# On the Complete Elliptic Integrals and Babylonian Identity I: The $\frac{1}{\pi}$ Formulaes Involving Gamma Functions and Summation

Edigles Guedes

edigles.guedes@gmail.com

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*And I will make an everlasting covenant with them, that I will not turn away from them, to do them good; but I will put my fear in their hearts, that they shall not depart from me.* — Jeremiah 32:40.

Abstract. I evaluate the constant $\frac{1}{\pi}$ using the Babylonian identity and complete elliptic integral of first kind. This resulted in two representations in terms of the Euler's gamma functions and summations.

## 1. Introduction

By means of the complete elliptic integral of the first kind and Babylonian identity, I demonstrated the identities following, among others:

$$\frac{1}{\pi}=\frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi}=4\sqrt{2}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2\Gamma\left(\frac{3}{4}\right)}.$$

## 2. Lemmas

Lemma 1. For $a$ and $b$ any number, then

(1) $$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}$$

and

(2) $$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=2\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}.$$

*Proof.* I know the Babylonian identity [1, page 119]

(3) $$ab=\frac{1}{4}[(a+b)^2-(a-b)^2].$$

Make the following algebraic manipulation in (3)

$$ab=\left(\frac{a+b}{2}\right)^2\left[1-\left(\frac{a-b}{a+b}\right)^2\right],$$

hence,

$$a^{\frac{1}{2}}b^{\frac{1}{2}}=\frac{a+b}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}\Leftrightarrow\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2},$$

and inverting both members, I have

$$a^{-\frac{1}{2}}b^{-\frac{1}{2}}=\frac{2}{a+b}\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}\Leftrightarrow\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=2\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}.\square$$

Lemma 2. For $a$ and $b$ any number, then

(4) $$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2\,(2k-1)}\left(\frac{a-b}{a+b}\right)^{2k}$$

and

(5) $$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=2\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}\left(\frac{a-b}{a+b}\right)^{2k}.$$

*Proof.* I calculate

(6) $$\sqrt{1-z^2}=-\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2\,(2k-1)}z^{2k}$$

and

(7) $$\frac{1}{\sqrt{1-z^2}}=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}z^{2k}$$

Take $z=\frac{a-b}{a+b}$ in (6) and (7), then replace in (1) and (2) respectively, completing the proof.□

## 3. THEOREMS

Theorem 1. *I have*

$$K(k)=\frac{\sqrt{2}}{2}\int_0^{\pi}\frac{1}{\sqrt{2-k^2(1+\cos t)}}\,dt,$$

*where* $K(k)$ *is the complete elliptic integral of first kind.*

*Proof.* Putting $\frac{a-b}{a+b}=t$ in (5), I encounter

(8) $$\frac{1}{\sqrt{1-t^2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}t^{2n}.$$

Multiplying (8) by $\frac{1}{\sqrt{1-k^2t^2}}$ and integrating from 0 at 1 in $t$, I find

$$\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \int_0^1 \frac{t^{2n}}{\sqrt{1-k^2t^2}}dt \Leftrightarrow$$

(9) $$K(k) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2}\, {}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; k^2\right).$$

On the one hand, in [2, page 21], I have

(10) $${}_2F_1(a,b;c;z) = \frac{2^{1-c}\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\pi} \frac{(\sin t)^{2b-1}(1+\cos t)^{c-2b}}{\left(1-\frac{1}{2}z+\frac{1}{2}z\cos t\right)^a}dt,$$

for $\Re(c) > \Re(b) > 0$. Substituting (10) in (9), I encounter

$$K(k) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{3n}(2n+1)n!^2} \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^{\pi} \frac{(\sin t)^{2n}(1+\cos t)^{-n+\frac{1}{2}}}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}}dt =$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi} \sqrt{\frac{1+\cos t}{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}} \sum_{n=0}^{\infty} \frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{3n}(2n+1)n!^2\,\Gamma\left(n+\frac{1}{2}\right)} (\sin t)^{2n}(1+\cos t)^{-n}dt =$$

$$= \frac{1}{2} \int_0^{\pi} \frac{1}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}}dt$$

$$= \frac{\sqrt{2}}{2} \int_0^{\pi} \frac{1}{\sqrt{2-k^2(1+\cos t)}}dt. \square$$

Theorem 2. *I have*

$$k'K(k) = K\left(-i\frac{k}{k'}\right).$$

*Proof.* I leave to the reader. □

Theorem 3. *For* $0 < k < 1$, *then*

$$\frac{K(k)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(\frac{2n+3}{2}\right)}{\left(\frac{3}{2}\right)_n} \frac{k^{2n}}{n!^2},$$

*where* $K(k)$ *is the complete elliptic integral of first kind.*

*Proof.* I consider

$$K(k) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2}\, {}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; k^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \left(n+\frac{1}{2}\right)_r}{\left(n+\frac{3}{2}\right)_r r!} k^{2r}$$

$$= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)_r \left( \sum_{n=0}^{\infty} \frac{(2n)! \left(n + \frac{1}{2}\right)_r}{2^{2n}(2n+1)n!^2 \left(n + \frac{3}{2}\right)_r} \right) \frac{k^{2r}}{r!}$$

$$= \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r \Gamma(r+1)} \frac{k^{2r}}{r!} = \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r} \frac{k^{2r}}{r!^2}.$$

Multiply both sides by $\frac{1}{\sqrt{\pi}}$ and let $r \to n$, so the result follows. □

**Corollary 1.** *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)}$$

*and*

$$\frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}.$$

*Proof.* Let $k = \frac{\sqrt{2}}{2}$ in Theorem 3

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r (r!)^2} \left(\frac{1}{2}\right)^r \tag{11}$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)}.$$

On the other hand, in [3], I find

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}. \tag{12}$$

I substitute (12) into (11) and obtain

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)} \Rightarrow \frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}. \square$$

**Corollary 2.** *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

*and*

$$\frac{1}{\pi} = 4\sqrt{2}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}.$$

*Proof.* From Theorem 2 and $k = \frac{\sqrt{2}}{2}$, I find

(13)
$$\frac{\sqrt{2}}{2} K\left(\frac{\sqrt{2}}{2}\right) = K\left(-i\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) \Rightarrow K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}\, K(-i)$$

Using the Theorem (3), I discover that

(14)
$$K(-i) = \sqrt{\pi}\sum_{r=0}^{\infty}(-1)^r \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r (r!)^2}$$

$$= \sqrt{\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

I set (14) in (13)

(15)
$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

I put (12) into (15), and have

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = \sqrt{2\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \Rightarrow \frac{1}{\pi} = 4\sqrt{2}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}. \square$$

Theorem 4. *For* $0 < k < 1$, *then*

$$K(k) = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{2n}(2n+1)n!^3}k^{2n},$$

*where* $K(k)$ *is the complete elliptic integral of first kind.*

*Proof.* I put $\frac{a-b}{a+b} = t$ in (5) and encounter

(16)
$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}t^{2n}.$$

Multiplying (16) by $\frac{1}{\sqrt{1-k^2t^2}}$ and integrating from 0 at 1 in $t$, I find

(17)
$$\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\int_0^1 \frac{t^{2n}}{\sqrt{1-k^2t^2}}dt.$$

Let $t \to kt$ in (16)

(18) $$\frac{1}{\sqrt{1-k^2t^2}}=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}k^{2m}t^{2m}.$$

I put (18) in (17)

(19) $$\int_0^1\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\int_0^1 t^{2n}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}k^{2m}t^{2m}\,dt$$

$$=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}\left[\int_0^1 t^{2(m+n)}\,dt\right]k^{2m}$$

$$=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}\frac{k^{2m}}{2m+2n+1}$$

$$=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}\left[\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(2m+2n+1)n!^2}\right]k^{2m}$$

$$=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{2m+3}{2}\right)}{(2m+1)\Gamma(m+1)}\right]k^{2m}$$

$$=\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,\Gamma\left(\frac{2m+3}{2}\right)}{2^{2m}(2m+1)m!^3}k^{2m}.$$

Let $m\to n$, this concludes the proof. □

**Corollary 3.** *I have*

$$K(k)=\sqrt{\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2\Gamma\left(n+\frac{3}{2}\right)}{(1)_n\left(\frac{3}{2}\right)_n n!}k^{2n},$$

*where $K(k)$ is the complete elliptic integral of first kind.*

*Proof.* I know [5, page 884] that

(20) $$\Gamma(z)\overset{\text{def}}{=}\int_0^{\infty}e^{-t}t^{z-1}\,dt,$$

for $\Re(z)>0$. I substitute (20) in Theorem 4

$$K(k)=\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2n)!}{2^{2n}(2n+1)n!^3}\int_0^{\infty}e^{-t}t^{n+\frac{3}{2}}\,dtk^{2m}$$

$$=\sqrt{\pi}\int_0^{\infty}e^{-t}t^{\frac{3}{2}}\sum_{m=0}^{\infty}\frac{(2n)!}{2^{2n}(2n+1)n!^3}(tk^2)^n\,dt$$

$$=\sqrt{\pi}\int_0^{\infty}e^{-t}t^{\frac{3}{2}}\,{}_2F_2\left(\frac{1}{2},\frac{1}{2};1,\frac{3}{2};k^2t\right)dt$$

$$= \sqrt{\pi} \int_0^\infty e^{-t} \sqrt{t} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} (tk^2)^n \, dt$$

$$= \sqrt{\pi} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} \left( \int_0^\infty e^{-t} \, t^{n+\frac{3}{2}} dt \right) k^{2n}$$

$$= \sqrt{\pi} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n}. \square$$

REFERENCES


[1] Havil, Julian, *Gamma: Exploring the Euler's Constant*, Princeton University Press, 2003.

[2] Slater, Lucy Joan, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.

[3] http://en.wikipedia.org/wiki/Elliptic_integral, available in July 02, 2012.

[4] Armitage, J. V. and Eberlein, W. F., *Elliptic Functions*, Cambridge University Press, 2006.

[5] Gradshteyn, I. S. and Ryzhik, I. M., Table of Integrals, Series and Products, Academic Press, 2000.