Schwarzschild's Metrical Model of a Liquid Sphere

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Abstract

We study Schwarzschild's metrical model of an incompressible (liquid) sphere of constant density and note the tremendous internal pressures described by the model when applied to a stellar body like the sun. We also study the relativistic radial motion of a photon and a massive particle in the associated gravitational field, with due regard to energy conservation. We note the similarities and the differences between this case and the case of a Schwarzschild singular source with special regard to repulsive effects and penetrability.

1 Introduction

In our study of the relativistic motion of particles in the gravity field of a spherically symmetric Schwarzschild metric[1], we have found that depending on the energy of the particle, there is a repulsive character of the Schwarzschild source, and that motion would cease at the Schwarzschild surface. The latter may be viewed as a reflecting boundary for massive as well as massless (photonic) particles. Whereas a Schwarzschild source may be regarded as an extreme limit of physical gravitational sources, it would be interesting to examine the relativistic motion of particles in the field of a source with some extended mass distribution. Besides his metric solution describing a very compact (a particle) source[2], Schwarzschild had given a metric describing a sphere of constant density (incompressible liquid)[3]. Our purpose in this article is to examine this metric describing a liquid sphere and its possible relevance to astrophysical bodies. Corresponding to an interesting proposal by Robitaille regarding a possible condensed state of the solar constitution based on liquid hydrogen[4], there has been a revival of Schwarzschild's metric for a liquid sphere. We shall verify and use the form of the metric presented by the recent work of Borissova and Rabounski[5],[6].

As we shall see, the Schwarzschild metric solution for a sphere of constant density is a discontinuous function, for it is an idealized system obtained by equating the metric at the surface of a sphere of constant density (essentially a de Sitter space metric) to the metric of empty space outside (the Schwarzschild metric for a central point). In a separate article[7] we propose and examine a corresponding modification that describes a continuous transition from a central spherical concentration of matter to an empty space at large distance. It seems to us that such a modification to the Schwarzschild metric solution would be a more realistic counterpart to the idealistic discontinuous metric. An astrophysical system like a star would better be described by a continuous transition from a higher-density incompressible core to a lower density gaseous atmosphere.

In this article, after introducing the Schwarzschild metric for an incompressible sphere, we shall examine its properties, and study the motion of a radial photon and a massive test particle in the associated gravitational field, making contrast with the corresponding motion in the field of a singular Schwarzschild source[1].

2 General Formalism

Let us consider a general spherical metric whose line element is desccribed by the line element

$$ds^2 = A(r)c^2dt^2 - B^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

Here $A(r)$ and $B(r)$ are two functions of the radial coordinate r. In Schwarzschild's metric for a central point source[2], both functions $A(r)$ and $B(r)$ are equal to the form $(1 - s/r)$ where $s = 2GM/c^2$ is the Schwarzschild's radius (or the Hilbert radius as some authors like to call it). Hence we should remember that in any new metric which is supposed to replace the above for an extended spherical source, the functions $A(r)$ and $B(r)$ must have the form $(1 - s/r)$ as their limit for radial distances that are large compared to the effective size of the spherical source.

Now corresponding to the above metric, we can compute the Ricci tensor $R_{\mu\nu}$, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, then derive the non-vanishing components of the energy-momentum-stress tensor T_μ^ν using Einstein's equations:

$$R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = -\frac{8\pi G}{c^4}T_\mu^\nu \tag{2}$$

The energy density ρc^2 of a gravitational source is given by the time component T_0^0, while other components of T_μ^ν give pressure, momentum, and stress quantities associated with the material source. For the mass density ρ, we have[1]

$$\rho = -\frac{c^2}{8\pi G}\frac{1}{r^2}(-1 + B + rB') \tag{3}$$

where the prime denotes differentiation with respect to the radial coordinate r. For three components of pressure, we obtain

$$P_1 = \frac{c^4}{8\pi G}\frac{1}{r^2}\frac{A(-1+B) + rA'B}{A} \tag{4}$$

$$P_3 = P_2 = \frac{c^4}{32\pi G}\frac{1}{r}\frac{(-rB(A')^2 + 2A^2B' + A(rA'B' + 2B(A' + rA'')))}{A^2} \tag{5}$$

[1]We shall present some time soon an article, *"Metrico: A Mathematica Package for General Relativity"*, describing the programming involved in the writing of a symbolic manipulation package (that, among other things, could take an arbitrary metric, compute all connections and tensor components rather quickly), giving definitions and useful formulae for many practical situations.

Notice that, in general, we have three different components of pressure. In the above case, we have $P_3 = P_2$ due to spherical symmetry. In the very special case of an *ideal fluid*, the three components of pressure are equal. In general, we can define the mean pressure as $P = (P_1 + P_2 + P_3)/3$.

With the above formalism we shall be able to analyze the properties of the spherical metrics that are the subject of this article, and others that are forthcoming. On the other hand, let us give some general formalism regarding the motion of a particle in the field of the general spherical metric given above.

The Lagrangian for a relativistic particle of mass m that corresponds to our general spherical metric is given by

$$\mathcal{L} = -mc^2\sqrt{A(r) - B^{-1}(r)\frac{v^2}{c^2} - r^2\frac{\omega^2}{c^2}} \tag{6}$$

Here v and ω are the radial and the angular velocities, respectively, and we have

$$v = \dot{r} \qquad \omega^2 = \dot{\theta}^2 + \sin^2(\theta)\,\dot{\phi}^2$$

with the dot representing differentiation with respect to time.

We proceed following the same elementary mechanical analysis used to handle particle motion in the field of a Schwarzschild source[1]. The above system is independent of the time coordinate, hence it is *conservative of energy*. However, it depends explicitly on the radial coordinate r and the angle θ, whose associated momenta are not conserved, while the angular momentum associated with ϕ is conserved. The radial and angular momenta are given by

$$p_r = \frac{\partial\mathcal{L}}{\partial v} = \frac{mv}{B\sqrt{A - \frac{v^2+r^2\omega^2 B}{c^2 B}}} \tag{7}$$

$$p_\omega = \frac{\partial\mathcal{L}}{\partial \omega} = \frac{mr^2\omega}{\sqrt{A - \frac{v^2+r^2\omega^2 B}{c^2 B}}} \tag{8}$$

The conserved energy is given by the Hamiltonian:

$$\varepsilon = p_r v + p_\omega \omega - \mathcal{L} = \frac{mc^2 A}{\sqrt{A - \frac{v^2+r^2\omega^2 B}{c^2 B}}} \tag{9}$$

Notice that the minimum value of ε is given by $mc^2\sqrt{A}$ corresponding to $v = \omega = 0$. This minimum value is smaller that mc^2 for all r, provided that the function $A(r)$ approximates to the Schwarzschild counterpart only for very large radial distance.

Solving for the velocity, we have the general relation

$$v^2 + r^2\omega^2 B = c^2 AB\left\{1 - \left(\frac{mc^2}{\varepsilon}\right)^2 A\right\} \tag{10}$$

When dealing strictly with radial motion ($\omega = 0$), we have for the radial velocity

$$v^2 = c^2 AB \left\{ 1 - \left(\frac{mc^2}{\varepsilon} \right)^2 A \right\} \tag{11}$$

The radial acceleration $a = dv/dt$ is given by $(dv/dr)v = (1/2)dv^2/dr$, or

$$a = \frac{c^2}{2} \left\{ \left(1 - 2 \left(\frac{mc^2}{\varepsilon} \right)^2 \right) BA' + A \left(1 - \left(\frac{mc^2}{\varepsilon} \right)^2 A \right) B' \right\} \tag{12}$$

where the prime denotes differentiation with respect to the radial coordinate r.

For strictly orbital motion ($v = 0$), we have

$$r^2 \omega^2 = c^2 A \left\{ 1 - \left(\frac{mc^2}{\varepsilon} \right)^2 A \right\} \tag{13}$$

For a massless particle like the photon with rest mass $m = 0$, we have the respective radial and angular speeds

$$v = c\sqrt{AB} \qquad r\omega = c\sqrt{A} \tag{14}$$

Notice that the radial and the orbital photonic speeds would differ at any radial position, in general, except for the limit $r \to \infty$, where the speed is c. For photonic radial acceleration, we have the symmetrical expression

$$a = \frac{dv}{dt} = \frac{c^2}{2}(A'B + AB') = \frac{c^2}{2} \frac{d}{dr}(AB) \tag{15}$$

In the following sections, after introducing the Schwarzschild model for a liquid sphere we shall remark on the astrophysical properties of such model, with reference to a stellar body like the sun. We shall consider the radial motion of a particle in the field of the associated metric, utilizing the above formalism, and giving due comments.

3 The Schwarzschild Model of a Liquid Sphere

For the Schwarzschild model of a liquid (incompressible) sphere, we have the associated spherical metric with the functions[5],[6]

$$\begin{cases} A(r) = \frac{1}{4} \left(3\sqrt{1 - \frac{s}{a}} - \sqrt{1 - \frac{sr^2}{a^3}} \right)^2 \\ \\ B(r) = \left(1 - \frac{sr^2}{a^3} \right) \end{cases} \qquad r \leq a \tag{16}$$

This is strictly for the metric inside the sphere of radius a, and where $s = 2GM/c^2$ is the Schwarzschild radius associated with the sphere of mass M. However, outside the sphere, we have the Schwarzschild metric for empty space, with

$$\begin{cases} A(r) = \left(1 - \frac{s}{r}\right) \\ B(r) = \left(1 - \frac{s}{r}\right) \end{cases} \qquad r \geq a \tag{17}$$

Notice that the above interior and exterior parts of the metric would match at the boundary $r = a$.

Now using the formalism of the preceding section, we proceed to compute the density and the pressure components of the energy-momentum-stress tensor. Other components are vanishing.

The mass density ρ given in the preceding section in terms of $A(r)$ and $B(r)$ would take on the following constant value inside the sphere

$$\rho = \frac{3c^2}{8\pi G}\frac{s}{a^3} \qquad r \leq a \tag{18}$$

while it is zero outside. Now with $s = 2GM/c^2$, we obtain

$$\rho = \frac{3M}{4\pi a^3} \tag{19}$$

This is just the usual mass density for a hard sphere of mass M and radius a. Taking $M = 1.98892 \times 10^{30}$ kilograms as the solar mass, and $a = 6.96 \times 10^8$ m, the solar radius, we obtain $\rho = 1408.32$ kg/m^3 or 1.40832 gm/cm^3 as the average solar density. Notice that this value is to be taken to be *constant throughout the solar body* in a liquid spherical model of the sun[4].

Turning to pressure, we utilize the formalism of the preceding section which gives the values of P_1, P_2, and P_3 in terms of the functions $A(r)$ and $B(r)$, and verify that the three pressure components are equal (ideal fluid) inside the sphere, with the common value

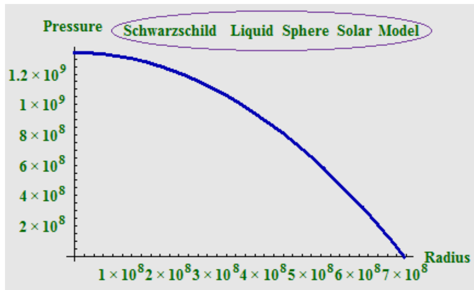
$$P = \frac{3c^4}{8\pi G}\left(\frac{s}{a^3}\right)\left\{\frac{\sqrt{1-\frac{s}{a}} - \sqrt{1-\frac{sr^2}{a^3}}}{-3\sqrt{1-\frac{s}{a}} + \sqrt{1-\frac{sr^2}{a^3}}}\right\} \qquad r \leq a \tag{20}$$

Moreover, it is clear that the pressure outside the sphere is zero.

Now using $s = 2GM/c^2$, then using the values $c = 2.99792458 \times 10^8$ m/sec for the speed of light, $G = 6.67259 \times 10^{-11}$ for the Newtonian constant in MKS units, together with the values of solar mass and radius given earlier, and dividing by 10^5 to get the value of pressure in *atmosphere*, we obtain the numeric expression for the pressure inside the sphere as a function of radial distance r:

$$\frac{1.26573 \times 10^{15}(0.999998 - \sqrt{1 - 8.75937 \times 10^{-24} r^2})}{-2.99999 + \sqrt{1 - 8.75937 \times 10^{-24} r^2}} \tag{21}$$

This expression gives zero for the value of the pressure when r is equal to the solar radius. However, for $r = 0$, at the center of the sun, the value of pressure comes out equal to 1.34269×10^9 atmosphere. The following is a plot of solar pressure as a function of radial distance, *if the sun is considered to be a Schwarzschild liquid sphere* of constant density:



With these tremendous values of pressures (hundreds of millions to billions of atmospheres) in most of the central part of the solar body, the question is, *would the liquid hydrogen model proposed by Robitaille be able to sustain these pressures*?[4]

Let us turn now for a consideration of the relativistic radial motion of a particle in the field of a Schwarzschild liquid sphere. Our presentation should be compared with the one given for the motion in the field of the singular Schwarzschild metric of central point source.[1]

3.1 Motion of a Photon

We shall begin with the radial motion of a massless photon. Whereas the radial speed squared of the photon is given by $v^2(r) = c^2 A(r) B(r)$, we have inside and outside the sphere, respectively:

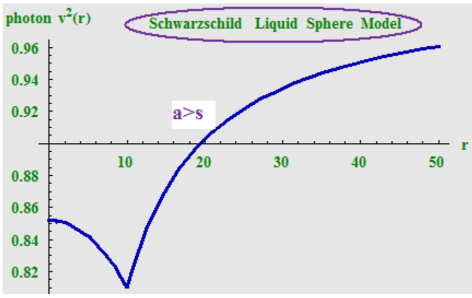
$$v^2(r) = \begin{cases} \frac{c^2}{4}\left(3\sqrt{1-\frac{s}{a}} - \sqrt{1-\frac{sr^2}{a^3}}\right)^2 \left(1-\frac{sr^2}{a^3}\right) & r \le a \\ \left(1-\frac{s}{r}\right)^2 c^2 & r \ge a \end{cases} \tag{22}$$

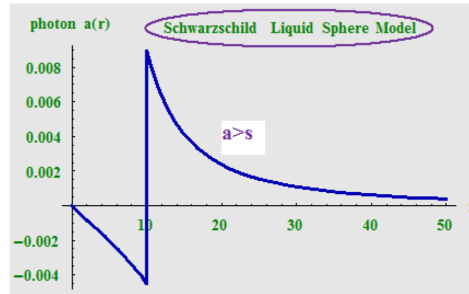
It is clear, in order to have real speed inside, we must have $s \le a$ and $sr^2 \le a^3$. This is satisfied inside the sphere[2] as long as $s \le a$. The following is the photon's radial acceleration:

$$a(r) = \begin{cases} \frac{rsc^2}{4a^6}\left\{9a^2 s + 2r^2 s + a^3\left(-11 + 9\sqrt{1-\frac{s}{a}}\sqrt{1-\frac{sr^2}{a^3}}\right)\right\} & r \le a \\ \frac{(r-s)sc^2}{r^3} & r \ge a \end{cases} \tag{23}$$

For the case $a > s$, these are schematic plots of $v^2(r)$ and $a(r)$, with the scales corresponding to, $c = 1$, $s = 1$ and $a = 10$:

[2] It should be clear that the singularity in the interior part of the metric (the vanishing of $B(r)$ and the blowing up of the radial part) when $sr^2 = a^3$ is of no consequence in the present Schwarzschild model since it lies outside the spherical surface. However, the observation[5],[6] that the value of $r = a^3/s$ corresponding to the sun does occur at the position of the asteroid belt is interesting.





We can see from the plots how an incoming photon's speed is decreasing (positive acceleration or *repulsion*) outside the sphere, and how the motion changes discontinuously inside the sphere, with the photon's speed increasing (negative acceleration or *attraction*) to reach a speed of $(3\sqrt{1-s/a}-1)c/2$. For a solar liquid sphere, the maximum speed of the photon reached at the center is $0.999997c$.

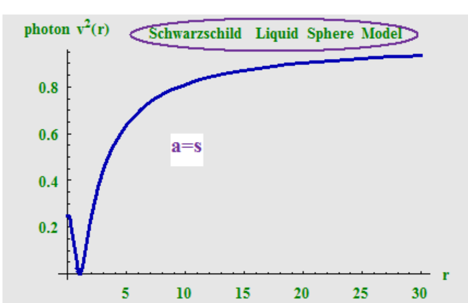
Turning to the case $a = s$, when *the radius of the sphere is equal to its Schwarzschild radius*, we have for the photon speed squared:

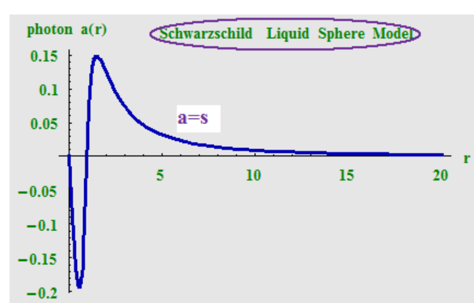
$$v^2(r) = \begin{cases} \frac{c^2}{4}\left(1-\frac{r^2}{s^2}\right)^2 & r \le s \\ \left(1-\frac{s}{r}\right)^2 c^2 & r \ge s \end{cases} \tag{24}$$

For the photon's acceleration, we have

$$a(r) = \begin{cases} \frac{r(r^2-s^2)c^2}{2s^4} & r \le s \\ \frac{(r-s)sc^2}{r^3} & r \ge s \end{cases} \tag{25}$$

Notice that both the speed and the acceleration of the photon are zero at the spherical surface. This situation is *exactly the same* as for the case of a point-particle Schwarzschild source[1]. However, whereas there couldn't be a penetration of the photon in the singular case of a point-particle Schwarzschild source (since the metric is valid only for $r > s$), in the present case of an extended Schwarzschild sphere penetration is allowed. Notice that for $r \to 0$ the speed of the photon becomes $c/2$, while the acceleration goes back to 0. These are the corresponding schematic plots:





It is clear that the curves for the $a = s$ case becomes much smoother.

3.2 Motion of a Massive Particle

From our general formalism with metric functions $A(r)$ and $B(r)$, we have for the radial speed squared of a particle, of mass m and energy ε, the expression:

$$v^2 = AB\left\{1 - \left(\frac{mc^2}{\varepsilon}\right)^2 A\right\}c^2 \tag{26}$$

and for the radial acceleration:

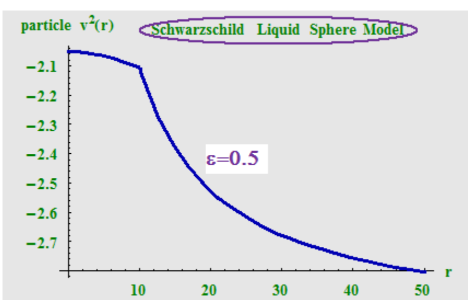
$$a = \frac{c^2}{2}\left\{\left(1 - 2\left(\frac{mc^2}{\varepsilon}\right)^2\right)BA' + A\left(1 - \left(\frac{mc^2}{\varepsilon}\right)^2\right)B'\right\} \tag{27}$$

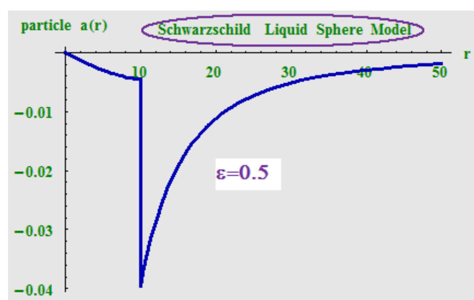
Now using the metric functions of the Schwarzschild model of a liquid sphere,

$$\begin{cases} \begin{cases} A(r) = \frac{1}{4}\left(3\sqrt{1 - s/a} - \sqrt{1 - sr^2/a}\right)^2 \\ B(r) = (1 - sr^2/a^3) \end{cases} & r \leq a \\ \\ \begin{cases} A(r) = (1 - s/r) \\ B(r) = (1 - s/r) \end{cases} & r \geq a \end{cases} \tag{28}$$

and choosing a unit value of mc^2, we shall present schematic plots corresponding to three values of ε, $\{0.5, 1, 1.5\}$.

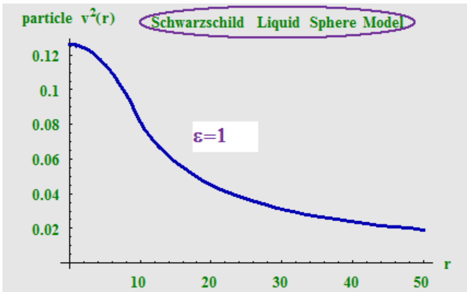
The followings are the schematic plots for particle speed squared $v^2(r)$ and acceleration $a(r)$ that correspond to $\varepsilon = 0.5$, with scales chosen such as $c = 1$, $s = 1$ and $a = 10$:

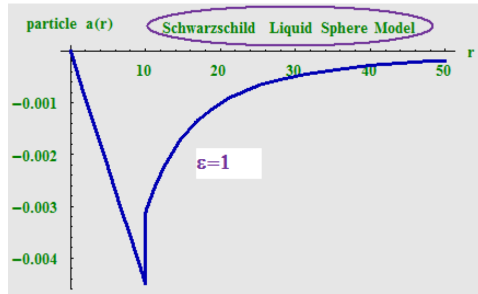




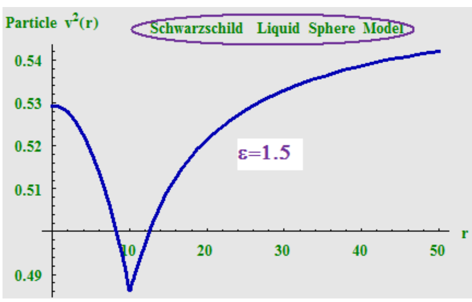
Hence we see how an incoming massive particle with energy $\varepsilon = 0.5mc^2$ has an increasing speed outside the sphere (negative acceleration or *attraction*), then there is a decrease in the rate of speed increase after penetrating the surface of the sphere. The negative acceleration (or the *attraction*) vanishes for $r \to 0$.

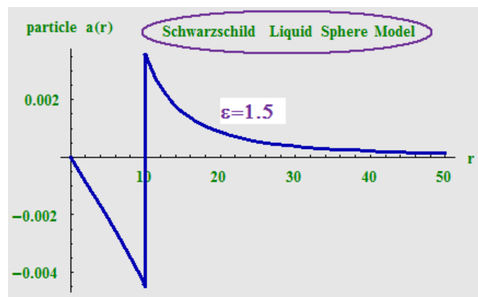
The followings are the schematic plots for particle speed squared $v^2(r)$ and acceleration $a(r)$ that correspond to $\varepsilon = 1$, with scales chosen such as $c = 1$, $s = 1$ and $a = 10$:





And, the followings are the schematic plots for particle speed squared $v^2(r)$ and acceleration $a(r)$ that correspond to $\varepsilon = 1.5$, with scales chosen such as $c = 1$, $s = 1$ and $a = 10$:





Notice that the case for which $\varepsilon = 1.5mc^2$ shows that such an incoming massive particle has a decreasing speed outside the sphere (positive acceleration or *repulsion*) then, as it penetrates the surface of the sphere, its inward speed starts increasing (negative acceleration or *attraction*). The maximum value reached at $r = 0$ is less than c, and can easily be obtained from the given formula.

4 Discussion

Our foregoing study of Schwarzschild's metrical model of a constant-density liquid sphere shows that, taken as an *idealistic liquid model of a star*, could only be possible if the underlying liquid could sustain the tremendous pressures (hundreds of millions to billions of atmospheres) that govern most of the central parts of the spherical body. A more realistic system would have to provide a continuous decrease of density as we go away from the center towards the external gaseous layers and beyond. We shall return to more realistic models in other articles. In particular, we shall present a continuous counterpart of Schwarzschild's metrical model which interpolates smoothly between the internal and the external parts.[7]

On the other hand, our study of the relativistic motion of a photon or a massive particle shows some similarities, and some differences, from the motion in the field of the singular Schwarzschild metric of a central point. The similarities pertain to the presence of a *repulsive effect* of the source in the face of an approaching particle, which effect also depends on the energy of the incoming particle. In general, *a photon or* *a highly energetic massive particle are faced with repulsion.* This relativistic repulsive effect seems to be a general property of gravitational sources which *must be exploited in several astrophysical situations.*

Of course, the main difference between the relativistic motion of a photon or a massive particle in the field of a Schwarzschild liquid sphere, and that in the field of a Schwarzschild singular source, concerns the *impenetrability* of the latter at the singular surface and the *penetrability* of the liquid sphere. This picture will be examined further in our forthcoming studies of extended bodies with *continuous* profiles.

References

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