A Continuous Counterpart to Schwarzschild's Liquid Sphere Model

N.S. Baaklini
nsbqft@aol.com

Abstract

We present a continuous counterpart to Schwarzschild's metrical model of a constant-density sphere. The new model interpolates between a central higher-density spherical concentration of mass and lower-density layers at large distance. Whereas the radial part of pressure shows a positive distribution for all values of radial distance, the angular part of pressure shows negative magnitudes in the upper layers of low density; both pressures vanishing for infinite distance. We speculate that the negative pressure effect might be connected with stellar winds. Studying the motions of photons and massive particles in the gravity field of the continuous model shows similar, however continuous, behaviors to those described before for Schwarzschild's constant-density model.

1 Introduction

The Schwarzschild relativistic model of an incompressible liquid sphere[1] of constant density may serve as an idealistic mathematical foundation[2],[3] for a liquid model of the sun[4] and other astrophysical bodies. However, it would be much more realistic if a model would account for a continuous transition from an internal higher-density incompressible spherical core to external lower-density gaseous layers. Our purpose in this article is to present such a counterpart to the Schwarzschild metric.

Let us begin by recalling that corresponding to a general metric described by the line element

$$ds^2 = A(r)c^2dt^2 - B^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

where r, θ, ϕ are the spherical coordinates, t is the time variable, c is the special relativistic speed of light constant, with $A(r)$ and $B(r)$ functions of the radial coordinate, the Schwarzschild metric of a sphere of constant density is given by such foregoing metric with the functions A and B given by separate expressions for the inside and the outside of the sphere. First for the internal part of the metric, we have

$$\begin{cases} A(r) = \frac{1}{4}\left(3\sqrt{1-\frac{s}{a}} - \sqrt{1-\frac{sr^2}{a^3}}\right)^2 \\ \\ B(r) = \left(1-\frac{sr^2}{a^3}\right) \end{cases} \qquad r \le a \tag{2}$$

while for the external part, we have the Schwarzschild metric for a central point, with

$$\begin{cases} A(r) = \left(1 - \frac{s}{r}\right) \\ B(r) = \left(1 - \frac{s}{r}\right) \end{cases} \qquad r \geq a \tag{3}$$

In the above expressions, the parameter a denotes the radius of the sphere, while the parameter s denotes the Schwarzschild (or Hilbert) radius given by $s = 2GM/c^2$, with G the Newtonian gravitational constant, and M is mass of the sphere. Notice that the values of the functions $A(r)$ and $B(r)$ for the inside of the sphere and for the outside, do in fact match at the boundary $r = a$.

Whereas Schwarzschild had given the above metric[1] in a much more complicated form and in terms of other coordinate variables, the above was given recently by Borissova and Rabounski[2],[3]. We have already examined the above metric[7] and verified the fact that it solves the Einstein equations with a constant mass density inside the sphere and zero outside:

$$\rho = \begin{cases} \frac{3c^2}{8\pi G}\frac{s}{a^3} & r \leq a \\ 0 & r > a \end{cases} \tag{4}$$

And with a pressure of an ideal fluid (radial and angular parts equal) given by

$$P = \begin{cases} \frac{3c^4}{8\pi G}\left(\frac{s}{a^3}\right)\left\{\frac{\sqrt{1-\frac{s}{a}}-\sqrt{1-\frac{sr^2}{a^3}}}{-3\sqrt{1-\frac{s}{a}}+\sqrt{1-\frac{sr^2}{a^3}}}\right\} & r \leq a \\ 0 & r \geq a \end{cases} \tag{5}$$

We also studied the relativistic motions[5] of a photon and a massive particle in the gravitational field of the above Schwarzschild metric for a sphere of constant density, and made contrast with the motions[6] in the field of a singular Schwarzschild source[7].

In the followings, after introducing our continuous counterpart to the Schwarzschild metric of a sphere of constant density, given above, we proceed to examine its properties keeping eye on the possibility of applying it as a stellar model, with particular reference to solar parameters. We shall also examine the radial motions of a photon and a massive particle in the associated gravity field, making comparisons and contrasts with the results of our similar study[5] with regard to the metric of a Schwarzschild liquid sphere.

2 The Continuous Counterpart

We now present the model which replaces Schwarzschild's discontinuous metric by a counterpart that interpolates continuously between the internal and external sections.

The metrical functions take the form:

$$\begin{cases} A(r) = \frac{1}{4}\left(3\sqrt{1-\frac{s}{(r^3+a^3)^{1/3}}} - \sqrt{1-\frac{sr^2}{r^3+a^3}}\right)^2 \\ \\ B(r) = \left(1-\frac{sr^2}{r^3+a^3}\right) \end{cases} \tag{6}$$

Notice that the above expressions reduce to the Schwarzschild internal form for $r \ll a$, and to the external form for $r \gg a$. The parameter a in our expressions would give a measure of the size that should be associated with the central mass distributions. To determine the value of a we can use its relationship to other quantities such as density and pressure, as we shall see. With regard to the description of a stellar body, the value of a need not be equal to the apparent value of the stellar radius. It could be smaller or greater depending on the state, structure and evolution of the system.

In the followings, we shall examine the properties of the above continuous metric, and show that it gives reasonable continuous expressions for density and pressure. We shall discuss implications for a possible associated stellar model, with particular reference to solar parameters. Subsequently, we shall depict the relativistic motion of a particle making contrast with Schwarzschild's model.

Recalling that our general expression[5] for the mass density is given by

$$\rho = -\frac{c^2}{8\pi G}\frac{(-1+B+rB')}{r^2} \tag{7}$$

where the prime denotes differentiation with respect to the radial coordinate r, we obtain corresponding to our continuous metric:

$$\rho = \frac{3c^2}{8\pi G}\frac{sa^3}{(r^3+a^3)^2} \tag{8}$$

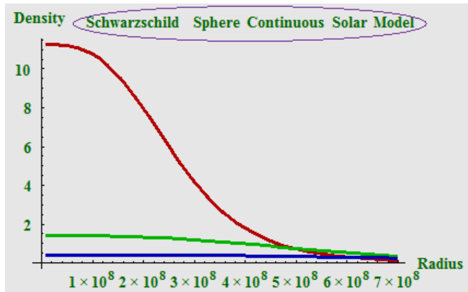
First note that the above expression is positive for all positive values of a and r. In order to make a realistic depiction of the above density distribution, we shall use solar parameters. Using $s = 2GM/c^2$, we obtain

$$\rho = \frac{3a^3M}{4\pi(r^3+a^3)^2} \tag{9}$$

Now using the solar mass value for M, we get the following expression (in gm/cm^3),

$$\rho = \frac{4.7482\times 10^{26}a^3}{(r^3+a^3)^2} \tag{10}$$

The following is a plot for three tentative values of $a = \{0.5, 1, 1.5\} \times R_\odot$, with $R_\odot$ the solar radius (respective colors, red, green and blue), with the radial scale in meters:



We see that the smaller the value of a, the more concentrated is the spherical mass distribution. Corresponding to the three respective values of a, we can compute that the respective values of the density at the center of a corresponding solar model are $\{11.2666, 1.40832, 0.41728\}$ gm/cm^3 while the values when r is equal to the visible solar radius are $\{0.139093, 0.35208, 0.248324\}$ gm/cm^3. In fact depending on the nature of the solar model, the needed density at the core, or the value needed at the photosphere, we might be able to determine the value of a more precisely. Notice that the above values of density would drop to $\{0.00266664, 0.0173867, 0.0367343\}$ gm/cm^3 at twice the solar radius and to $\{1.75996 \times 10^{-7}, 1.40551 \times 10^{-6}, 4.72115 \times 10^{-6}\}$ gm/cm^3 at ten solar radii.

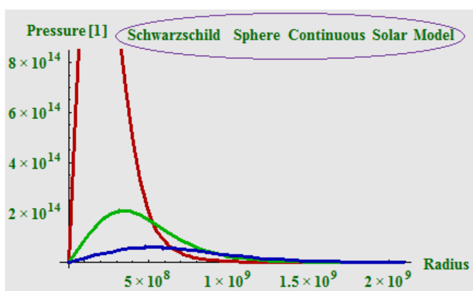
Turning to the pressure parts, we first note the the radial pressure part P_1 and the angular pressure part P_2 (which is equal to P_3) are not equal in this model (which means that the underlying distribution does not correspond to an ideal fluid). The expressions for P_1 and P_2 are too complicated, so we shall present them as images from our symbolic computational system. This gives the expression for P_1:

$$\left(c^4 s\left(-\sqrt{1-\frac{r^2 s}{a^3+r^3}}+3\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right)\left(\left(1-\frac{r^2 s}{a^3+r^3}\right)\left(\frac{2a^3-r^3}{(a^3+r^3)^2\sqrt{1-\frac{r^2 s}{a^3+r^3}}}+\frac{3r}{(a^3+r^3)^{4/3}\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}}\right)+\frac{\sqrt{1-\frac{r^2 s}{a^3+r^3}}-3\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}}{a^3+r^3}\right)\right)\Big/\left(8G\pi\left(\sqrt{1-\frac{r^2 s}{a^3+r^3}}-3\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right)^2\right) \quad (11)$$

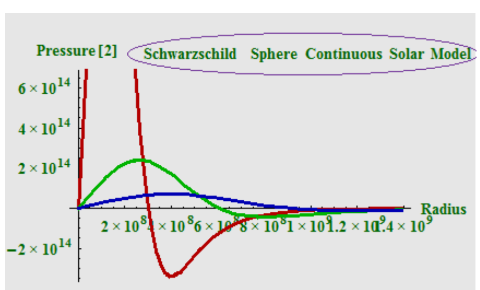
This gives the expression for P_2:

$$
\begin{aligned}
&\Bigg(3c^4 s\Bigg(-r^9\left(r-(a^3+r^3)^{1/3}\right)\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}\left(-2(a^3+r^3)^{1/3}+s\right)+\\
&\quad 2a^9\left((a^3+r^3)^{1/3}-s\right)\left(-3r\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}+2(a^3+r^3)^{1/3}\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}-2\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}\,s-2(a^3+r^3)^{1/3}\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right)+\\
&\quad a^6r^2\Bigg(r^2\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}\left(-10(a^3+r^3)^{1/3}+11s\right)+4(a^3+r^3)^{1/3}\left((a^3+r^3)^{1/3}-s\right)s\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}+\\
&\qquad 2r\left((a^3+r^3)^{1/3}-s\right)\left(3(a^3+r^3)^{1/3}\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}+\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}\,s+2(a^3+r^3)^{1/3}\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right)\Bigg)+\\
&\quad a^3r^5\Bigg(2r^2\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}\left(-(a^3+r^3)^{1/3}+2s\right)+8(a^3+r^3)^{1/3}s\left(-(a^3+r^3)^{1/3}+s\right)\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}+\\
&\qquad r\left(5(a^3+r^3)^{1/3}\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}\,s-6\sqrt{\frac{a^3+r^2(r-s)}{a^3+r^3}}\,s^2+8(a^3+r^3)^{2/3}\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}-8(a^3+r^3)^{1/3}s\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right)\Bigg)\Bigg)\Bigg)\Bigg/\\
&\left(32G\pi(a^3+r^3)^{11/3}\sqrt{1-\frac{r^2s}{a^3+r^3}}\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\left(a^3+r^3-(a^3+r^3)^{2/3}s\right)\left(\sqrt{1-\frac{r^2s}{a^3+r^3}}-3\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right)\right)
\end{aligned}
\tag{12}
$$

Now using the solar parameters we obtain numeric expressions for P_1 and P_2. The following is a plot of P_1 corresponding to our three values of $a = \{0.5, 1, 1.5\} \times R_\odot$ (respective colors red, green and blue). The radial scale is given in meters, and the pressure scale in atmospheres:

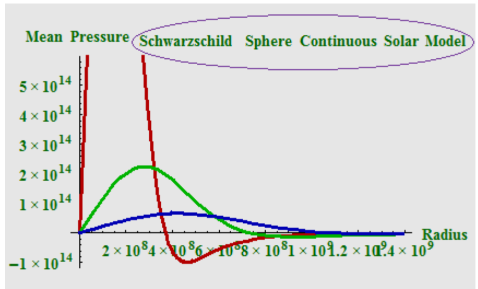


Likewise, this is the plot for P_2:



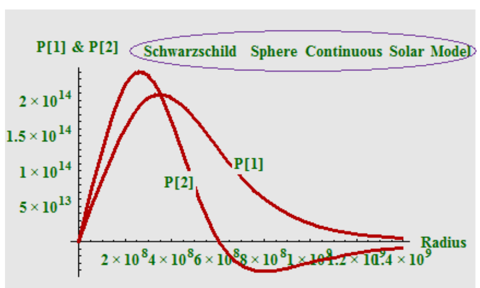
At this point, we note an *important observation*. Whereas the radial part of pressure P_1 is positive for all values of the radial coordinate, the angular part P_2 exhibits negative values in upper layers of the solar sphere. However, notice that both parts tend to zero for large radial distance.

For the mean pressure $P = (P_1 + P_2 + P_3)/3$, we have the following plot:



We can see that the effect of a negative angular part of pressure does show up in the depiction of the mean pressure. The physical meaning of negative pressure, we believe, is that of *inflation*, to be contrasted with *compression*. What would this mean with respect to a solar model? This is not a question which we can answer immediately. However, we ask whether this could have any relationship with the phenomenon of *stellar winds*! The outer layer of a star would seem to be characterized by negative pressure, and this seems to come from the angular part of pressure rather than from the radial, according to the continuous metrical model at hand. Any other suggestions?

It would be interesting to compare the radial part of pressure with the angular part at the same radial distance. The following is a plot of both P_1 and P_2 for the value of $a = 0.5 \times R_\odot$:



Let us turn now to an analysis of the relativistic radial motion of a particle in the field of our continuous metrical counterpart to the Schwarzschild liquid sphere. We shall begin with the motion of a photon followed by the motion of a massive particle.

3 Motion of a Photon

The radial speed squared of the photon is given[5] by $v^2(r) = c^2 A(r)B(r)$, and we have correspondingly:

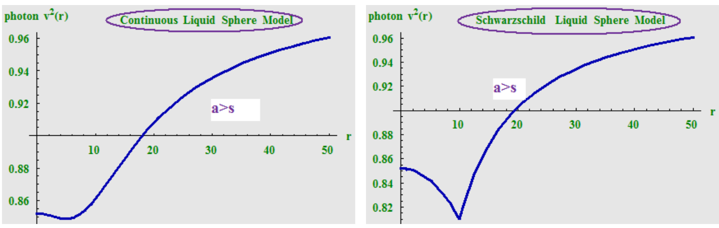
$$v^2(r) = \frac{c^2}{4}\left(3\sqrt{1-\frac{s}{(r^3+a^3)^{1/3}}} - \sqrt{1-\frac{sr^2}{r^3+a^3}}\right)^2\left(1-\frac{sr^2}{r^3+a^3}\right) \qquad (13)$$

The following is the photon's radial acceleration divided by c^2:

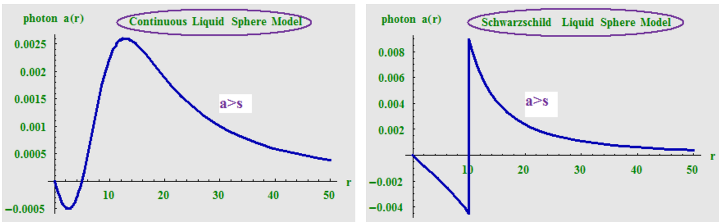
$$\frac{1}{8} r s \left(-\sqrt{1-\frac{r^2 s}{a^3+r^3}}+3\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right) \left(\left(1-\frac{r^2 s}{a^3+r^3}\right)\left(\frac{2a^3-r^3}{(a^3+r^3)^2\sqrt{1-\frac{r^2 s}{a^3+r^3}}}+\frac{3r}{(a^3+r^3)^{4/3}\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}}\right)+\frac{(2a^3-r^3)\left(\sqrt{1-\frac{r^2 s}{a^3+r^3}}-3\sqrt{1-\frac{s}{(a^3+r^3)^{1/3}}}\right)}{(a^3+r^3)^2}\right) \tag{14}$$

Notice, from the above expression, that a radial photon's acceleration vanishes for $r \to 0$.

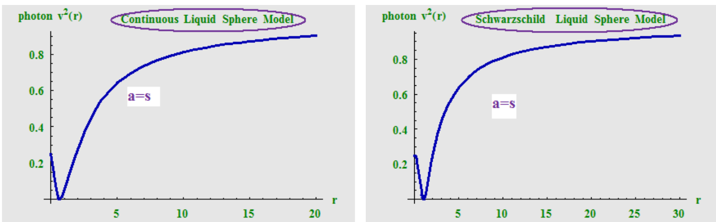
The following is a schematic plot of $v^2(r)$ for the case $a > s$, in our continuous model, compared with the corresponding plot[5] in the Schwarzschild model[5] (the scales are the same for the two cases with $c = 1$, $s = 1$, and $a = 10$):



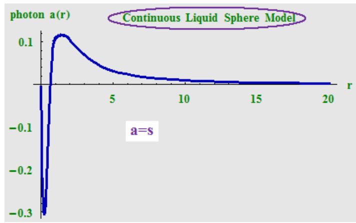
And these give the plots for the acceleration functions $a(r)$:

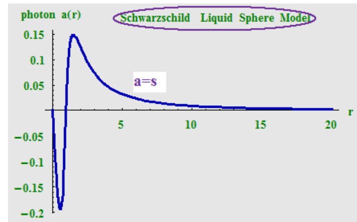


Turning to the case $a = s$, when the radius of the sphere is equal to its Schwarzschild radius, we have for the photon speed squared:



For the photon's acceleration, we have





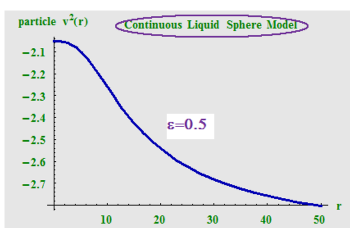
4 Motion of a Massive Particle

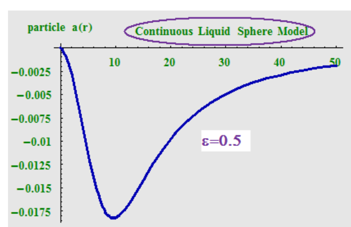
The expression for the speed squared $v^2(r)$ of a massive particle of mass m and energy ε, moving radially, as well as the expression for the acceleration $a(r)$, are given by the general formulae[5], in terms of the metric functions $A(r)$ and $B(r)$:

$$\begin{cases} v^2(r) = c^2 AB \left\{ 1 - \left(\frac{mc^2}{\varepsilon} \right)^2 A \right\} \\ \\ a(r) = \frac{1}{2} \frac{d}{dr} v^2(r) \end{cases} \tag{15}$$

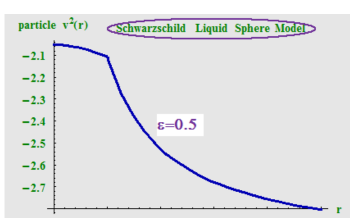
With the forms given for A and B in our continuous model, we can work with the corresponding numeric expressions, and move directly to the schematic plots.

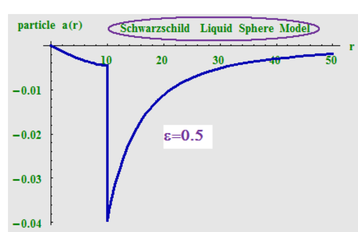
The followings are the schematic plots for particle speed squared $v^2(r)$ and acceleration $a(r)$ that correspond to $\varepsilon = 0.5mc^2$, with scales chosen such as $c = 1$, $s = 1$ and $a = 10$:



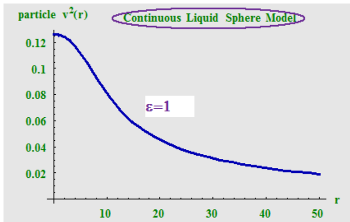


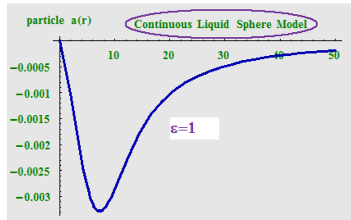
to be compared with their Schwarzschild counterparts[5]:



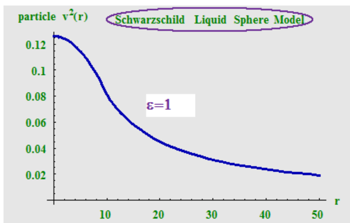


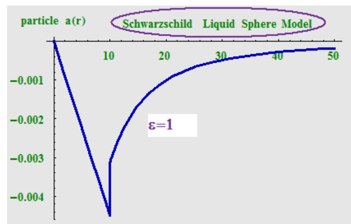
The followings are the schematic plots for particle speed squared $v^2(r)$ and acceleration $a(r)$ that correspond to $\varepsilon = mc^2$, with scales chosen such as $c = 1$, $s = 1$ and $a = 10$:



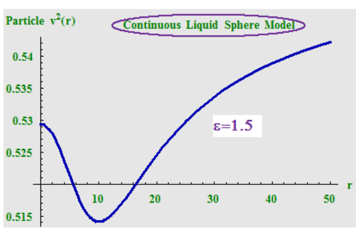


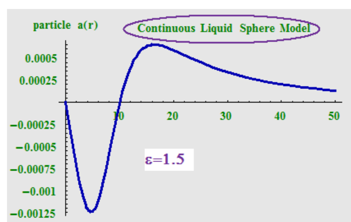
to be compared with their Schwarzschild counterparts[5]:



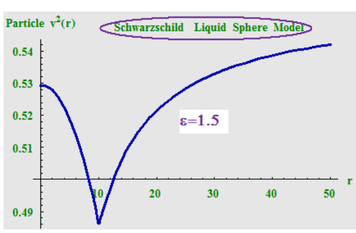


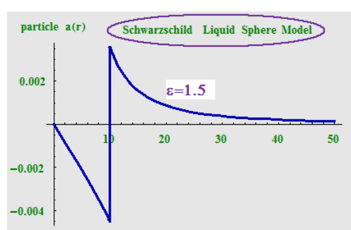
And, the followings are the schematic plots for particle speed squared $v^2(r)$ and acceleration $a(r)$ that correspond to $\varepsilon = 1.5mc^2$, with scales chosen such as $c = 1$, $s = 1$ and $a = 10$:





to be compared with their Schwarzschild counterparts[5]:





The reader can learn a lot by studying carefully all the preceding plots.

5 Discussion

The relativistic metrical model presented in this article offers a continuous counterpart to Schwarzschild's model of a constant-density sphere. Along with continuity, our model brings about a *new effect* that is manifest in the plot of pressure as a function of radial distance. The new effect is that of negative magnitude associated with the angular part of pressure. Whether this effect of negative pressure is a *universal feature* of continuous models in general, and whether it is of any important *physical significance* such as being connected with stellar winds, is the subject of our current investigations. However, we have found that many metrical models with acceptable density distributions do in fact demonstrate the negative pressure effects in the *central part* of the radial range, as well as in the external part (seen here).

On the other hand, our study of the motions of photons, and massive particles with large energy, in the gravity field of the continuous model, does show the effect of *gravitational repulsion*, as we have seen before in connection with singular Schwarzschild sources[6], and with the constant-density Schwarzschild sphere[5].

Other examples of continuous metrical models, that are not related to the Schwarzschild model at all, will be presented in other articles. These models will also help to clarify the question of the *interior gravity* and the repulsive effect, together with the novel effect of negative pressure.

References

[1] K. Schwarzschild, "On the Gravitational Field of a Sphere of Incompressible Liquid, According to Einstein's Theory", 1916, (reproduced in *The Abraham Zelmanov Journal*, **Vol. 1** (2008); http://zelmanov.ptep-online.com/)

[2] Larissa Borissova, "The Gravitational Field of a Condensed Matter Model of the Sun: The Space Breaking Meets the Asteroid Strip", *The Abraham Zelmanov Journal*, **Vol. 2** (2009); http://zelmanov.ptep-online.com/

[3] L. Borissova and D. Rabounski, ***Inside Stars****: A Theory of the Internal Constitution of Stars, and the Sources of Stellar Energy According to General Relativity*, (American Research Press, Rehoboth, New Mexico, USA, 2013); http://ptep-online.com/index files/books.html

[4] Pierre-Marie Robitaille, "The Liquid Metallic Hydrogen Model of the Sun and the Solar Atmosphere", *Progress in Physics*, July, 2013; http://ptep-online.com/

[5] N.S. Baaklini, "Schwarzschild's Metrical Model of a Liquid Sphere", *N.S.B. Letters*, **NSBL-RT-017**

[6] N.S. Baaklini, "Relativistic Motion and Schwarzschild Sources", *N.S.B. Letters*, **NSBL-RT-015**

[7] K. Schwarzschild, "On the Gravitational Field of a Point-Mass According to Einstein's Theory", 1916, (reproduced in *The Abraham Zelmanov Journal*, **Vol. 1** (2008); http://zelmanov.ptep-online.com/)

For Those Who Seek True Comprehension of Fundamental Theoretical Physics