived from coprime. Of course c may be d^z above.

So putting all together we have all solutions for:

$$\boldsymbol{a^2 - b^2 = c^z}$$

And all non coprime solutions for:

$$\boldsymbol{a^2 + b^2 = c^z}$$

And we knot that:

$$\boldsymbol{a^2 \pm b^2 = c^2}$$

has no non coprime not derived solutions.

QED.

Appendix 3 – Content of email to the full professor in University of Warsaw Edmund Puczylowski (10/26/2011)

Theorem 1: $ab = t * \prod_{i=1}^{n} c_i + x$, has integer solution for every a for given c_i, and given x, where $gcd(a, \prod_{i=1}^{n} c_i) = 1$.

[-

of course Chinese remainder theorem solves this problem

$$\begin{cases} w = x\left(mod\left(\prod_{i=1}^{n} c_i\right)\right) \\ w = 0(mod\ a) \end{cases}$$

$$w_k = w + k * a * \prod_{i=1}^{n} c_i = \left(\frac{w-x}{\prod_{i=1}^{n} c_i} + k * a\right) * \prod_{i=1}^{n} c_i + x$$

-]

For:

$$a^x + b^y = c^z, where\ \gcd(xy, z) = 1$$

for every

$$d = k^x + l^y = p1^{q1} * \ldots * pn^{qn} * m^{f*z}$$

where f could be 0, we have only solutions:

$$\left(p1^{(t1+f1*z)*y} * \ldots * pn^{(tn+fn*z)*y} * k\right)^x + \left(p1^{(t1+f1*z)*x} * \ldots * pn^{(tn+fn*z)*x} * l\right)^y$$
$$= p1^{(t1+f1*z)*xy+q1} * \ldots * pn^{(tn+fn*z)*xy+qn} * m^{f*z} = c^z$$

So for every k,l we have as much subclasses of solutions as much “images of divisibility” exists, in form:

$$p1^{q1} * \ldots * pn^{qn} * m^{f*z}\ of\ given\ k^x + l^y.$$

So for given $gcd(a, b, c) = p1^{t1xy} * \ldots * pn^{tnxy} * m^{f*z}$

there is only one image of divisibility $p1^{q1} * \ldots * pn^{qn} * m^{f*z}$ for which are constant numbers of k, l pairs such that $k^x + l^y = p1^{q1} * \ldots * pn^{qn} * m^{f*z}$, which has only above solutions.

And if equation has one solution : $k^x + l^y = m^z$, then it has infinitely many solutions:

$\left(g^{\frac{tzy}{fgcd(zy,x)}} * k\right)^x + \left(g^{\frac{tzx}{fgcd(zy,x)}} * l\right)^y = (k^x + l^y)\left(g^{\frac{txy}{fgcd(zy,x)}}\right)^z$, for every g,t

fgcd(zy,x) could be also fgcd(zx,y)

where fgcd(x,y) could be every selected divisor of gcd(x,y)

And those are all solutions that can be derived from $k^x + l^y = m^z$.

Gcd(a)=1 – not complex solutions

Gcd(a)>1 – complex solutions

Where a is variables set.

And those are all solutions (derived from all not complex solutions) when there are not complex not derived solutions (when for w=x,y,z : gcd(xyz/w,w) > 1).

So putting both together, when we know all gcd(a)=1 solutions (that the amount of is constant number or zero), we know all solutions of Diophantine equation.

And if there is a solution for: $(n[i,1] - n[j,1])k^x + (n[i,2] - n[j,2])l^y = 0$, and a!=b!=c and gcd(a,b,c)> 1 then and only then (because otherwise c[i]!=c[j]) simultaneous equation has infinitely many solutions:

$$n[i,1](P1 * k)^x + n[i,2](P2 * l)^y = c^z, where\, n[i,j]\, are\, rationals.$$

There is also a solution for eg.:

$$a^x + b^y c^z = d^w, where \gcd(xyz, w) = 1$$

$$k^x + l^y m^z = p1^{q1} * \dots * pn^{qn} * m^{f*w}$$

$$\begin{aligned}
&\left(p1^{(t1+f1*z)*y*z} *..* pn^{(tn+fn*z)*y*z} * k\right)^x \\
&\quad + \left(subset\left(p1^{(t1+f1*z)*x*z} * \dots * pn^{(tn+fn*z)*x*z}\right) * l\right)^y \\
&\quad * \left(subset\left(p1^{(t1+f1*z)*x*y} * \dots * pn^{(tn+fn*z)*x*y}\right) * m\right)^z \\
&\quad = (p1)^{(t1+f1*z)*xyz+q1} * \dots * pn^{(tn+fn*z)*xyz+qn} * m^{f*w} = c^z
\end{aligned}$$

For example:

$$2^2 + 2^3 * 2^5 = 260 = 26 * 10$$

$$t1 * (2 * 3 * 5) + 1 = 7q1$$

$$t2 * (2 * 3 * 5) + 1 = 7q2$$

$$t1 = 3, t2 = 3 + 7 = 10$$

so:

$$\begin{aligned}
&(26^{3*3*5} * 10^{10*3*5} * 2)^2 + (26^{3*2*3} * 2)^3 * (10^{10*2*3} * 2)^5 = (26)^{3*30+1} * 10^{10*30+1} \\
&\quad = (26^{13} * 10^{43})^7
\end{aligned}$$

For d^w it will give all complex solutions.

$b^y c^z$ can be calculated as f^{y+z}, but it will not give all possible solutions, but there still is a way to calculate them:

$$d^w - a^x = b^y c^z, where \gcd(wx, yz) = 1$$

$$k^w - l^x = p1^{q1} * \ldots * pn^{qn} * m^{f*y} * n^{g*z}$$

So p, t have to be selected such a way to contruct b^yc^z.

For example :

$$d^7 - a^2 = b^3 c^5$$

$$2^7 - 2^2 = 124 = 2^2 * 31$$

$$2 * 7 * t1 + 2 = 3q1$$

$$2 * 7 * t2 + 1 = 5q2$$

$$t1 = 2, t2 = 1$$

$$\left((61^{1*2} * 2^{2*2}) * 2\right)^7 - \left((61^{1*7} * 2^{2*7}) * 2\right)^2 = (2^7 - 2^2) * (2^{14} * 61^{14})$$
$$= (2^2 * 61) * (2^{28} * 61^{14}) = 2^{30} * 61^{15} = (2^{10})^3 * (61^3)^5$$

The same is for derivation:

$$\left(g^{\frac{t1xyz}{fgcd(xyz,w)}} * h^{\frac{t2xyz}{fgcd(xyz,w)}} k\right)^w - \left(g^{\frac{t1yzw}{fgcd(xyz,w)}} h^{\frac{t2yzw}{fgcd(xyz,w)}} * l\right)^x$$
$$= (k^w - l^x)\left(g^{\frac{t1xzw}{fgcd(xyz,w)}}\right)^y * \left(h^{\frac{t2xyw}{fgcd(xyz,w)}}\right)^z$$

And the same is for combinations when there exist partial solved solution:

$$d^7 - a^2 = b^2 c^5$$

$$2^7 - 2^2 = 124 = (2^2) * 31$$

So to calucalte all solutions you need to know only not complex solutions which are few or zero.

So there always is infinitely many complex not derived solutions only when $gcd\left(\frac{Mul(x)}{s[j]}, s[j]\right) = 1$, where Mul(x) is multiplication of all powers, and s[j] is a multiplication of powers at position j; or there exists combination (there exist partial solved solution, eg.: $d^7 - a^2 = b^2 c^5, 2^7 - 2^2 = 124 = (2^2) * (31)^1$, where the condition should be sufficed only for those x[k] that are not

solved; of course in this case $(\boldsymbol{b^2c^5}, (\mathbf{3^2}) * (\mathbf{14^1}))$ divisibilities could be exchanged 2->5, 1->2) – proved; and there exist always infinitely many complex derived solutions if there exist at least one solution – proved.

So in general this is the way to calculate all rational complex solutiuons of Diophantine equations where there exist such j that $\boldsymbol{gcd}\left(\frac{\boldsymbol{Mul}(x)}{\boldsymbol{x}[j]}, \boldsymbol{x}[\boldsymbol{j}]\right) = \mathbf{1}$, where x[j] is a multiplication of powers at position j in equation, eg.: $\mathbf{2}\boldsymbol{x}^{\mathbf{3}} + \mathbf{3}\boldsymbol{y}^{\mathbf{5}}\boldsymbol{v}^{\mathbf{3}} = \mathbf{5}\boldsymbol{z}^{\mathbf{7}}\boldsymbol{w}^{\mathbf{2}}$, etc.