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Relativistic Motion and Schwarzschild Sources

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Abstract

We give an elementary analysis of the classical motion of a particle in the spherically symmetric gravitational field of a Schwarzschild source, with due regard to energy conservation. We observe that whereas a massive particle at large distances could be attracted towards the central source, it would however encounter repulsion as it comes close to the Schwarzschild surface. We also note that there is a limited energy range for which the radial motion is ruled by attraction. An attracted incoming particle reaches a maximum speed at a specific distance greater than the Schwarzschild radius, before decelerating to zero, then bouncing back. Like the radial motion, the orbital motion around a Schwarzschild source would stop at the Schwarzschild radius. A massless photon would always be repelled, with its speed decreasing as it approaches the source, ultimately getting reflected at the Schwarzschild surface. The timing problem associated with surface singularity is resolved by regarding particles as Schwarzschild sources themselves. We depict a picture of ideal Schwarzschild sources as mutually repulsive bubbles endowed with reflecting surfaces.

1 Introduction

It is well-known that the speed of light in Einstein's special theory of relativity, or in flat spacetime, is a universal constant. The general theory of relativity, besides providing a geometrical description of gravitation, it also provides a framework for the variation of the speed of light either as a function of space or of time. In the following, we shall begin by reviewing the simple motion of a particle in special relativity, that is, in the absence of a gravitational field. This demonstrates the well-known constraint on particle speeds, with the speed of light being the upper bound. It also shows the constancy of the speed of the photon.

The Lagrangian of a relativistic particle of mass m in a flat Minkowskian metric is:

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} \tag{1}$$

Here c is a universal constant speed, and $\vec{v} = d\vec{r}/dt$ represents the three components of Cartesian velocity, $v^2 = \vec{v}\cdot\vec{v}$, and $\vec{r}(t)$ is the position vector as a function of time. The above Lagrangian does not depend explicitly on the time variable, hence it is conservative of energy. Also, the Lagrangian does not depend explicitly on the coordinate variables, hence it is conservative of momentum, as well. The momentum is given by:

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{2}$$

The total energy ε is given by the Hamiltonian:

$$\varepsilon = \vec{p}\cdot\vec{v} - \mathcal{L} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3}$$

Notice from the above expressions that an upper bound on speeds is given by the constant c. Notice as well, that the minimum value of p is 0, and the minimum value of ε is mc^2 (the rest energy), corresponding to $\vec{v} = 0$. And we have the relativistic equation relating energy and momentum:

$$\varepsilon^2 = p^2c^2 + m^2c^4 \tag{4}$$

We can solve for particle speed in terms of either the momentum or the energy:

$$\begin{cases} v = \frac{c}{\sqrt{1+(\frac{mc}{p})^2}} \\ \\ v = c\sqrt{1-(\frac{mc^2}{\varepsilon})^2} \end{cases} \tag{5}$$

Notice that corresponding to the the range of values taken on by momentum (0 to ∞) and energy (mc^2 to ∞), the values of speed are real, between 0 and c. Both energy and momentum being conserved, the motion of the particle is that of constant velocity. There is is no acceleration.

For a massless particle, like the photon, the above formulae between energy, momentum, and speed, become (with $m = 0$): $\varepsilon = pc$ and $v = c$. Hence we have the important result of special relativity (in the absence of gravitational fields): *The speed of the photon* (or any massless particle) *is given by the universal constant c*. This result concerning the motion of the photon could easily have been deduced by requiring that the associated line element should vanish:

$$c^2dt^2 - dr^2 = 0 \qquad \Rightarrow \qquad v = \frac{dr}{dt} = \pm c \tag{6}$$

In the following developments, we shall see how the above special relativistic results should be modified in a metric describing a spherically symmetric gravitational field which depends on radial coordinate.

2 Particle Mechanics in a Schwarzschild Field

A spherically symmetric metric solution of Einstein's equations of general relativity is given by the line element

$$ds^2 = \left(1-\frac{s}{r}\right)c^2dt^2 - \left(1-\frac{s}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{7}$$

Here $s = 2GM/c^2$ is the socalled Schwarzschild, or gravitational, radius of the source of mass M, G is the Newtonian constant associated with the force of gravity, r is the radial distance, θ and ϕ are the spherical angles, t is time. The form of the above metric solution was first obtained by Schwarzschild.[1] However, Schwarzschild's solution is given in terms of a variable $R = (r^3 + s^3)^{1/3}$ instead of r. We are told that the above form is given by Hilbert.[2] It is clear that Schwarzschild's insistence on the use of his variable R was just to emphasize the fact that the above metric solution is only valid for $r > s$ (his $r = 0$ corresponds to our $r = s$), where the source density and other components of the energy-momentum tensor must vanish. For $r \leq s$, the theory breaks down, for we must know the nature of the matter distribution in order to complete the solution down to the origin. In fact it is easier to work directly with the above form of solution, with the natural radial variable r, provided that we should remember that it is only valid for $r > s$.

Correspondingly, the Lagrangian of a relativistic particle of mass m in a spherical metric produced by a particle of mass M at the origin is:

$$\mathcal{L} = -mc^2\sqrt{(1-\frac{s}{r}) - (1-\frac{s}{r})^{-1}\frac{v^2}{c^2} - r^2\frac{\omega^2}{c^2}} \tag{8}$$

where v and ω are the radial and angular speeds respectively. More explicitly, we have

$$v = \dot{r} \qquad \omega^2 = \dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2 \tag{9}$$

where the dot represents differentiation with respect to t.

The above system is independent of the time coordinate, hence it is conservative of energy. However, it depends explicitly on the radial coordinate r and the angle θ, whose associated momenta are not conserved, while the angular momentum associated with ϕ is conserved. The radial and angular momenta are given by:

$$p_r = \frac{\partial \mathcal{L}}{\partial v} = \frac{mv(1-\frac{s}{r})^{-1}}{\sqrt{(1-\frac{s}{r}) - (1-\frac{s}{r})^{-1}\frac{v^2}{c^2} - r^2\frac{\omega^2}{c^2}}} \tag{10}$$

$$p_\omega = \frac{\partial \mathcal{L}}{\partial \omega} = \frac{mr^2\omega}{\sqrt{(1-\frac{s}{r}) - (1-\frac{s}{r})^{-1}\frac{v^2}{c^2} - r^2\frac{\omega^2}{c^2}}} \tag{11}$$

The conserved energy ε is given by the Hamiltonian:

$$\varepsilon = p_r v + p_\omega \omega - \mathcal{L} = \frac{mc^2(1-\frac{s}{r})}{\sqrt{(1-\frac{s}{r}) - (1-\frac{s}{r})^{-1}\frac{v^2}{c^2} - r^2\frac{\omega^2}{c^2}}} \tag{12}$$

Notice that the minimum value of ε for all $r > s$ is given by $mc^2\sqrt{1-s/r}$ corresponding to $v = \omega = 0$. This minimum value is *smaller* than mc^2 for all $r > s$. The value mc^2 is the rest energy of the particle at infinite radial distance.

3 Radial Motion

Let us consider radial motion corresponding to $\omega = 0$, where we have

$$\varepsilon = \frac{mc^2\sqrt{1-\frac{s}{r}}}{\sqrt{1-(1-\frac{s}{r})^{-2}\frac{v^2}{c^2}}} \tag{13}$$

Solving this equation for the radial velocity, we obtain:

$$v = \pm c(1-\frac{s}{r})\sqrt{1-(\frac{mc^2}{\varepsilon})^2(1-\frac{s}{r})} \tag{14}$$

Here, the $\pm$ sign corresponds to radially *outgoing* $(+)$ or *incoming* $(-)$ particle. Notice that the radial velocity is zero for $r = s$, while for infinite radial distance, it is given by the special relativistic value $\pm c\sqrt{1-(mc^2/\varepsilon)^2}$.

That the radial velocity should tend to zero as $r \to s$, even if the particle was accelerating in free fall from large distances, indicates that there is pronounced *gravitational repulsion* near that radius. The radial acceleration is given by:

$$a_r = \frac{dv}{dr}v = \frac{sc^2}{r^2}(1-\frac{s}{r})\left\{1-\left(\frac{mc^2}{\varepsilon}\right)^2\frac{3}{2}(1-\frac{s}{r})\right\} \tag{15}$$

Notice that as $r \to s$, to first order in $(1-s/r)$, the acceleration becomes

$$a_r \approx \frac{sc^2}{r^2}\left(1-\frac{s}{r}\right)$$

which is positive for $r > s$, indicating *repulsion*. Notice as well, that while the energy satisfies

$$\varepsilon > mc^2\sqrt{\frac{3}{2}(1-\frac{s}{r})} \qquad r > s \tag{16}$$

the radial acceleration a_r is always positive, which means *repulsion*. Only in the range

$$mc^2\sqrt{1-\frac{s}{r}} < \varepsilon < mc^2\sqrt{\frac{3}{2}(1-\frac{s}{r})} \tag{17}$$

we do have negative radial acceleration or *attraction*.

Starting with an incoming particle having a minimum value of energy $\varepsilon = mc^2\sqrt{1-s/r}$, the particle's speed *increases to a maximum value before decreasing to zero* at the Schwarzschild radius $s = 2GM/c^2$. In fact, the maximum of speed occurs at radial distance

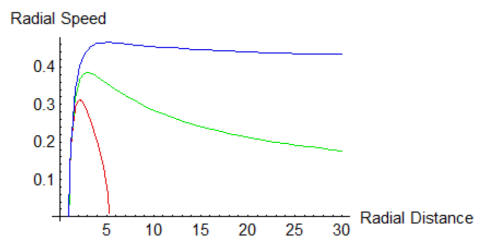
$$r = \frac{s}{1-\frac{2}{3}(\frac{\varepsilon}{mc^2})^2} \tag{18}$$

And the corresponding value of the maximum speed is

$$v_{\max} = \frac{2c}{3\sqrt{3}}\left(\frac{\varepsilon}{mc^2}\right)^2 \tag{19}$$

Hence for a particle *starting from rest at infinity*, with energy $\varepsilon = mc^2$, the maximum of speed achieved is $(2c/3\sqrt{3})$, and this occurs at radial distance $r = 3s$, *three times the Schwarzschild radius.*

The following is a plot of the radial speed against radial distance for three values of the ratio $\varepsilon/mc^2 = \{0.9, 1, 1.1\}$, with three corresponding colors (red, green, blue), the radial scale is given in units of the Schwarzschild distance s, and the speed scale in units of c:

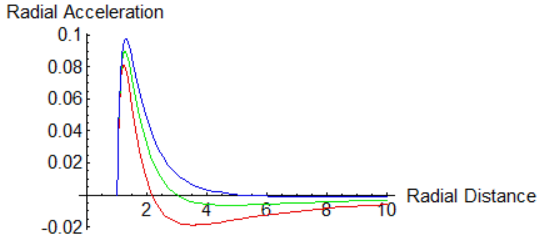


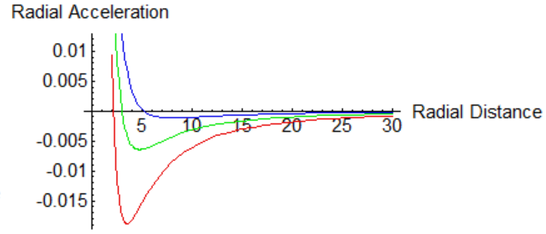
For the lowest value $\varepsilon/mc^2 = 0.9$ (red color), we see that the motion could exist only in a limited radial range. The lower value is the Schwarzschild radius $r = s$, and the upper value ($r \sim 5.26s$) corresponds to the solution of the equation

$$1 - (\frac{mc^2}{\varepsilon})^2(1 - \frac{s}{r}) = 0 \tag{20}$$

A particle could oscillate between these two radii. This case corresponds to a particle starting to fall from some finite distance where its energy is less than mc^2. When it reaches the Schwarzschild surface it gets *reflected*, and the motion continues in an oscillatory manner. For the other two values $\varepsilon/mc^2 = 1, 1.1$, however, we see that the motion has no upper radial limit. We also see that the higher the energy, the higher is the speed maximum, and the location of the maximum gets further away from the center. For the intermediate value, corresponding to a particle falling from rest at infinity, the location of the maximum is at *three times* the Schwarzschild radius.

The followings are two corresponding plots of radial acceleration in two ranges of radial distance, the radial scale being in units of the Schwarzschild distance s, and the acceleration scale in units of sc^2:





It is clearly seen, for each energy value, how the radial acceleration is negative at large distances (attraction), then becoming zero at centain values for $r > s$, becoming positive (repulsion), getting to a maximum value, then descending to zero as $r \to s$.

4 Orbital Motion

Let us consider orbital motion corresponding to the radial speed $v = 0$. The energy is given by

$$\varepsilon = \frac{mc^2(1 - \frac{s}{r})}{\sqrt{(1 - \frac{s}{r}) - r^2 \frac{\omega^2}{c^2}}} \tag{21}$$

where $r\omega$ is the orbital speed. Solving for $r\omega$, we have:

$$r\omega = c\sqrt{1 - \frac{s}{r}}\sqrt{1 - (\frac{mc^2}{\varepsilon})^2(1 - \frac{s}{r})} \tag{22}$$

Again we notice that this vanishes as $r \to s$, and there is a maximum of orbital speed equal to

$$(r\omega)_{\max} = \frac{\varepsilon}{2mc} \tag{23}$$

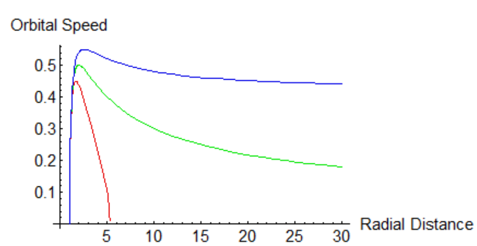
The location of the maximum is at

$$r = \frac{s}{(1 - \frac{\varepsilon^2}{2m^2c^4})} \tag{24}$$

This location is greater than the Schwarzschild radius for any acceptable value of energy

$$0 < \varepsilon < \sqrt{2}mc^2$$

The following is a plot of the orbital speed against radial distance for three values of the ratio $\varepsilon/mc^2 = \{0.9, 1, 1.1\}$, with three corresponding colors (red, green, blue), the radial scale is given in units of the Schwarzschild distance s, and the speed scale in units of c: :



For the lowest value $\varepsilon/mc^2 = 0.9$ (red color), we see that the orbital motion could exist only in a limited radial range. The lower value is the Schwarzschild radius $r = s$, and the upper value corresponds to the solution of the equation

$$1 - (\frac{mc^2}{\varepsilon})^2(1 - \frac{s}{r}) = 0 \tag{25}$$

A particle could oscillate between these two radii *while executing its orbital motion*. For the other two values $\varepsilon/mc^2 = 1, 1.1$, however, we see that the orbital motion has no upper radial limit. We also see that the higher the energy, the higher is the maximum of orbital speed, and the location of the maximum gets further away from the center. The situation is *not very different* from the case of radial motion.

We can say, in general, that the motion of a massive particle (radial as well as orbital) in the Schwarzschild field becomes very fast in the vicinity of the gravitational radius $s = 2GM/c^2$ and *ceases, or gets reversed*, as $r \to s$.

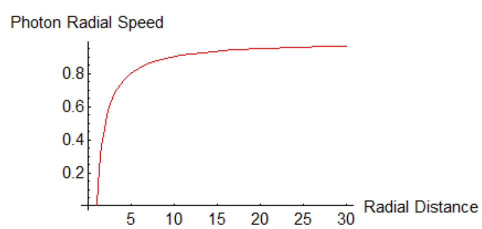
5 Photon Motion

For a massless particle, like the photon, the formulae obtained in the preceding section for the radial velocity and the radial acceleration become (with $m = 0$):

$$v = c\left(1 - \frac{s}{r}\right) \tag{26}$$

$$a_r = \frac{dv}{dr}v = \frac{sc^2}{r^2}\left(1 - \frac{s}{r}\right) \tag{27}$$

Notice that $a_r > 0$ for all $r > s$ (repulsion). The following figure depicts the radial speed of the photon as a function of radial distance. The radial scale is given in units of the Schwarzschild distance s, and the speed scale in units of c:



Hence we have the striking result: The speed of a radial photon would decrease as the Schwarzschild surface is approached from outside. The acceleration is always positive for $r > s$. An incoming photon *decelerates*, and an outgoing photon *accelerates*. Effectively, this means that a radial photon is *always repelled by the central source of a gravitational field*. This result concerning the motion of a radial photon could easily have been deduced by requiring that the line element for a radial photon should vanish, as in special relativity,

$$\left(1 - \frac{s}{r}\right)c^2dt^2 - \left(1 - \frac{s}{r}\right)^{-1}dr^2 = 0 \qquad \Rightarrow \qquad \frac{dr}{dt} = \pm c\left(1 - \frac{s}{r}\right) \tag{28}$$

This tells that the speed of a radial photon at a distance r is $c(1 - s/r)$ as before. Likewise, we may obtain the speed of an orbital photon, either from the orbital equation of a massive particle setting $m = 0$, or by equating the corresponding line element to zero:

$$\left(1 - \frac{s}{r}\right) c^2 dt^2 - r^2 d\omega^2 = 0 \qquad \Rightarrow \qquad r\frac{d\omega}{dt} = \pm c\sqrt{1 - \frac{s}{r}} \tag{29}$$

Hence the speed of an orbital photon at a distance r is given by $c\sqrt{1 - \frac{s}{r}}$. Notice that the speed of an orbital photon at some radial distance is *different* from the speed of a radial counterpart. Of course, we must distinguish these speeds of the photon from the universal constant c which gives the speed at infinite distance from the gravitational center.

The propagation of light at constant speed is one of the cornerstones of Einstein's theory of special relativity. In his theory, even the classical mechanics of particles is formulated in such a way that a fundamental constant is introduced and identified with the special relativistic speed of light.[3] Subsequently the quantum field theoretic description of point particles is built such as the associated nature of propagation, in the massless limit, is identical to that of light particles (or photons).[4]

Whereas Einstein has formulated his general theory of relativity in order to provide a geometric description of gravitation, it is however natural to regard that construction as a generalized framework for the propagation of light and other particles, and to regard gravitation itself as a mere outcome of this viewpoint. In contrast to the metric of special relativity, which implies the constancy of the speed of light, an arbitrary spacetime metric would imply the variation of the speed of light in space (for instance, in a local gravitational field) or in time (for instance, in cosmology).

6 The Singularity & the Timing Problem

Our preceding analysis in terms of particle speeds and accelerations gives us a nice description of the relativistic motion of a particle in the field of a central Schwarzschild source. However, we should now confront the singularity problem which is associated with the breakdown of the metric when the radial distance approaches the Schwarzschild surface, $r \to s$. As a matter of fact, dealing with speeds and accelerations, this problem did not show up in our preceding analysis at all. We shall face the problem when we try to integrate the differential equation of speed in order to obtain the distance traversed by the particle as a function of time, or the time taken to reach the surface at $r = s$.

Let us first consider the differential equation pertaining to an incoming radial photon's speed:

$$\frac{dr}{dt} = -c\left(1 - \frac{s}{r}\right) \tag{30}$$

Solving the above equation for dt, and integrating from radial distance R down to a radial distance R_0, we obtain for the *time taken by the photon*:

$$\frac{R - R_0}{c} + \frac{s}{c}\ln\left(\frac{R - s}{R_0 - s}\right) \tag{31}$$

The first term, in the above expression, is the normal time taken by a photon which travels at free speed c. The second term is the *delay* due to the deceleration of the photon as it comes down closer to the central source. The problem lies with this latter term when the approach gets closer to the Schwarzschild surface, $R_0 \to s$. Here the singularity problem shows up, giving positive infinity. One would attempt to say that the photon would never reach the singular surface at $R_0 = s$, unless one waits for infinite time!

On the other hand, the equation of radial speed associated with an incoming massive test particle is

$$\frac{dr}{dt} = -c\left(1 - \frac{s}{r}\right)\sqrt{1 - (\frac{mc^2}{\varepsilon})^2\left(1 - \frac{s}{r}\right)} \tag{32}$$

This can also be integrated obtaining a much more complicated expression for the time taken by the massive particle to go from radial distance R to radial distance R_0. However, we find that the singular term as $R_0 \to s$ is exactly the same one obtained above for the photon, namely,

$$\frac{s}{c}\ln\left(\frac{R - s}{R_0 - s}\right) \tag{33}$$

However, a moment's thought would tell us that the photon, as well as any massive test particle, should have their own Schwarzschild surfaces, and these surfaces should be impenetrable just like the Schwarzschild surface associated with the central source. Hence the length $(R_0 - s)$ should be no less than σ, the Schwarzschild radius associated with the test particle. Whereas the value of $\sigma = 2Gm/c^2$ for a particle of mass m, the value of σ associated with a photon would be the same using $m = \varepsilon/c^2$, the mass equivalent of a photon's energy ε. The reader could verify that, with this prescription, the value of $(s/c)\ln[(R - s)/\sigma]$ would always give a reasonable value for the delay time, whatever reasonable values we give to R, s and σ. For example a compact Schwarzschild source with mass of solar order would have $s \sim 3 \times 10^3$ meters. An MeV particle or a photon would have a corresponding $\sigma \sim 3 \times 10^{-57}$ meter. Consequently, if such a particle or photon descends towards the Schwarzschild source from a distance $R \sim 10^9$ meters, the above delay time would be of the order of ~ 0.001 second rather than infinity! The actual time for such a real photon to reach the Schwarzschild surface is not practically different from $(R - s)/c \sim 3.3$ seconds.

7 Discussion

Our elementary analysis of particle motion in the spherically symmetric gravitational field of a Schwarzschild source gives remarkable insight regarding the relativistic theory of gravitation. Whereas gravitation is thought to be always attractive, we have seen that a Schwarzschild source would repel particles that come close to the Schwarzschild surface. This together with our resolution to the timing problem associated with the singularity at the Schwarzschild surface, makes us depict a fascinating picture regarding the nature of Schwarzschild sources as ***impenetrable bubbles which reflect photons***

and other particles, and bounce from each other, if they get in contact. The question is whether this picture can be tested.

As a matter of fact, the Schwarzschild metric which we have used in our present analysis can be regarded as describing the gravitational field of a very compact central source, or an elementary particle, if such a thing does exist in reality. Consequently the gravitational interactions of elementary particles may be regarded as being repulsive at distances approaching their Schwarzschild radii. However, we should remember that elementary particles have other forces (electromagnetic and nuclear) that enter the game. The forgoing picture for particles *must be extended* taking into account the effects of other interactions.

However, for a description of celestial bodies like stars, we must consider metrics that can describe extended spherical mass distributions. A stellar body, however dense and compact, cannot be described by an ideal compact Schwarzschild source unless it manages to contain within its Schwarzschild surface all the surrounding mass, without a trace of gas outside! Is this possible? We shall return to the treatment of stellar systems in other articles.

For the moment, supposing that the extreme case of a collapsed star, whose actual sharp radius falls below its Schwarzschild radius, does in fact exist, then this would be a place to test our picture regarding the ***repulsive nature of gravity near the Schwarzschild surface***. Would the tremendous activity in the nuclei of some galaxies and other astrophysical systems be a manifestation of gravitational *repulsion* rather than attraction?

References

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[3] N.S. Baaklini, "Introduction to Relativity Theory", *N.S.B. Letters*, **NSBL-RT-010**

[4] N.S. Baaklini, "Introduction to Relativistic Fields", *N.S.B. Letters*, **NSBL-RT-020**

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