# NOTES ON NONCOMMUTATIVE GEOMETRY

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# Foreword

These notes are neither an introduction nor a survey (if only a brief) of noncommutative geometry – later NCG; rather, they strive to answer some naive but vital questions:

*What is the purpose of NCG and what is it good for? Why a number theorist or an algebraic geometer should care about the NCG? Can NCG solve open problems of classical geometry inaccessible otherwise? In other words, why does NCG matter? What is it anyway?*

Good answer means good examples. A sweetheart of NCG called noncommutative torus captures classical geometry of elliptic curves because such a torus is a coordinate ring for elliptic curves. In other words, one deals with a functor from algebraic geometry to the NCG; such functors are at the heart of our book.

What is NCG anyway? It is a calculus of functors on the classical spaces (e.g. algebraic, geometric, topological, etc) with the values in NCG. Such an approach departs from the tradition of recasting geometry of the classical space $X$ in terms of the $C^*$-algebra $C(X)$ of continuous complex-valued functions on $X$, see the monograph by [Connes 1994] [16].

*Boston, April 2014* *Igor Nikolaev*

# Introduction

It is not easy to write an elementary introduction to the NCG because the simplest non-trivial examples are mind-blowing and involve the $KK$-groups, subfactors, Sklyanin algebras, etc. Such a material cannot be shrink-wrapped into a single graduate course being the result of a slow roasting of ideas from different (and distant) mathematical areas. This universality of NCG is exactly its traction force giving the reader a long lost sense of unity of mathematics. In writing these notes the author had in mind a graduate student in (say) number theory eager to learn something new, e.g. a noncommutative torus with real multiplication; it will soon transpire that such an object is linked to the $K$-rational points of elliptic curves and the Langlands program.

The book has three parts. Part I is preparatory: Chapter 1 deals with the simplest examples of functors arising in algebraic geometry, number theory and topology; the functors take value in a category of the $C^*$-algebras known as noncommutative tori. Using these functors one gets a set of noncommutative invariants for elliptic curves and Anosov's automorphisms of the two-dimensional torus. Chapter 2 is a brief introduction to the categories, functors and natural transformations; they will be used throughout the book. Chapter 3 covers an essential information about the category of $C^*$-algebras and their $K$-theory; we introduce certain important classes of the $C^*$-algebras: the AF-algebras, the UHF-algebras and the Cuntz-Krieger algebras. Our choice of the $C^*$-algebras is motivated by their applications in Part II.

Part II deals with the noncommutative invariants obtained from the functors acting on various classical spaces. Chapter 4 is devoted to such functors on the topological spaces with values in the category of the so-called stationary AF-algebras; the noncommutative invariants are the Handelman triples $(\Lambda, [I], K)$, where $\Lambda$ is an order in a real algebraic number field $K$ and $[I]$ an equivalence class of the ideals of $\Lambda$. Chapter 5 deals with the examples of functors arising in projective algebraic geometry and their noncommutative invariants. Finally, Chapter 6 covers functors in number theory and the corresponding invariants.

Part III is a brief survey of the NCG; the survey is cursory yet an extensive guide to the literature has been compiled at the end of each chapter. We hope that the reader can instruct himself by looking at the original publications; we owe an apology to the authors whose works are not on the list.

There exist several excellent textbooks on the NCG. The first and foremost is the monograph by A. Connes *"Géométrie Non Commutative"*, Paris, 1990 and its English edition *"Noncommutative Geometry"*, Academic Press, 1994. The books by J. Madore *"An Introduction to Noncommutative Differential Geometry & its Applications"*, Cambridge Univ. Press, 1995, by J. M. Gracia-Bondia, J. C. Varilly and H. Figueroa *"Elements of Noncommutative Geometry"*, Birkhäuser, 2000 and by M. Khalkhali *"Basic Noncommutative Geometry"*, EMS Series of Lectures in Mathematics, 2007 treat particular aspects of Connes' monograph. A different approach to the NCG is covered in a small but instructive book by Yu. I. Manin *"Topics in Noncommutative Geometry"*, Princeton Univ. Press, 1991. Finally, a more specialized *"Noncommutative Geometry, Quantum Fields and Motives"*, AMS Colloquium Publications, 2008 by A. Connes and M. Marcolli is devoted to the links to physics and number theory. None of these books treat the NCG as a functor [63].

I thank the organizers, participants and sponsors of the Spring Institute on *Noncommutative Geometry and Operator Algebras (NCGOA)* held annually at the Vanderbilt University in Nashville, Tennessee; these notes grew from efforts to find out what is going on there. (I still don't have an answer.) I am grateful to folks who helped me with the project; among them are P. Baum, D. Bisch, B. Blackadar, O. Bratteli, A. Connes, J. Cuntz, G. Elliott, K. Goodearl, D. Handelman, N. Higson, B. Hughes, V. F. R. Jones, M. Kapranov, M. Khalkhali, W. Krieger, Yu. Manin, V. Manuilov, M. Marcolli, V. Mathai, A. Mishchenko, S. Novikov, M. Rieffel, W. Thurston, V. Troitsky, G. Yu and others.

# Contents

# Part I

# BASICS

# Chapter 1

# Model Examples

We shall start with the simplest functors arising in algebraic geometry, number theory and topology; all these functors range in a category of the $C^*$-algebras called noncommutative tori. We strongly believe that a handful of simple examples tell more than lengthy theories based on them; we encourage the reader to keep these model functors in mind for the rest of the book. No special knowledge of the $C^*$-algebras, elliptic curves or Anosov automorphisms (beyond an intuitive level) is required at this point; the interested reader can look up the missing definitions in the standard literature indicated at the end of each section.

## 1.1 Noncommutative torus

The noncommutative torus is an associative algebra over $\mathbb{C}$ of particular simplicity and beauty; such an algebra can be defined in several equivalent ways, e.g. as the universal algebra $\mathbb{C}\langle u, v\rangle$ on two unitary generators $u$ and $v$ satisfying the unique commutation relation $vu = e^{2\pi i\theta}uv$, where $\theta$ is a real number. There is a more geometric introduction as a deformation of the commutative algebra $C^\infty(T^2)$ of smooth complex-valued functions on the two-dimensional torus $T^2$; we shall pick up the latter because it clarifies the origin and notation for such algebras. Roughly speaking, one starts with the commutative algebra $C^\infty(T^2)$ of infinitely differentiable complex-valued functions on $T^2$ endowed with the usual pointwise sum and product of two functions. The idea is to replace the commutative product $f(x)g(x)$ of functions $f, g \in C^\infty(T^2)$ by a non-commutative product $f(x) *_\hbar g(x)$ depending on a continuous deformation parameter $\hbar$, so that $\hbar = 0$ corresponds to the usual product $f(x)g(x)$; the product $f(x) *_\hbar g(x)$ must be associative for each value of $\hbar$. To achieve the goal, it is sufficient to construct the Poisson bracket $\{f, g\}$ on $C^\infty(T^2)$, i.e. a binary operation satisfying the identities $\{f, f\} = 0$ and $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$; the claim is a special case of Kontsevich's Theorem for the Poisson manifolds, see Section 14.3. The algebra $C^\infty_\hbar(T^2)$ equipped with the usual sum $f(x) + g(x)$ and a non-commutative associative product $f(x) *_\hbar g(x)$ is called a *deformation quantization* of algebra $C^\infty(T^2)$.

The required Poisson bracket can be constructed as follows. For a real number $\theta$ define a bracket on $C^\infty(T^2)$ by the formula

$$\{f, g\}_\theta := \theta \left( \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right).$$

The reader is encouraged to verify that the bracket satisfies the identities $\{f, f\}_\theta = 0$ and $\{f, \{g, h\}_\theta\}_\theta + \{h, \{f, g\}_\theta\}_\theta + \{g, \{h, f\}_\theta\}_\theta = 0$, i.e. is the Poisson bracket. The Kontsevich Theorem says that there exists an associative product $f *_\hbar g$ on $C^\infty(T^2)$ obtained from the bracket $\{f, g\}_\theta$. Namely, let $\varphi$ and $\psi$ denote the Fourier transform of functions $f$ and $g$ respectively; one can define an $\hbar$-family of products between the Fourier transforms according to the formula

$$(\varphi *_\hbar \psi)(p) = \sum_{q \in \mathbb{Z}^2} \varphi(q)\psi(p - q)e^{-\pi i\hbar\ k(p,q)},$$

where $k(p, q) = \theta(pq - qp)$ is the kernel of the Fourier transform of the Poisson bracket $\{f, g\}_\theta$, i.e. an expression defined by the formula

$$\{\varphi, \psi\}_\theta = -4\pi^2 \sum_{q \in \mathbb{Z}^2} \varphi(q)\psi(p - q)k(q, p - q).$$

The product $f *_\hbar g$ is defined as a pull back of the product $(\varphi *_\hbar \psi)$; the resulting associative algebra $C^\infty_{\hbar,\theta}(T^2)$ is called the deformation quantization of $C^\infty(T^2)$ in the *direction* $\theta$ defined by the Poisson bracket $\{f, g\}_\theta$, see Fig. 1.1.

**Remark 1.1.1** The algebra $C^\infty_{\hbar,\theta}(T^2)$ is endowed with the natural involution coming from the complex conjugation on $C^\infty(T^2)$, so that $\varphi^*(p) := \bar{\varphi}(-p)$ for all $p \in \mathbb{Z}^2$. The natural norm on $C^\infty_{\hbar,\theta}(T^2)$ comes from the operator norm of the Schwartz functions $\varphi, \psi \in \mathcal{S}(\mathbb{Z}^2)$ acting on the Hilbert space $\ell^2(\mathbb{Z}^2)$.

**Definition 1.1.1** *By a noncommutative torus* $\mathcal{A}_\theta^{geometric}$ *one understands the* $C^*$*-algebra obtained from the norm closure of the* $^*$*-algebra* $C^\infty_{1,\ \theta}(T^2)$.

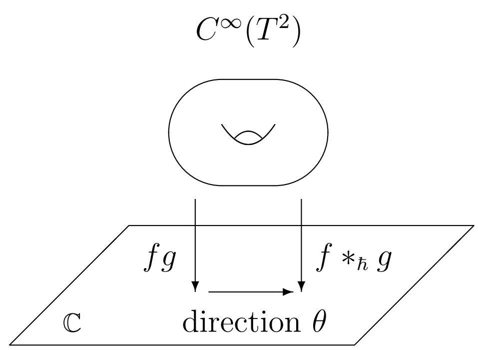



Figure 1.1: Deformation of algebra $C^\infty(T^2)$.

The less visual *analytic* definition of the noncommutative torus involves bounded linear operators acting on the Hilbert space $\mathcal{H}$; the reader can think of the operators as the infinite-dimensional matrices over $\mathbb{C}$. Namely, let $S^1$ be the unit circle; denote by $L^2(S^1)$ the Hilbert space of the square integrable complex valued functions on $S^1$. Fix a real number $\theta \in [0,1)$; for every $f(e^{2\pi it}) \in L^2(S^1)$ we shall consider two bounded linear operators $U$ and $V$ acting by the formula

$$\left\{\begin{array}{ll} U[f(e^{2\pi it})] & = f(e^{2\pi i(t-\theta)}) \\ V[f(e^{2\pi it})] & = e^{2\pi it} f(e^{2\pi it}). \end{array}\right.$$

It is verified directly that

$$\left\{\begin{array}{ll} VU & = e^{2\pi i\theta} UV, \\ UU^* & = U^*U = E, \\ VV^* & = V^*V = E, \end{array}\right.$$

where $U^*$ and $V^*$ are the adjoint operators of $U$ and $V$, respectively, and $E$ is the identity operator.

**Definition 1.1.2** *By a noncommutative torus $\mathcal{A}_\theta^{analytic}$ one understands the $C^*$-algebra generated by the operators $U$ and $V$ acting on the Hilbert space $L^2(S^1)$.*

The *algebraic* definition of the noncommutative torus is the shortest; it involves the universal algebras, i.e. the associative algebras given by the generators and relations. Namely, let $\mathbb{C}\langle x_1, x_2, x_3, x_4\rangle$ be the polynomial ring in four non-commuting variables $x_1, x_2, x_3$ and $x_4$. Consider a two-sided ideal, $I_\theta$, generated by the relations

$$\begin{cases} x_3x_1 &= e^{2\pi i\theta}x_1x_3,\\ x_1x_2 &= x_2x_1 = e,\\ x_3x_4 &= x_4x_3 = e.\end{cases}$$

**Definition 1.1.3** *By a noncommutative torus $\mathcal{A}_\theta^{algebraic}$ one understands the $C^*$-algebra given by the norm closure of the $*$-algebra $\mathbb{C}\langle x_1, x_2, x_3, x_4\rangle/I_\theta$, where the involution acts on the generators according to the formula $x_1^* = x_2$ and $x_3^* = x_4$.*

**Theorem 1.1.1** $\mathcal{A}_\theta^{geometric} \cong \mathcal{A}_\theta^{analytic} \cong \mathcal{A}_\theta^{algebraic}$

*Proof.* The isomorphism $\mathcal{A}_\theta^{analytic} \cong \mathcal{A}_\theta^{algebraic}$ is obvious, because one can write $x_1 = U, x_2 = U^*, x_3 = V$ and $x_4 = V^*$. The isomorphism $\mathcal{A}_\theta^{geometric} \cong \mathcal{A}_\theta^{analytic}$ is established by the identification of functions $t \mapsto e^{2\pi itp}$ of $\mathcal{A}_\theta^{geometric}$ with the unitary operators $U_p$ for each $p \in \mathbb{Z}^2$; then the generators of $\mathbb{Z}^2$ will correspond to the operators $U$ and $V$. $\square$

**Remark 1.1.2** We shall write $\mathcal{A}_\theta$ to denote an abstract noncommutative torus independent of its geometric, analytic or algebraic realization.

The noncommutative torus $\mathcal{A}_\theta$ has a plethora of remarkable properties; for the moment we shall dwell on the most fundamental: *Morita equivalence* and *real multiplication*. Roughly speaking, the first property presents a basic equivalence relation in the category of noncommutative tori; such a relation indicates that $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are identical from the standpoint of noncommutative geometry. The second property is rare; only a countable family of non-equivalent $\mathcal{A}_\theta$ can have real multiplication. The property means that the ring of endomorphisms of $\mathcal{A}_\theta$ is non-trivial, i.e. it exceeds the ring $\mathbb{Z}$. To give an exact definition, denote by $\mathcal{K}$ the $C^*$-algebra of all compact operators.

**Definition 1.1.4** *The noncommutative torus $\mathcal{A}_\theta$ is said to be stably isomorphic (Morita equivalent) to a noncommutative torus $\mathcal{A}_{\theta'}$ whenever $\mathcal{A}_\theta \otimes \mathcal{K} \cong \mathcal{A}_{\theta'} \otimes \mathcal{K}$.*

Recall that the Morita equivalence means that the associative algebras $A$ and $A'$ have the same category of projective modules, i.e. **Mod** $(A) \cong$ **Mod** $(A')$. It is notoriously hard to tell (in intrinsic terms) when two non-isomorphic algebras are Morita equivalent; of course, if $A \cong A'$ then $A$ is Morita equivalent to $A'$. The following remarkable result provides a clear and definitive solution to the Morita equivalence problem for the noncommutative tori; it would be futile to talk about any links to the elliptic curves (complex tori) if a weaker or fuzzier result were true.

**Theorem 1.1.2 (Rieffel)** *The noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are stably isomorphic (Morita equivalent) if and only if*

$$\theta' = \frac{a\theta + b}{c\theta + d} \quad \textit{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The second fundamental property of the algebra $\mathcal{A}_\theta$ is the so-called *real multiplication*; such a multiplication signals exceptional symmetry of $\mathcal{A}_\theta$. Recall that the Weierstrass uniformization of elliptic curves by the lattices $L_\tau := \mathbb{Z} + \mathbb{Z}\tau$ gives rise to the *complex multiplication*, i.e. phenomenon of an unusual behavior of the endomorphism ring of $L_\tau$; the noncommutative torus $\mathcal{A}_\theta$ demonstrates the same behavior with (almost) the same name. To introduce real multiplication, denote by $\mathcal{Q}$ the set of all quadratic irrational numbers, i.e. the irrational roots of all quadratic polynomials with integer coefficients.

**Theorem 1.1.3 (Manin)** *The endomorphism ring of a noncommutative torus $\mathcal{A}_\theta$ is given by the formula*

$$End\ (\mathcal{A}_\theta) \cong \begin{cases} \mathbb{Z}, & \textit{if } \ \theta \in \mathbb{R} - (\mathcal{Q} \cup \mathbb{Q}) \\ \mathbb{Z} + fO_k, & \textit{if } \ \theta \in \mathcal{Q}, \end{cases}$$

*where integer $f \geq 1$ is conductor of an order in the ring of integers $O_k$ of the real quadratic field $k = \mathbb{Q}(\sqrt{D})$.*

**Definition 1.1.5** *The noncommutative torus $\mathcal{A}_\theta$ is said to have real multiplication if End $(\mathcal{A}_\theta)$ is bigger than $\mathbb{Z}$, i.e. $\theta$ is a quadratic irrationality; we shall write $\mathcal{A}_{RM}^{(D,f)}$ to denote noncommutative tori with real multiplication by an order of conductor $f$ in the quadratic field $\mathbb{Q}(\sqrt{D})$,*

**Remark 1.1.3** It is easy to see that real multiplication is an invariant of the stable isomorphism (Morita equivalence) class of noncommutative torus $\mathcal{A}_\theta$.

**Guide to the literature.** For an authentic introduction to the noncommutative tori we encourage the reader to start with the survey paper by [Rieffel 1990] [89]. The noncommutative torus is also known as the irrational rotation algebra, see [Pimsner & Voiculescu 1980] [83] and [Rieffel 1981] [88]. The real multiplication has been introduced and studied by [Manin 2003] [52].

## 1.2 Elliptic curves

Elliptic curves are so fundamental that they hardly need any introduction; many basic facts and open problems of complex analysis, algebraic geometry and number theory can be reformulated in terms of such curves. Perhaps it is the single most ancient mathematical object so well explored yet hiding the deepest unsolved problems, e.g. the Birch and Swinnerton-Dyer Conjecture. Unless otherwise stated, we deal with the elliptic curves over the field $\mathbb{C}$ of complex numbers; recall that an *elliptic curve* is the subset of the complex projective plane of the form

$$\mathcal{E}(\mathbb{C}) = \{(x, y, z) \in \mathbb{C}P^2 \mid y^2 z = 4x^3 + axz^2 + bz^3\},$$

where $a$ and $b$ are some constant complex numbers. The real points of $\mathcal{E}(\mathbb{C})$ are depicted in Figure 1.2.

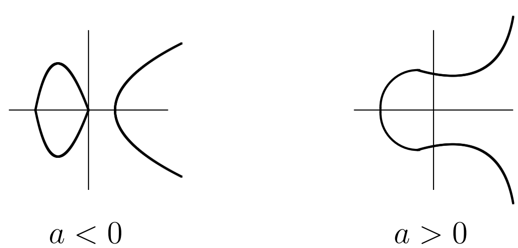



Figure 1.2: The real points of an affine elliptic curve $y^2 = 4x^3 + ax$.

**Remark 1.2.1** It is known that each elliptic curve $\mathcal{E}(\mathbb{C})$ is isomorphic to the set of points of intersection of two *quadric surfaces* in the complex projective space $\mathbb{C}P^3$ given by the system of homogeneous equations

$$\begin{cases} u^2+v^2+w^2+z^2 &= 0, \\ Av^2+Bw^2+z^2 &= 0, \end{cases}$$

where $A$ and $B$ are some constant complex numbers and $(u,v,w,z)\in\mathbb{C}P^3$; the system is called the *Jacobi form* of elliptic curve $\mathcal{E}(\mathbb{C})$.

**Definition 1.2.1** *By a complex torus one understands the space $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$, where $\omega_1$ and $\omega_2$ are linearly independent vectors in the complex plane $\mathbb{C}$, see Fig. 1.3; the ratio $\tau=\omega_2/\omega_1$ is called a complex modulus.*

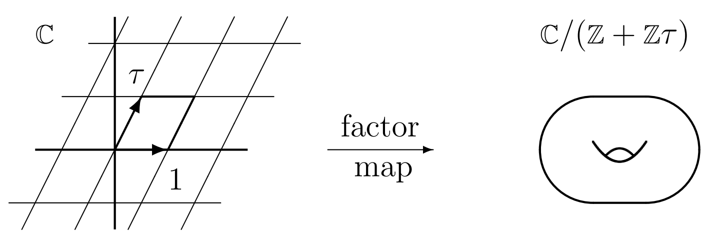



Figure 1.3: Complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$.

**Remark 1.2.2** Two complex tori $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ and $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau')$ are isomorphic if and only if

$$\tau'=\frac{a\tau+b}{c\tau+d}\quad\text{for some matrix}\quad\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}).$$

(We leave the proof to the reader. Hint: notice that $z\mapsto\alpha z$ is an invertible holomorphic map for each $\alpha\in\mathbb{C}-\{0\}$.)

One may wonder if the complex analytic manifold $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ can be embedded into an $n$-dimensional complex projective space as an algebraic variety; it turns out that the answer is emphatically yes even for the case $n=2$. The following classical result relates complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with an elliptic curve $\mathcal{E}(\mathbb{C})$ in the projective plane $\mathbb{C}P^2$.

**Theorem 1.2.1 (Weierstrass)** *There exists a holomorphic embedding*

$$\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \hookrightarrow \mathbb{C}P^2$$

*given by the formula*

$$z \mapsto \begin{cases} (\wp(z), \wp'(z), 1) & \text{for } z \notin L_\tau := \mathbb{Z}+\mathbb{Z}\tau, \\ (0,1,0) & \text{for } z \in L_\tau \end{cases},$$

*which is an isomorphism between complex torus* $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ *and elliptic curve*

$$\mathcal{E}(\mathbb{C}) = \{(x,y,z) \in \mathbb{C}P^2 \mid y^2 z = 4x^3 + axz^2 + bz^3\},$$

*where* $\wp(z)$ *is the Weierstrass function defined by the convergent series*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L_\tau - \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

*and*

$$\begin{cases} a = -60 \sum_{\omega \in L_\tau - \{0\}} \frac{1}{\omega^4}, \\ b = -140 \sum_{\omega \in L_\tau - \{0\}} \frac{1}{\omega^6}. \end{cases}$$

**Remark 1.2.3** Roughly speaking, the Weierstrass Theorem identifies elliptic curves $\mathcal{E}(\mathbb{C})$ and complex tori $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$; we shall write $\mathcal{E}_\tau$ to denote elliptic curve corresponding to the complex torus of modulus $\tau = \omega_2/\omega_1$.

**Remark 1.2.4 (First encounter with functors)** The declared purpose of our notes were functors with the range in category of associative algebras; as a model example we picked the category of noncommutative tori. Based on what is known about the algebra $\mathcal{A}_\theta$, one cannot avoid the following fundamental question: *Why the isomorphisms of elliptic curves look exactly the same as the stable isomorphisms (Morita equivalences) of noncommutative tori? In other words, what makes the diagram in Fig. 1.4 commute?*

To settle the problem, we presume that the observed phenomenon is the part of a categorical correspondence between elliptic curves and noncommutative tori; the following theorem says that it is indeed so.

**Theorem 1.2.2** *There exists a covariant functor* $F$ *from the category of all elliptic curves* $\mathcal{E}_\tau$ *to the category of noncommutative tori* $\mathcal{A}_\theta$*, such that if* $\mathcal{E}_\tau$ *is isomorphic to* $\mathcal{E}_{\tau'}$ *then* $\mathcal{A}_\theta = F(\mathcal{E}_\tau)$ *is stably isomorphic (Morita equivalent) to* $\mathcal{A}_{\theta'} = F(\mathcal{E}_{\tau'})$*.*

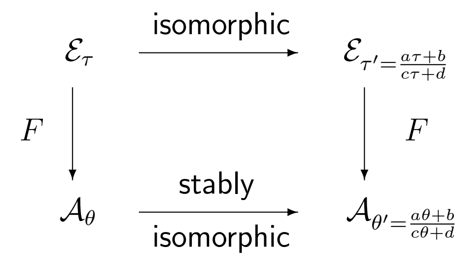



Figure 1.4: Fundamental phenomenon.

*Proof.* We shall give an algebraic proof of this fact based on the notion of a *Sklyanin algebra*; there exists a geometric proof using the notion of measured foliations and the Teichmüller theory, see Section 5.1.2. Recall that the Sklyanin algebra $S(\alpha, \beta, \gamma)$ is a free $\mathbb{C}$-algebra on four generators $x_1, \dots, x_4$ and six quadratic relations:

$$\left\{\begin{array}{lcc} x_1x_2 - x_2x_1 & = & \alpha(x_3x_4 + x_4x_3), \\ x_1x_2 + x_2x_1 & = & x_3x_4 - x_4x_3, \\ x_1x_3 - x_3x_1 & = & \beta(x_4x_2 + x_2x_4), \\ x_1x_3 + x_3x_1 & = & x_4x_2 - x_2x_4, \\ x_1x_4 - x_4x_1 & = & \gamma(x_2x_3 + x_3x_2), \\ x_1x_4 + x_4x_1 & = & x_2x_3 - x_3x_2, \end{array}\right.$$

where $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$, see e.g. [Smith & Stafford 1992] [99], p. 260. The algebra $S(\alpha, \beta, \gamma)$ is isomorphic to a *(twisted homogeneous) coordinate ring* of elliptic curve $\mathcal{E}_\tau \subset \mathbb{C}P^3$ given in its Jacobi form

$$\left\{\begin{array}{ccc} u^2 + v^2 + w^2 + z^2 & = & 0, \\ \frac{1-\alpha}{1+\beta}v^2 + \frac{1+\alpha}{1-\gamma}w^2 + z^2 & = & 0; \end{array}\right.$$

the latter means that $S(\alpha, \beta, \gamma)$ satisfies the fundamental isomorphism

$$\mathbf{Mod}\ (S(\alpha, \beta, \gamma))/\mathbf{Tors} \cong \mathbf{Coh}\ (\mathcal{E}_\tau),$$

where **Coh** is the category of quasi-coherent sheaves on $\mathcal{E}_\tau$, **Mod** the category of graded left modules over the graded ring $S(\alpha, \beta, \gamma)$ and **Tors** the full subcategory of **Mod** consisting of the torsion modules, see [Serre 1955] [91]. The algebra $S(\alpha, \beta, \gamma)$ defines a natural *automorphism* $\sigma : \mathcal{E}_\tau \to \mathcal{E}_\tau$ of the elliptic curve $\mathcal{E}_\tau$, see e.g. [Stafford & van den Bergh 2001] [100], p. 173. Fix an automorphism $\sigma$ of order 4, i.e. $\sigma^4 = 1$; in this case $\beta = 1,\ \gamma = -1$ and it is known that system of quadratic relations for the Sklyanin algebra $S(\alpha, \beta, \gamma)$ can be brought to a skew symmetric form

$$\begin{cases} x_3x_1 & = \mu e^{2\pi i\theta} x_1x_3, \\ x_4x_2 & = \frac{1}{\mu} e^{2\pi i\theta} x_2x_4, \\ x_4x_1 & = \mu e^{-2\pi i\theta} x_1x_4, \\ x_3x_2 & = \frac{1}{\mu} e^{-2\pi i\theta} x_2x_3, \\ x_2x_1 & = x_1x_2, \\ x_4x_3 & = x_3x_4, \end{cases}$$

where $\theta = Arg\ (q)$ and $\mu = |q|$ for some complex number $q \in \mathbb{C} - \{0\}$, see [Feigin & Odesskii 1989] [26], Remark 1.

On the other hand, the system of relations involved in the algebraic definition of noncommutative torus $\mathcal{A}_\theta$ is equivalent to the following system of quadratic relations

$$\begin{cases} x_3x_1 & = e^{2\pi i\theta} x_1x_3, \\ x_4x_2 & = e^{2\pi i\theta} x_2x_4, \\ x_4x_1 & = e^{-2\pi i\theta} x_1x_4, \\ x_3x_2 & = e^{-2\pi i\theta} x_2x_3, \\ x_2x_1 & = x_1x_2 = e, \\ x_4x_3 & = x_3x_4 = e. \end{cases}$$

(We leave the proof to the reader as an exercise in non-commutative algebra.) Comparing these relations with the skew symmetric relations for the Sklyanin algebra $S(\alpha, 1, -1)$, one concludes that they are almost identical; to pin down the difference we shall add two extra relations

$$x_1x_3 = x_3x_4 = \frac{1}{\mu} e$$

to relations of the Sklyanin algebra and bring it (by multiplication and cancellations) to the following equivalent form

$$\begin{cases} x_3x_1x_4 & = e^{2\pi i\theta} x_1, \\ x_4 & = e^{2\pi i\theta} x_2x_4x_1, \\ x_4x_1x_3 & = e^{-2\pi i\theta} x_1, \\ x_2 & = e^{-2\pi i\theta} x_4x_2x_3, \\ x_2x_1 & = x_1x_2 = \frac{1}{\mu} e, \\ x_4x_3 & = x_3x_4 = \frac{1}{\mu} e. \end{cases}$$

Doing the same type of equivalent transformations to the system of relations for the noncommutative torus, one brings the system to the form

$$\left\{\begin{array}{cc} x_3x_1x_4 & = e^{2\pi i\theta}x_1, \\ x_4 & = e^{2\pi i\theta}x_2x_4x_1, \\ x_4x_1x_3 & = e^{-2\pi i\theta}x_1, \\ x_2 & = e^{-2\pi i\theta}x_4x_2x_3, \\ x_2x_1 & = x_1x_2 = e, \\ x_4x_3 & = x_3x_4 = e. \end{array}\right.$$

Thus the only difference between relations for the Sklyanin algebra (modulo the ideal $I_\mu$ generated by relations $x_1x_3 = x_3x_4 = \frac{1}{\mu}e$) and such for the noncommutative torus $\mathcal{A}_\theta$ is a *scaling of the unit* $\frac{1}{\mu}e$. Thus one obtains the following remarkable isomorphism

$$\mathcal{A}_\theta \cong S(\alpha, 1, -1) \ / \ I_\mu.$$

**Remark 1.2.5 (Noncommutative torus as coordinate ring of $\mathcal{E}_\tau$)** Roughly speaking, the above formula says that modulo the ideal $I_\mu$ the noncommutative torus $\mathcal{A}_\theta$ is a coordinate ring of elliptic curve $\mathcal{E}_\tau$.

The required functor $F$ can be obtained as a quotient map of the fundamental (Serre) isomorphism

$$I_\mu \backslash \mathbf{Coh}\ (\mathcal{E}_\tau) \cong \mathbf{Mod}\ (I_\mu \backslash S(\alpha, 1, -1))/\mathbf{Tors} \cong \mathbf{Mod}\ (\mathcal{A}_\theta)/\mathbf{Tors}$$

and the fact that the isomorphisms in category $\mathbf{Mod}\ (\mathcal{A}_\theta)$ correspond to the stable isomorphisms (Morita equivalences) of category $\mathcal{A}_\theta$. $\square$

**Guide to the literature.** The reader can enjoy a plenty of excellent literature introducing elliptic curves; see e.g. [Husemöller 1986] [42], [Knapp 1992] [44], [Koblitz 1984] [46], [Silverman 1985] [93], [Silverman 1994] [94], [Silverman & Tate 1992] [95] and others. More advanced topics are covered in the survey papers [Cassels 1966] [14], [Mazur 1986] [53] and [Tate 1974] [103]. Noncommutative tori as coordinate rings of elliptic curves were studied in [62] and [64]; the higher genus curves were considered in [65].

## 1.3 Complex multiplication

A central problem of algebraic number theory is to give an explicit construction for the abelian extensions of a given field $k$. For instance, if $k \cong \mathbb{Q}$ is the field of rational numbers, the Kronecker-Weber Theorem says that the maximal abelian extension of $\mathbb{Q}$ is the union of all cyclotomic extensions; thus we have an explicit *class field theory* over $\mathbb{Q}$. If $k = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field, then *complex multiplication* realizes the class field theory for $k$. Let us recall some useful definition.

**Definition 1.3.1** *Elliptic curve $\mathcal{E}_\tau$ is said to have complex multiplication if the endomorphism ring of $\mathcal{E}_\tau$ is bigger than $\mathbb{Z}$, i.e. $\tau$ is a quadratic irrationality (see below); we shall write $\mathcal{E}_{CM}^{(-D,f)}$ to denote elliptic curves with complex multiplication by an order of conductor $f$ in the quadratic field $\mathbb{Q}(\sqrt{-D})$.*

**Remark 1.3.1** The endomorphism ring

$$End\ (\mathcal{E}_\tau) := \{\alpha \in \mathbb{C} \ : \ \alpha L_\tau \subseteq L_\tau\}$$

of elliptic curve $\mathcal{E}_\tau = \mathbb{C}/L_\tau$ is given by the formula

$$End\ (\mathcal{E}_\tau) \cong \begin{cases} \mathbb{Z}, & \text{if } \ \tau \in \mathbb{C} - \mathcal{Q} \\ \mathbb{Z} + fO_k, & \text{if } \ \tau \in \mathcal{Q}, \end{cases}$$

where $\mathcal{Q}$ is the set of all imaginary quadratic numbers and integer $f \geq 1$ is conductor of an order in the ring of integers $O_k$ of the imaginary quadratic number field $k = \mathbb{Q}(\sqrt{-D})$.

In previous section we constructed a functor $F$ on elliptic curves $\mathcal{E}_\tau$ with the range in the category of noncommutative tori $\mathcal{A}_\theta$; roughly speaking the following theorem characterizes the restriction of $F$ to elliptic curves with complex multiplication.

**Theorem 1.3.1 ([66], [71])** $F(\mathcal{E}_{CM}^{(-D,f)}) = \mathcal{A}_{RM}^{(D,f)}$.

**Remark 1.3.2** It follows from the theorem that the associative algebra $\mathcal{A}_{RM}^{(D,f)}$ is a coordinate ring of elliptic curve $\mathcal{E}_{CM}^{(-D,f)}$; in other words, each geometric property of curve $\mathcal{E}_{CM}^{(-D,f)}$ can be expressed in terms of the noncommutative torus $\mathcal{A}_{RM}^{(D,f)}$.

The noncommutative invariants of algebra $\mathcal{A}_{RM}^{(D,f)}$ are linked to *ranks* of the $K$-rational elliptic curves, i.e. elliptic curves $\mathcal{E}(K)$ over an algebraic field $K$; to illustrate the claim, let us recall some basic definitions and facts. Let $k = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic number field and let $j(\mathcal{E}_{CM}^{(-D,f)})$ be the $j$-invariant of elliptic curve $\mathcal{E}_{CM}^{(-D,f)}$; it is well known from complex multiplication, that

$$\mathcal{E}_{CM}^{(-D,f)} \cong \mathcal{E}(K),$$

where $K = k(j(\mathcal{E}_{CM}^{(-D,f)}))$ is the Hilbert class field of $k$, see e.g. [Silverman 1994] [94], p. 95. The Mordell-Weil theorem says that the set of the $K$-rational points of $\mathcal{E}_{CM}^{(-D,f)}$ is a finitely generated abelian group, see [Tate 1974] [103], p. 192; the rank of such a group will be denoted by $rk\ (\mathcal{E}_{CM}^{(-D,f)})$. For the sake of simplicity, we further restrict to the following class of curves. If $(\mathcal{E}_{CM}^{(-D,f)})^{\sigma}$, $\sigma \in Gal\ (K|\mathbb{Q})$ is the Galois conjugate of the curve $\mathcal{E}_{CM}^{(-D,f)}$, then by a $\mathbb{Q}$-*curve* one understands elliptic curve $\mathcal{E}_{CM}^{(-D,f)}$, such that there exists an isogeny between $(\mathcal{E}_{CM}^{(-D,f)})^{\sigma}$ and $\mathcal{E}_{CM}^{(-D,f)}$ for each $\sigma \in Gal\ (K|\mathbb{Q})$, see e.g. [Gross 1980] [29]. Let $\mathfrak{P}_{3\ \mathbf{mod}\ 4}$ be the set of all primes $p = 3\ \mathbf{mod}\ 4$; it is known that $\mathcal{E}_{CM}^{(-p,1)}$ is a $\mathbb{Q}$-curve whenever $p \in \mathfrak{P}_{3\ \mathbf{mod}\ 4}$, see [Gross 1980] [29], p. 33. The rank of $\mathcal{E}_{CM}^{(-p,1)}$ is always divisible by $2h_k$, where $h_k$ is the class number of field $k = \mathbb{Q}(\sqrt{-p})$, see [Gross 1980] [29], p. 49; by a $\mathbb{Q}$-*rank* of $\mathcal{E}_{CM}^{(-p,1)}$ one understands the integer

$$rk_{\mathbb{Q}}(\mathcal{E}_{CM}^{(-p,1)}) := \frac{1}{2h_k}\ rk\ (\mathcal{E}_{CM}^{(-p,1)}).$$

**Definition 1.3.2** *Suppose that* $[a_0, \overline{a_1, a_2, \dots, a_2, a_1, 2a_0}]$ *is the periodic continued fraction of* $\sqrt{D}$*, see e.g. [Perron 1954] [82], p.83; then by an arithmetic complexity* $c(\mathcal{A}_{RM}^{(D,f)})$ *of torus* $\mathcal{A}_{RM}^{(D,f)}$ *one understands the total number of independent* $a_i$ *in its period* $(a_1, a_2, \dots, a_2, a_1, 2a_0)$*, see Section 6.3.2 for the details.*

**Remark 1.3.3** It is easy to see that arithmetic complexity $c(\mathcal{A}_{RM}^{(D,f)})$ is an invariant of the stable isomorphism (Morita equivalence) class of the noncommutative torus $\mathcal{A}_{RM}^{(D,f)}$; in other words, $c(\mathcal{A}_{RM}^{(D,f)})$ is a *noncommutative invariant* of torus $\mathcal{A}_{RM}^{(D,f)}$.

The declared purpose of our notes were noncommutative invariants related to the classical geometry of elliptic curves; theorem below is one of such statements for the $\mathbb{Q}$-curves.

**Theorem 1.3.2 ([72])** *$rk_{\mathbb{Q}}\ (\mathcal{E}_{CM}^{(-p,1)}) + 1 = c(\mathcal{A}_{RM}^{(p,1)})$, whenever $p = 3$ **mod** 4.*

**Remark 1.3.4** It is known that there are infinitely many pairwise non-isomorphic $\mathbb{Q}$-curves, see e.g. [Gross 1980] [29]; all pairwise non-isomorphic $\mathbb{Q}$-curves $\mathcal{E}_{CM}^{(-p,1)}$ with $p < 100$ and their noncommutative invariant $c(\mathcal{A}_{RM}^{(p,1)})$ are calculated in Fig.1.5.

| $p \equiv 3$ **mod** 4 | $rk_{\mathbb{Q}}(\mathcal{E}_{CM}^{(-p,1)})$ | $\sqrt{p}$ | $c(\mathcal{A}_{RM}^{(p,1)})$ |
|---|---|---|---|
| 3 | 1 | $[1, \overline{1, 2}]$ | 2 |
| 7 | 0 | $[2, \overline{1, 1, 1, 4}]$ | 1 |
| 11 | 1 | $[3, \overline{3, 6}]$ | 2 |
| 19 | 1 | $[4, \overline{2, 1, 3, 1, 2, 8}]$ | 2 |
| 23 | 0 | $[4, \overline{1, 3, 1, 8}]$ | 1 |
| 31 | 0 | $[5, \overline{1, 1, 3, 5, 3, 1, 1, 10}]$ | 1 |
| 43 | 1 | $[6, \overline{1, 1, 3, 1, 5, 1, 3, 1, 1, 12}]$ | 2 |
| 47 | 0 | $[6, \overline{1, 5, 1, 12}]$ | 1 |
| 59 | 1 | $[7, \overline{1, 2, 7, 2, 1, 14}]$ | 2 |
| 67 | 1 | $[8, \overline{5, 2, 1, 1, 7, 1, 1, 2, 5, 16}]$ | 2 |
| 71 | 0 | $[8, \overline{2, 2, 1, 7, 1, 2, 2, 16}]$ | 1 |
| 79 | 0 | $[8, \overline{1, 7, 1, 16}]$ | 1 |
| 83 | 1 | $[9, \overline{9, 18}]$ | 2 |

Figure 1.5: The $\mathbb{Q}$-curves $\mathcal{E}_{CM}^{(-p,1)}$ with $p < 100$.

**Guide to the literature.** D. Hilbert counted complex multiplication as not only the most beautiful part of mathematics but also of entire science; it surely does as it links complex analysis and number theory. One cannot beat [Serre 1967] [92] for an introduction, but more comprehensive [Silverman 1994] [94], Ch. 2 is the must. Real multiplication has been introduced in [Manin 2004] [52]. The link between the two was the subject of [66].

## 1.4 Anosov automorphisms

Roughly speaking topology studies invariants of continuous maps $f : X \to Y$ between the topological spaces $X$ and $Y$; if $f$ is invertible and $X = Y$ we shall call it an *automorphism*. The automorphisms $f, f' : X \to X$ are said to be *conjugate* if there exists an automorphism $h : X \to X$ such that $f' = h \circ f \circ h^{-1}$, where $f \circ f'$ means the composition of $f$ and $f'$; the conjugation means a "change of coordinate system" for the topological space $X$ and each property of $f$ invariant under the conjugation is *intrinsic*, i.e. a topological invariant of $f$. The conjugation problem is unsolved even when $X$ is a topological surface (compact two-dimensional manifold); such a solution would imply topological classification of the three-dimensional manifolds, see e.g. [Hemion 1979] [39]. The automorphism $f : X \to X$ is said to have an *infinite order* if $f^n \neq Id$ for each $n \in \mathbb{Z}$. Further we shall focus on the topological invariants of automorphisms $f$ when $X = T^2$ is the two-dimensional torus. Because $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, each automorphism of $T^2$ can be given by an invertible map (isomorphism) of lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, see Fig. 1.6; in other words, the automorphism $f : T^2 \to T^2$ can be written in the matrix form

$$A_f = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2, \mathbb{Z}).$$

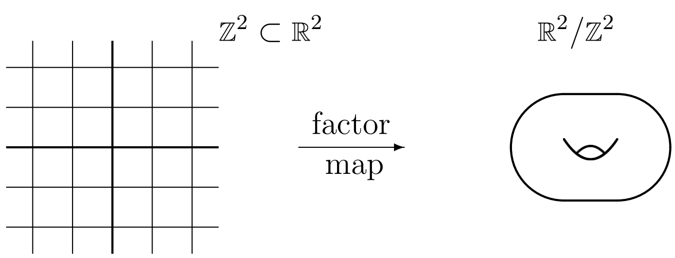



Figure 1.6: Topological torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$.

**Definition 1.4.1** *An infinite order automorphism $f : T^2 \to T^2$ is called Anosov if its matrix form $A_f$ satisfies the inequality $|a_{11} + a_{22}| > 2$.*

**Remark 1.4.1** The definition of Anosov's automorphism does not depend on the conjugation, because the trace $a_{11} + a_{22}$ is an invariant of the latter. Moreover, it is easy to see that "almost all" automorphisms of $T^2$ are

Anosov's; they constitute the most interesting part among all automorphisms of the torus.

To study topological invariants we shall construct a functor $F$ on the set of all Anosov automorphisms with the values in a category of the noncommutative tori such that the diagram in Fig. 1.7 is commutative; in other words, if $f$ and $f'$ are conjugate Anosov automorphisms, then the corresponding noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are stably isomorphic (Morita equivalent). The required map $F : A_f \mapsto \mathcal{A}_\theta$ can be constructed as follows. For simplic-

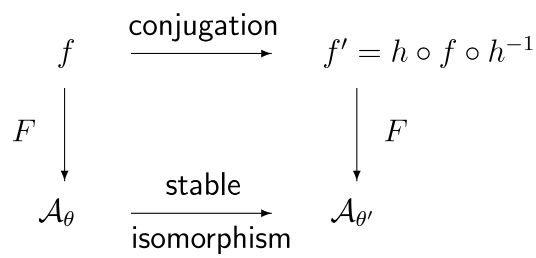



Figure 1.7: Functor $F$.

ity, we shall assume that $a_{11} + a_{22} > 2$; the case $a_{11} + a_{22} < -2$ is treated similarly. Moreover, we can assume that $A_f$ is a positive matrix since each class of conjugation of the Anosov automorphism $f$ contains such a representative; denote by $\lambda_{A_f}$ the Perron-Frobenius eigenvalue of positive matrix $A_f$. The noncommutative torus $\mathcal{A}_\theta = F(A_f)$ is defined by the normalized Perron-Frobenius eigenvector $(1, \theta)$ of the matrix $A_f$, i.e.

$$A_f \begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lambda_{A_f} \begin{pmatrix} 1 \\ \theta \end{pmatrix}.$$

**Remark 1.4.2** We leave it to the reader to prove that if $f$ is Anosov's, then $\theta$ is a quadratic irrationality given by the formula

$$\theta = \frac{a_{22} - a_{11} + \sqrt{(a_{11} + a_{22})^2 - 4}}{2a_{12}}.$$

It follows from the above formula that map $F$ takes values in the noncommutative tori with real multiplication; the following theorem says that our map $F : A_f \mapsto \mathcal{A}_\theta$ is actually a functor. (The proof of this fact is an easy exercise for anyone familiar with the notion of the $AF$-algebra of stationary type, the Bratteli diagram, etc.; we refer the interested reader to [63], p.153 for a short proof.)

**Theorem 1.4.1 ([69])** *If $f$ and $f'$ are conjugate Anosov automorphisms, then the noncommutative torus $\mathcal{A}_\theta = F(A_f)$ is stably isomorphic (Morita equivalent) to $\mathcal{A}_{\theta'} = F(A_{f'})$.*

Thus the problem of conjugation for the Anosov automorphisms can be recast in terms of the noncommutative tori; namely, one needs to find invariants of the stable isomorphism class of a noncommutative torus with real multilplication. Such a noncommutative invariant has been calculated in [Handelman 1981] [32]; namely, consider the eigenvalue problem for a matrix $A_f \in GL(2,\mathbb{Z})$, i.e. $A_f v_A = \lambda_{A_f} v_A$, where $\lambda_{A_f} > 1$ is the Perron-Frobenius eigenvalue and $v_A = (v_A^{(1)}, v_A^{(2)})$ the corresponding eigenvector with the positive entries normalized so that $v_A^{(i)} \in K = \mathbb{Q}(\lambda_{A_f})$. Denote by $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \mathbb{Z}v_A^{(2)}$ a $\mathbb{Z}$-module in the number field $K$. Recall that the coefficient ring, $\Lambda$, of module $\mathfrak{m}$ consists of the elements $\alpha \in K$ such that $\alpha\mathfrak{m} \subseteq \mathfrak{m}$. It is known that $\Lambda$ is an order in $K$ (i.e. a subring of $K$ containing 1) and, with no restriction, one can assume that $\mathfrak{m} \subseteq \Lambda$. It follows from the definition, that $\mathfrak{m}$ coincides with an ideal, $I$, whose equivalence class in $\Lambda$ we shall denote by $[I]$.

**Theorem 1.4.2 (Handelman's noncommutative invariant)** *The triple $(\Lambda, [I], K)$ is an arithmetic invariant of the stable isomorphism class of the noncommutative torus $\mathcal{A}_\theta$ with real multiplication, i.e. the tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are stably isomorphic (Morita equivalent) if and only if $\Lambda = \Lambda', [I] = [I']$ and $K = K'$.*

**Remark 1.4.3** The Handelman Theorem was proved for the so-called $AF$-algebras of a stationary type; such algebras and the noncommutative tori with real multiplication are known to have the same $K_0^+$ semigroup and therefore the same classes of stable isomorphisms. Similar problem for matrices was solved in [Latimer & MacDuffee 1933] [49] and [Wallace 1984] [107].

Handelman's Invariant $(\Lambda, [I], K)$ gives rise to a series of numerical invariants of the conjugation class of Anosov's automorphisms; we shall consider one such invariant called *module determinant* $\Delta(\mathfrak{m})$. Let $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \mathbb{Z}v_A^{(2)}$ be the module attached to $\Lambda$; consider the symmetric bilinear form

$$q(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} Tr\ (v_A^{(i)} v_A^{(j)}) x_i x_j,$$

where $Tr\ (v_A^{(i)} v_A^{(j)})$ is the trace of the algebraic number $v_A^{(i)} v_A^{(j)}$.

**Definition 1.4.2** *By a determinant of module* $\mathfrak{m}$ *one understands the determinant of the bilinear form* $q(x,y)$*, i.e. the rational integer*

$$\Delta(\mathfrak{m}) := Tr\ (v_A^{(1)} v_A^{(1)})\ Tr\ (v_A^{(2)} v_A^{(2)}) - Tr^2\ (v_A^{(1)} v_A^{(2)}).$$

**Remark 1.4.4** The rational integer $\Delta(\mathfrak{m})$ is a numerical invariant of Anosov's automorphisms, because it does not depend on the basis of module $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \mathbb{Z}v_A^{(2)}$; we leave the proof to the reader.

In conclusion, we calculate the noncommutative invariant $\Delta(\mathfrak{m})$ for the concrete automorphisms $f$ of $T^2$; the reader can see that in both cases our invariant $\Delta(\mathfrak{m})$ is *stronger* than the classical Alexander polynomial $\Delta(t)$, i.e. $\Delta(\mathfrak{m})$ detects the topological classes of $f$ which invariant $\Delta(t)$ cannot see.

**Example 1.4.1** Consider Anosov's automorphisms $f_A, f_B : T^2 \to T^2$ given by matrices

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix},$$

respectively. The Alexander polynomials of $f_A$ and $f_B$ are identical $\Delta_A(t) = \Delta_B(t) = t^2 - 6t + 1$; yet the automorphisms $f_A$ and $f_B$ are *not* conjugate. Indeed, the Perron-Frobenius eigenvector of matrix $A$ is $v_A = (1, \sqrt{2} - 1)$ while of the matrix $B$ is $v_B = (1, 2\sqrt{2} - 2)$. The bilinear forms for the modules $\mathfrak{m}_A = \mathbb{Z} + (\sqrt{2} - 1)\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (2\sqrt{2} - 2)\mathbb{Z}$ can be written as

$$q_A(x,y) = 2x^2 - 4xy + 6y^2, \qquad q_B(x,y) = 2x^2 - 8xy + 24y^2,$$

respectively. The modules $\mathfrak{m}_A, \mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{2})$, since their determinants $\Delta(\mathfrak{m}_A) = 8$ and $\Delta(\mathfrak{m}_B) = 32$ are not equal. Therefore, matrices $A$ and $B$ are not similar in the group $GL(2, \mathbb{Z})$.

**Example 1.4.2** Consider Anosov's automorphisms $f_A, f_B : T^2 \to T^2$ given by matrices

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix},$$

respectively. The Alexander polynomials of $f_A$ and $f_B$ are identical $\Delta_A(t) = \Delta_B(t) = t^2 - 8t + 1$; yet the automorphisms $f_A$ and $f_B$ are not conjugate. Indeed, the Perron-Frobenius eigenvector of matrix $A$ is $v_A = (1, \frac{1}{3}\sqrt{15})$ while of the matrix $B$ is $v_B = (1, \frac{1}{15}\sqrt{15})$. The corresponding modules are $\mathfrak{m}_A = \mathbb{Z} + (\frac{1}{3}\sqrt{15})\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (\frac{1}{15}\sqrt{15})\mathbb{Z}$; therefore

$$q_A(x, y) = 2x^2 + 18y^2, \qquad q_B(x, y) = 2x^2 + 450y^2,$$

respectively. The modules $\mathfrak{m}_A, \mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{15})$, since the module determinants $\Delta(\mathfrak{m}_A) = 36$ and $\Delta(\mathfrak{m}_B) = 900$ are not equal. Therefore, matrices $A$ and $B$ are not similar in the group $GL(2, \mathbb{Z})$.

**Guide to the literature.** The topology of surface automorphisms is the fundamental and the oldest part of geometric topology; it dates back to the works of J. Nielsen [Nielsen 1927; 1929; 1932] [61] and M. Dehn [Dehn 1938] [19]. W. Thurston proved that that there are only three types of such automorphisms: they are either of finite order, or of the Anosov type (called *pseudo-Anosov*) or else a mixture of the two, see e.g. [Thurston 1988] [105]; the topological classification of pseudo-Anosov automorphisms is the next problem after the *Geometrization Conjecture* proved by G. Perelman, see [Thurston 1982] [104]. An excellent introduction to the subject are the books [Fathi, Laudenbach & Poénaru 1979] [24] and [Casson & Bleiler 1988] [15]. The noncommutative invariants of pseudo-Anosov automorphisms were constructed in [69].

# Exercises

1. Show that the bracket on $C^\infty(T^2)$ defined by the formula

$$\{f, g\}_\theta := \theta \left( \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right).$$

is the Poisson bracket, i.e. satisfies the identities $\{f, f\}_\theta = 0$ and $\{f, \{g, h\}_\theta\}_\theta + \{h, \{f, g\}_\theta\}_\theta + \{g, \{h, f\}_\theta\}_\theta = 0$.

2. Prove that real multiplication is an invariant of the stable isomorphism (Morita equivalence) class of noncommutative torus $\mathcal{A}_\theta$.

3. Prove that complex tori $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ a $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau')$ are isomorphic if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. (Hint: notice that $z \mapsto \alpha z$ is an invertible holomorphic map for each $\alpha \in \mathbb{C} - \{0\}$.)

4. Prove that the system of relations

$$\begin{cases} x_3x_1 & = e^{2\pi i\theta}x_1x_3, \\ x_1x_2 & = x_2x_1 = e, \\ x_3x_4 & = x_4x_3 = e. \end{cases}$$

involved in the algebraic definition of noncommutative torus $\mathcal{A}_\theta$ is equivalent to the following system of quadratic relations

$$\begin{cases} x_3x_1 & = e^{2\pi i\theta}x_1x_3, \\ x_4x_2 & = e^{2\pi i\theta}x_2x_4, \\ x_4x_1 & = e^{-2\pi i\theta}x_1x_4, \\ x_3x_2 & = e^{-2\pi i\theta}x_2x_3, \\ x_2x_1 & = x_1x_2 = e, \\ x_4x_3 & = x_3x_4 = e. \end{cases}$$

5. Prove that elliptic curve $\mathcal{E}_\tau$ has complex multiplication if and only if the complex modulus $\tau \in \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic number. (Hint: Let $\alpha \in \mathbb{C}$ be such that $\alpha(\mathbb{Z}+\mathbb{Z}\tau) \subseteq \mathbb{Z}+\mathbb{Z}\tau$. That is there exist $m, n, r, s \in \mathbb{Z}$, such that

$$\begin{cases} \alpha & = m + n\tau, \\ \alpha\tau & = r + s\tau. \end{cases}$$

One can divide the second equation by the first, so that one gets $\tau = \frac{r+s\tau}{m+n\tau}$. Thus $n\tau^2 + (m-s)\tau - r = 0$; in other words, $\tau$ is an imaginary quadratic number. Conversely, if $\tau$ is the imaginary quadratic number, then it is easy to see that $End\ (\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau))$ is non-trivial.)

6. Prove that the period $\overline{(a_1, a_2, \dots, a_2, a_1, 2a_0)}$ (and the arithmetic complexity) is an invariant of the stable isomorphism (Morita equivalence) class of the noncommutative torus $\mathcal{A}_{RM}^{(D,f)}$.

7. Prove that the rational integer $\Delta(\mathfrak{m})$ is a numerical invariant of Anosov's automorphisms. (Hint: $\Delta(\mathfrak{m})$ does not depend on the basis of module $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \mathbb{Z}v_A^{(2)}$.)

8. Prove that the matrices

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}$$

are not similar by using the *Gauss method*, i.e. the method of continued fractions. (Hint: Find the fixed points $Ax = x$ and $Bx = x$, which gives us $x_A = 1 + \sqrt{2}$ and $x_B = \frac{1+\sqrt{2}}{2}$, respectively. Then one unfolds the fixed points into a periodic continued fraction, which gives us $x_A = [2, 2, 2, \ldots]$ and $x_B = [1, 4, 1, 4, \ldots]$. Since the period $(\overline{2})$ of $x_A$ differs from the period $(\overline{1, 4})$ of $B$, one concludes that matrices $A$ and $B$ belong to different similarity classes in $GL(2, \mathbb{Z})$.)

9. Repeat the exercise for matrices

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix}.$$

# Part II

# NONCOMMUTATIVE INVARIANTS

# Chapter 2

# Topology

In this chapter we shall construct several functors on the topological spaces with values in the category of stationary AF-algebras or the Cuntz-Krieger algebras. The functors give rise to a set of noncommutative invariants some of which can be explicitly calculated; all the invariants are homotopy invariants of the corresponding topological space. The chapter is written for a topologist and we assume that all topological facts are known to the reader; for otherwise, a reference list is compiled at the end of each section.

## 2.1 Classification of the surface maps

We assume that $X$ is a compact oriented surface of genus $g \geq 1$; we shall be interested in the continuous invertible self-maps (automorphisms) of $X$, i.e.

$$\phi : X \to X.$$

As it was shown in the model example for $X \cong T^2$, there exists a functor on the set of all Anosov's maps $\phi$ with values in the category of noncommutative tori with real multiplication; the functor sends the conjugate Anosov's maps to the stably isomorphic (Morita equivalent) noncommutative tori $\mathcal{A}_\theta$. Roughly speaking, in this section we extend this result to the higher genus surfaces, i.e for $g \geq 2$. However, instead of using $\mathcal{A}_\theta$'s as the target category, we shall use the category of stationary AF-algebras introduced in Section 3.5.2; in the case $g = 1$ the two categories are order-isomorphic because their $K_0^+$ semigroups are, see the end of Section 3.5.1.

### 2.1.1 Pseudo-Anosov maps of a surface

Let $Mod\ (X)$ be the mapping class group of a compact surface $X$, i.e. the group of orientation preserving automorphisms of $X$ modulo the trivial ones. Recall that $\phi, \phi' \in Mod\ (X)$ are conjugate automorphisms, whenever $\phi' = h \circ \phi \circ h^{-1}$ for an $h \in Mod\ (X)$. It is not hard to see that conjugation is an equivalence relation which splits the mapping class group into disjoint classes of conjugate automorphisms. The construction of invariants of the conjugacy classes in $Mod\ (X)$ is an important and difficult problem studied by [Hemion 1979] [39], [Mosher 1986] [57], and others; it is important to understand that any knowledge of such invariants leads to a topological classification of three-dimensional manifolds [Thurston 1982] [104]. It is known that any $\phi \in Mod\ (X)$ is isotopic to an automorphism $\phi'$, such that either (i) $\phi'$ has a finite order, or (ii) $\phi'$ is a *pseudo-Anosov* (aperiodic) automorphism, or else (iii) $\phi'$ is reducible by a system of curves $\Gamma$ surrounded by the small tubular neighborhoods $N(\Gamma)$, such that on $X \setminus N(\Gamma)$ $\phi'$ satisfies either (i) or (ii). Let $\phi$ be a representative of the equivalence class of a pseudo-Anosov automorphism. Then there exist a pair consisting of the stable $\mathcal{F}_s$ and unstable $\mathcal{F}_u$ mutually orthogonal measured foliations on the surface $X$, such that $\phi(\mathcal{F}_s) = \frac{1}{\lambda_\phi}\mathcal{F}_s$ and $\phi(\mathcal{F}_u) = \lambda_\phi \mathcal{F}_u$, where $\lambda_\phi > 1$ is called a dilatation of $\phi$. The foliations $\mathcal{F}_s, \mathcal{F}_u$ are minimal, uniquely ergodic and describe the automorphism $\phi$ up to a power. In the sequel, we shall focus on the conjugacy problem for the pseudo-Anosov automorphisms of a surface $X$; we shall try to solve the problem using functors with values in the NCG. Namely, we shall assign to each pseudo-Anosov map $\phi$ an AF-algebra, $\mathbb{A}_\phi$, so that for every $h \in Mod\ (X)$ the diagram in Fig. 4.1 is commutative. In words, if $\phi$ and $\phi'$ are conjugate pseudo-Anosov automorphisms, then the AF-algebras $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$ are stably isomorphic. For the sake of clarity, we shall consider an example illustrating the idea in the case $X \cong T^2$.

**Example 2.1.1 (case $X \cong T^2$)** Let $\phi \in Mod\ (T^2)$ be the Anosov automorphism given by a non-negative matrix $A_\phi \in SL(2, \mathbb{Z})$. Consider a stationary AF-algebra, $\mathbb{A}_\phi$, given by the periodic Bratteli diagram shown in Fig. 4.2, where $a_{ij}$ indicate the multiplicity of the respective edges of the graph. (We encourage the reader to verify that $F : \phi \mapsto \mathbb{A}_\phi$ is a well-defined function on the set of Anosov automorphisms given by the hyperbolic matrices with the non-negative entries.) Let us show that if $\phi, \phi' \in Mod\ (T^2)$ are conjugate Anosov automorphisms, then $\mathbb{A}_\phi, \mathbb{A}_{\phi'}$ are stably isomorphic AF-algebras. Indeed, let $\phi' = h \circ \phi \circ h^{-1}$ for an $h \in Mod\ (X)$. Then $A_{\phi'} = TA_\phi T^{-1}$ for

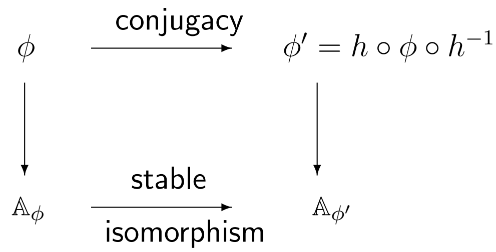
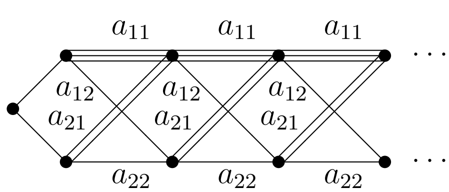



Figure 2.1: Conjugation of the pseudo-Anosov maps.

a matrix $T \in SL_2(\mathbb{Z})$. Note that $(A'_\phi)^n = (TA_\phi T^{-1})^n = TA_\phi^n T^{-1}$, where $n \in \mathbb{N}$. We shall use the following criterion: the AF-algebras $\mathbb{A}, \mathbb{A}'$ are stably isomorphic if and only if their Bratteli diagrams contain a common block of an arbitrary length, see [Effros 1981] [21], Theorem 2.3 and recall that an order-isomorphism mentioned in the theorem is equivalent to the condition that the corresponding Bratteli diagrams have the same infinite tails – i.e. a common block of infinite length. Consider the following sequences of matrices

$$\begin{cases} \underbrace{A_\phi A_\phi \dots A_\phi}_{n} \\ T \underbrace{A_\phi A_\phi \dots A_\phi}_{n} T^{-1}, \end{cases}$$

which mimic the Bratteli diagrams of $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$. Letting $n \to \infty$, we conclude that $\mathbb{A}_\phi \otimes \mathcal{K} \cong \mathbb{A}_{\phi'} \otimes \mathcal{K}$.




$$A_\phi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Figure 2.2: The AF-algebra $\mathbb{A}_\phi$.

**Remark 2.1.1 (Handelman's invariant of the AF-algebra $\mathbb{A}_\phi$)** One can reformulate the conjugacy problem for the automorphisms $\phi : T^2 \to T^2$

in terms of the AF-algebras $\mathbb{A}_\phi$; namely, one needs to find invariants of the stable isomorphism (Morita equivalence) classes of the stationary AF-algebras $\mathbb{A}_\phi$. One such invariant was introduced in Section 1.4; let us recall its definition and properties. Consider an eigenvalue problem for the matrix $A_\phi \in SL(2,\mathbb{Z})$, i.e. $A_\phi v_A = \lambda_A v_A$, where $\lambda_A > 1$ is the Perron-Frobenius eigenvalue and $v_A = (v_A^{(1)}, v_A^{(2)})$ the corresponding eigenvector with the positive entries normalized so that $v_A^{(i)} \in K = \mathbb{Q}(\lambda_A)$. Denote by $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \mathbb{Z}v_A^{(2)}$ the $\mathbb{Z}$-module in the number field $K$. The coefficient ring, $\Lambda$, of module $\mathfrak{m}$ consists of the elements $\alpha \in K$ such that $\alpha\mathfrak{m} \subseteq \mathfrak{m}$. It is known that $\Lambda$ is an order in $K$ (i.e. a subring of $K$ containing 1) and, with no restriction, one can assume that $\mathfrak{m} \subseteq \Lambda$. It follows from the definition, that $\mathfrak{m}$ coincides with an ideal, $I$, whose equivalence class in $\Lambda$ we shall denote by $[I]$. The triple $(\Lambda, [I], K)$ is an arithmetic invariant of the stable isomorphism class of $\mathbb{A}_\phi$: the $\mathbb{A}_\phi, \mathbb{A}_{\phi'}$ are stably isomorphic AF-algebras if and only if $\Lambda = \Lambda'$, $[I] = [I']$ and $K = K'$, see [Handelman 1981] [32].

### 2.1.2 Functors and invariants

Denote by $\mathcal{F}_\phi$ the stable foliation of a pseudo-Anosov automorphism $\phi \in Mod\ (X)$. For brevity, we assume that $\mathcal{F}_\phi$ is an oriented foliation given by the trajectories of a closed 1-form $\omega \in H^1(X;\mathbb{R})$. Let $v^{(i)} = \int_{\gamma_i} \omega$, where $\{\gamma_1, \dots, \gamma_n\}$ is a basis in the relative homology $H_1(X, \mathbf{Sing}\ \mathcal{F}_\phi; \mathbb{Z})$, such that $\theta = (\theta_1, \dots, \theta_{n-1})$ is a vector with positive coordinates $\theta_i = v^{(i+1)}/v^{(1)}$.

**Remark 2.1.2** The constants $\theta_i$ depend on a basis in the homology group, but the $\mathbb{Z}$-module generated by the $\theta_i$ does not.

Consider the infinite Jacobi-Perron continued fraction of $\theta$:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$ is a vector of the nonnegative integers, $I$ the unit matrix and $\mathbb{I} = (0, \dots, 0, 1)^T$; we refer the reader to [Bernstein 1971] [7] for the definition of the Jacobi-Perron algorithm and related continued fractions.

**Definition 2.1.1** *By $\mathbb{A}_\phi$ one understands the AF-algebra given by the Bratteli diagram defined by the incidence matrices $B_k = \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}$ for $k = 1, \dots, \infty$.*

**Remark 2.1.3** We encourage the reader to verify, that $\mathbb{A}_\phi$ coincides with the one for the Anosov maps. (Hint: the Jacobi-Perron fractions of dimension $n = 2$ coincide with the regular continued fractions.)

**Definition 2.1.2** *For a matrix $A \in GL_n(\mathbb{Z})$ with positive entries, we shall denote by $\lambda_A$ the Perron-Frobenius eigenvalue and let $(v_A^{(1)}, \dots, v_A^{(n)})$ be the corresponding normalized eigenvector such that $v_A^{(i)} \in K = \mathbb{Q}(\lambda_A)$. The coefficient (endomorphism) ring of the module $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \dots + \mathbb{Z}v_A^{(n)}$ will shall write as $\Lambda$; the equivalence class of ideal $I$ in $\Lambda$ will be written as $[I]$. We shall denote by $\Delta =$ **det** $(a_{ij})$ and $\Sigma$ the determinant and signature of the symmetric bilinear form $q(x,y) = \sum_{i,j}^n a_{ij}x_ix_j$, where $a_{ij} =$ **Tr** $(v_A^{(i)}v_A^{(j)})$ and **Tr** $(\bullet)$ the trace function.*

**Theorem 2.1.1** *$\mathbb{A}_\phi$ is a stationary AF-algebra.*

Let $\Phi$ be a category of all pseudo-Anosov (Anosov, resp.) automorphisms of a surface of the genus $g \ge 2$ ($g = 1$, resp.); the arrows (morphisms) are conjugations between the automorphisms. Likewise, let $\mathcal{A}$ be the category of all stationary AF-algebras $\mathbb{A}_\phi$, where $\phi$ runs over the set $\Phi$; the arrows of $\mathcal{A}$ are stable isomorphisms among the algebras $\mathbb{A}_\phi$.

**Theorem 2.1.2 (Functor on pseudo-Anosov maps)** *Let $F : \Phi \to \mathcal{A}$ be a map given by the formula $\phi \mapsto \mathbb{A}_\phi$. Then:*

*(i) $F$ is a functor which maps conjugate pseudo-Anosov automorphisms to stably isomorphic AF-algebras;*

*(ii) $Ker\ F = [\phi]$, where $[\phi] = \{\phi' \in \Phi \mid (\phi')^m = \phi^n,\ m,n \in \mathbb{N}\}$ is the commensurability class of the pseudo-Anosov automorphism $\phi$.*

**Corollary 2.1.1 (Noncommutative invariants)** *The following are invariants of the conjugacy classes of the pseudo-Anosov automorphisms:*

*(i) triples $(\Lambda, [I], K)$;*

*(ii) integers $\Delta$ and $\Sigma$.*

**Remark 2.1.4 (Effectiveness of invariants $(\Lambda, [I], K)$, $\Delta$ and $\Sigma$)** How to calculate invariants $(\Lambda, [I], K)$, $\Delta$ and $\Sigma$? There is no obvious way; the problem is similar to that of numerical invariants of the fundamental group of a knot. A step in this direction would be computation of the matrix

$A$; the latter is similar to the matrix $\rho(\phi)$, where $\rho : Mod\ (X) \to PIL$ is a faithful representation of the mapping class group as a group of the piecewise-integral-linear (PIL) transformations [Penner 1984] [80], p.45. The entries of $\rho(\phi)$ are the linear combinations of the Dehn twists along the $(3g-1)$ (Lickorish) curves on the surface $X$. Then one can effectively determine whether the $\rho(\phi)$ and $A$ are similar matrices (over $\mathbb{Z}$) by bringing the polynomial matrices $\rho(\phi) - xI$ and $A - xI$ to the Smith normal form; when the similarity is established, the numerical invariants $\Delta$ and $\Sigma$ become the polynomials in the Dehn twists. A tabulation of the simplest elements of $Mod\ (X)$ is possible in terms of $\Delta$ and $\Sigma$.

Theorems 4.1.1, 4.1.2 and Corollary 4.1.1 will be proved in Section 4.1.5; the necessary background is developed in the sections below.

### 2.1.3 Jacobian of measured foliations

Let $\mathcal{F}$ be a measured foliation on a compact surface $X$ [105]. For the sake of brevity, we shall always assume that $\mathcal{F}$ is an oriented foliation, i.e. given by the trajectories of a closed 1-form $\omega$ on $X$. (The assumption is not a restriction – each measured foliation is oriented on a surface $\widetilde{X}$, which is a double cover of $X$ ramified at the singular points of the half-integer index of the non-oriented foliation [Hubbard & Masur 1979] [41].) Let $\{\gamma_1, \dots, \gamma_n\}$ be a basis in the relative homology group $H_1(X, \mathbf{Sing}\ \mathcal{F}; \mathbb{Z})$, where $\mathbf{Sing}\ \mathcal{F}$ is the set of singular points of the foliation $\mathcal{F}$. It is well known that $n = 2g+m-1$, where $g$ is the genus of $X$ and $m = |\mathbf{Sing}\ (\mathcal{F})|$. The periods of $\omega$ in the above basis will be written

$$\lambda_i = \int_{\gamma_i} \omega.$$

The real numbers $\lambda_i$ are coordinates of $\mathcal{F}$ in the space of all measured foliations on $X$ with the fixed set of singular points, see e.g. [Douady & Hubbard 1975] [20].

**Definition 2.1.3** *By a jacobian Jac* $(\mathcal{F})$ *of the measured foliation* $\mathcal{F}$*, we understand a* $\mathbb{Z}$*-module* $\mathfrak{m} = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ *regarded as a subset of the real line* $\mathbb{R}$*.*

An importance of the jacobians stems from an observation that although the periods, $\lambda_i$, depend on the choice of basis in $H_1(X, \mathbf{Sing}\ \mathcal{F}; \mathbb{Z})$, the jacobian does not. Moreover, up to a scalar multiple, the jacobian is an invariant of the equivalence class of the foliation $\mathcal{F}$. We formalize these observations in the following two lemmas.

**Lemma 2.1.1** *The $\mathbb{Z}$-module $\mathfrak{m}$ is independent of choice of basis in $H_1(X, \mathbf{Sing}\ \mathcal{F}; \mathbb{Z})$ and depends solely on the foliation $\mathcal{F}$.*

*Proof.* Indeed, let $A = (a_{ij}) \in GL(n, \mathbb{Z})$ and let

$$\gamma_i' = \sum_{j=1}^{n} a_{ij}\gamma_j$$

be a new basis in $H_1(X, \mathbf{Sing}\ \mathcal{F}; \mathbb{Z})$. Then using the integration rules:

$$\begin{aligned} \lambda_i' &= \textstyle\int_{\gamma_i'} \omega &&= \int_{\sum_{j=1}^{n} a_{ij}\gamma_j} \omega = \\ &= \textstyle\sum_{j=1}^{n} \int_{\gamma_j} \omega &&= \sum_{j=1}^{n} a_{ij}\lambda_j. \end{aligned}$$

To prove that $\mathfrak{m} = \mathfrak{m}'$, consider the following equations:

$$\begin{aligned} \mathfrak{m}' &= \textstyle\sum_{i=1}^{n} \mathbb{Z}\lambda_i' &&= \sum_{i=1}^{n} \mathbb{Z} \sum_{j=1}^{n} a_{ij}\lambda_j = \\ &= \textstyle\sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}\mathbb{Z}\right) \lambda_j &&\subseteq \mathfrak{m}. \end{aligned}$$

Let $A^{-1} = (b_{ij}) \in GL(n, \mathbb{Z})$ be an inverse to the matrix $A$. Then $\lambda_i = \sum_{j=1}^{n} b_{ij}\lambda_j'$ and

$$\begin{aligned} \mathfrak{m} &= \textstyle\sum_{i=1}^{n} \mathbb{Z}\lambda_i &&= \sum_{i=1}^{n} \mathbb{Z} \sum_{j=1}^{n} b_{ij}\lambda_j' = \\ &= \textstyle\sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ij}\mathbb{Z}\right) \lambda_j' &&\subseteq \mathfrak{m}'. \end{aligned}$$

Since both $\mathfrak{m}' \subseteq \mathfrak{m}$ and $\mathfrak{m} \subseteq \mathfrak{m}'$, we conclude that $\mathfrak{m}' = \mathfrak{m}$. Lemma 4.1.1 follows. □

**Definition 2.1.4** *Two measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are said to equivalent, if there exists an automorphism $h \in Mod\ (X)$, which sends the leaves of the foliation $\mathcal{F}$ to the leaves of the foliation $\mathcal{F}'$.*

**Remark 2.1.5** The equivalence relation involves the topological foliations, i.e. projective classes of the measured foliations, see [Thurston 1988] [105] for the details.

**Lemma 2.1.2** *Let $\mathcal{F}, \mathcal{F}'$ be the equivalent measured foliations on a surface $X$. Then*

$$Jac\ (\mathcal{F}') = \mu\ Jac\ (\mathcal{F}),$$

*where $\mu > 0$ is a real number.*

*Proof.* Let $h : X \to X$ be an automorphism of the surface $X$. Denote by $h_*$ its action on $H_1(X, \mathbf{Sing}\ (\mathcal{F}); \mathbb{Z})$ and by $h^*$ on $H^1(X; \mathbb{R})$ connected by the formula:

$$\int_{h_*(\gamma)} \omega = \int_\gamma h^*(\omega), \qquad \forall \gamma \in H_1(X, \mathbf{Sing}\ (\mathcal{F}); \mathbb{Z}), \qquad \forall \omega \in H^1(X; \mathbb{R}).$$

Let $\omega, \omega' \in H^1(X; \mathbb{R})$ be the closed 1-forms whose trajectories define the foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Since $\mathcal{F}, \mathcal{F}'$ are equivalent measured foliations,

$$\omega' = \mu\ h^*(\omega)$$

for a $\mu > 0$.

Let $Jac\ (\mathcal{F}) = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $Jac\ (\mathcal{F}') = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$. Then:

$$\lambda'_i = \int_{\gamma_i} \omega' = \mu \int_{\gamma_i} h^*(\omega) = \mu \int_{h_*(\gamma_i)} \omega, \qquad 1 \le i \le n.$$

By lemma 4.1.1, it holds:

$$Jac\ (\mathcal{F}) = \sum_{i=1}^n \mathbb{Z} \int_{\gamma_i} \omega = \sum_{i=1}^n \mathbb{Z} \int_{h_*(\gamma_i)} \omega.$$

Therefore:

$$Jac\ (\mathcal{F}') = \sum_{i=1}^n \mathbb{Z} \int_{\gamma_i} \omega' = \mu \sum_{i=1}^n \mathbb{Z} \int_{h_*(\gamma_i)} \omega = \mu\ Jac\ (\mathcal{F}).$$

Lemma 4.1.2 follows. □

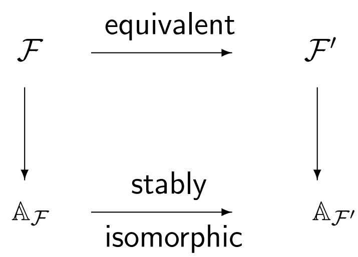



Figure 2.3: Functor on measured foliations.

### 2.1.4 Equivalent foliations

Recall that for a measured foliation $\mathcal{F}$, we constructed an AF-algebra, $\mathbb{A}_{\mathcal{F}}$. Our goal is to prove commutativity of the diagram in Fig. 4.3.; in other words, two equivalent measured foliations map to the stably isomorphic (Morita equivalent) AF-algebras $\mathbb{A}_{\mathcal{F}}$.

**Lemma 2.1.3 (Perron)** *Let $\mathfrak{m} = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $\mathfrak{m}' = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$ be two $\mathbb{Z}$-modules, such that $\mathfrak{m}' = \mu\mathfrak{m}$ for a $\mu > 0$. Then the Jacobi-Perron continued fractions of the vectors $\lambda$ and $\lambda'$ coincide except, possibly, at a finite number of terms.*

*Proof.* Let $\mathfrak{m} = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $\mathfrak{m}' = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$. Since $\mathfrak{m}' = \mu\mathfrak{m}$, where $\mu$ is a positive real, one gets the following identity of the $\mathbb{Z}$-modules:

$$\mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n = \mathbb{Z}(\mu\lambda_1) + \ldots + \mathbb{Z}(\mu\lambda_n).$$

One can always assume that $\lambda_i$ and $\lambda'_i$ are positive reals. For obvious reasons, there exists a basis $\{\lambda''_1, \ldots, \lambda''_n\}$ of the module $\mathfrak{m}'$, such that:

$$\left\{\begin{array}{ll} \lambda'' & = A(\mu\lambda) \\ \lambda'' & = A'\lambda', \end{array}\right.$$

where $A, A' \in GL^+(n, \mathbb{Z})$ are the matrices, whose entries are non-negative integers. In view of [Bauer 1996] [6], Proposition 3, we have

$$\left\{\begin{array}{ll} A & = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \\ A' & = \begin{pmatrix} 0 & 1 \\ I & b'_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b'_l \end{pmatrix}, \end{array}\right.$$

where $b_i, b_i'$ are non-negative integer vectors. Since the Jacobi-Perron continued fraction for the vectors $\lambda$ and $\mu\lambda$ coincide for any $\mu > 0$ (see e.g. [Bernstein 1971] [7]), we conclude that:

$$\begin{cases} \begin{pmatrix} 1 \\ \theta \end{pmatrix} & = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \dots \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} \\ \begin{pmatrix} 1 \\ \theta' \end{pmatrix} & = \begin{pmatrix} 0 & 1 \\ I & b_1' \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ I & b_l' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \dots \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \end{cases}$$

where

$$\begin{pmatrix} 1 \\ \theta'' \end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ I & a_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}.$$

In other words, the continued fractions of the vectors $\lambda$ and $\lambda'$ coincide but at a finite number of terms. Lemma 4.1.3 follows. $\square$

**Lemma 2.1.4 (Basic lemma)** *Let $\mathcal{F}$ and $\mathcal{F}'$ be equivalent measured foliations on a surface $X$. Then the AF-algebras $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are stably isomorphic.*

*Proof.* Notice that lemma 4.1.2 implies that equivalent measured foliations $\mathcal{F}, \mathcal{F}'$ have proportional jacobians, i.e. $\mathfrak{m}' = \mu\mathfrak{m}$ for a $\mu > 0$. On the other hand, by lemma 4.1.3 the continued fraction expansion of the basis vectors of the proportional jacobians must coincide, except a finite number of terms. Thus, the AF-algebras $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are given by the Bratteli diagrams, which are identical, except a finite part of the diagram. It is well known (see e.g. [Effros 1981] [21], Theorem 2.3), that the AF-algebras, which have such a property, are stably isomorphic. Lemma 4.1.4 follows. $\square$

### 2.1.5 Proofs

#### Proof of theorem 4.1.1

Let $\phi \in Mod\ (X)$ be a pseudo-Anosov automorphism of the surface $X$. Denote by $\mathcal{F}_\phi$ the invariant foliation of $\phi$. By definition of such a foliation, $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$, where $\lambda_\phi > 1$ is the dilatation of $\phi$. Consider the jacobian $Jac\ (\mathcal{F}_\phi) = \mathfrak{m}_\phi$ of foliation $\mathcal{F}_\phi$. Since $\mathcal{F}_\phi$ is an invariant foliation of the pseudo-Anosov automorphism $\phi$, one gets the following equality of the $\mathbb{Z}$-modules:

$$\mathfrak{m}_\phi = \lambda_\phi \mathfrak{m}_\phi, \qquad \lambda_\phi \neq \pm 1.$$

Let $\{v^{(1)}, \ldots, v^{(n)}\}$ be a basis in module $\mathfrak{m}_\phi$, such that $v^{(i)} > 0$; from the above equation, one obtains the following system of linear equations:

$$\left\{\begin{array}{lll}\lambda_\phi v^{(1)} & = & a_{11}v^{(1)} + a_{12}v^{(2)} + \ldots + a_{1n}v^{(n)}\\ \lambda_\phi v^{(2)} & = & a_{21}v^{(1)} + a_{22}v^{(2)} + \ldots + a_{2n}v^{(n)}\\ \quad\vdots & & \\ \lambda_\phi v^{(n)} & = & a_{n1}v^{(1)} + a_{n2}v^{(2)} + \ldots + a_{nn}v^{(n)},\end{array}\right.$$

where $a_{ij} \in \mathbb{Z}$. The matrix $A = (a_{ij})$ is invertible. Indeed, since foliation $\mathcal{F}_\phi$ is minimal, real numbers $v^{(1)}, \ldots, v^{(n)}$ are linearly independent over $\mathbb{Q}$. So do numbers $\lambda_\phi v^{(1)}, \ldots, \lambda_\phi v^{(n)}$, which therefore can be taken for a basis of the module $\mathfrak{m}_\phi$. Thus, there exists an integer matrix $B = (b_{ij})$, such that $v^{(j)} = \sum_{i,j} w^{(i)}$, where $w^{(i)} = \lambda_\phi v^{(i)}$. Clearly, $B$ is an inverse to matrix $A$. Therefore, $A \in GL(n, \mathbb{Z})$.

Moreover, without loss of the generality one can assume that $a_{ij} \geq 0$. Indeed, if it is not yet the case, consider the conjugacy class $[A]$ of the matrix $A$. Since $v^{(i)} > 0$, there exists a matrix $A^+ \in [A]$ whose entries are non-negative integers. One has to replace basis $v = (v^{(1)}, \ldots, v^{(n)})$ in the module $\mathfrak{m}_\phi$ by a basis $Tv$, where $A^+ = TAT^{-1}$. It will be further assumed that $A = A^+$.

**Lemma 2.1.5** *Vector $(v^{(1)}, \ldots, v^{(n)})$ is the limit of a periodic Jacobi-Perron continued fraction.*

*Proof.* It follows from the discussion above, that there exists a non-negative integer matrix $A$, such that $Av = \lambda_\phi v$. In view of [Bauer 1996] [6], Proposition 3, matrix $A$ admits the unique factorization

$$A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix},$$

where $b_i = (b_1^{(i)}, \ldots, b_n^{(i)})^T$ are vectors of the non-negative integers. Let us consider the periodic Jacobi-Perron continued fraction

$$Per\ \overline{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}.$$

According to [Perron 1907] [81], **Satz XII**, the above fraction converges to vector $w = (w^{(1)}, \ldots, w^{(n)})$, such that $w$ satisfies equation $(B_1 B_2 \ldots B_k)w = Aw = \lambda_\phi w$. In view of equation $Av = \lambda_\phi v$, we conclude that vectors $v$ and

$w$ are collinear. Therefore, the Jacobi-Perron continued fractions of $v$ and $w$ must coincide. Lemma 4.1.5 follows. $\square$

It is easy to see, that the AF-algebra attached to foliation $\mathcal{F}_\phi$ is stationary. Indeed, by lemma 4.1.5, the vector of periods $v^{(i)} = \int_{\gamma_i} \omega$ unfolds into a periodic Jacobi-Perron continued fraction. By definition, the Bratteli diagram of the AF-algebra $\mathbb{A}_\phi$ is periodic as well. In other words, the AF-algebra $\mathbb{A}_\phi$ is stationary. Theorem 4.1.1 is proved. $\square$

### Proof of theorem 4.1.2

(i) Let us prove the first statement. For the sake of completeness, let us give a proof of the following well-known lemma.

**Lemma 2.1.6** *If $\phi$ and $\phi'$ are conjugate pseudo-Anosov automorphisms of a surface $X$, then their invariant measured foliations $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi'}$ are equivalent.*

*Proof.* Let $\phi, \phi' \in Mod\ (X)$ be conjugate, i.e $\phi' = h \circ \phi \circ h^{-1}$ for an automorphism $h \in Mod\ (X)$. Since $\phi$ is the pseudo-Anosov automorphism, there exists a measured foliation $\mathcal{F}_\phi$, such that $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$. Let us evaluate the automorphism $\phi'$ on the foliation $h(\mathcal{F}_\phi)$:

$$\begin{aligned} \phi'(h(\mathcal{F}_\phi)) &= h\phi h^{-1}(h(\mathcal{F}_\phi)) &&= h\phi(\mathcal{F}_\phi) = \\ &= h\lambda_\phi \mathcal{F}_\phi &&= \lambda_\phi(h(\mathcal{F}_\phi)). \end{aligned}$$

Thus, $\mathcal{F}_{\phi'} = h(\mathcal{F}_\phi)$ is the invariant foliation for the pseudo-Anosov automorphism $\phi'$ and $\mathcal{F}_\phi, \mathcal{F}_{\phi'}$ are equivalent foliations. Note also that the pseudo-Anosov automorphism $\phi'$ has the same dilatation as the automorphism $\phi$. Lemma 4.1.6 follows. $\square$

To finish the proof of item (i), suppose that $\phi$ and $\phi'$ are conjugate pseudo-Anosov automorphisms. Functor $F$ acts by the formulas $\phi \mapsto \mathbb{A}_\phi$ and $\phi' \mapsto \mathbb{A}_{\phi'}$, where $\mathbb{A}_\phi, \mathbb{A}_{\phi'}$ are the AF-algebras corresponding to invariant foliations $\mathcal{F}_\phi, \mathcal{F}_{\phi'}$. In view of lemma 4.1.6, $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi'}$ are equivalent measured foliations. Then, by lemma 4.1.4, the AF-algebras $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$ are stably isomorphic AF-algebras. Item (i) follows.

(ii) Let us prove the second statement. We start with an elementary observation. Let $\phi \in Mod\ (X)$ be a pseudo-Anosov automorphism. Then there exists a unique measured foliation, $\mathcal{F}_\phi$, such that $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$, where

$\lambda_\phi > 1$ is an algebraic integer. Let us evaluate automorphism $\phi^2 \in Mod\ (X)$ on the foliation $\mathcal{F}_\phi$:

$$\begin{array}{lll} \phi^2(\mathcal{F}_\phi) & = \phi(\phi(\mathcal{F}_\phi)) & = \phi(\lambda_\phi \mathcal{F}_\phi) = \\ = \lambda_\phi \phi(\mathcal{F}_\phi) & = \lambda_\phi^2 \mathcal{F}_\phi & = \lambda_{\phi^2} \mathcal{F}_\phi, \end{array}$$

where $\lambda_{\phi^2} := \lambda_\phi^2$. Thus, foliation $\mathcal{F}_\phi$ is an invariant foliation for the automorphism $\phi^2$ as well. By induction, one concludes that $\mathcal{F}_\phi$ is an invariant foliation of the automorphism $\phi^n$ for any $n \geq 1$.

Even more is true. Suppose that $\psi \in Mod\ (X)$ is a pseudo-Anosov automorphism, such that $\psi^m = \phi^n$ for some $m \geq 1$ and $\psi \neq \phi$. Then $\mathcal{F}_\phi$ is an invariant foliation for the automorphism $\psi$. Indeed, $\mathcal{F}_\phi$ is invariant foliation of the automorphism $\psi^m$. If there exists $\mathcal{F}' \neq \mathcal{F}_\phi$ such that the foliation $\mathcal{F}'$ is an invariant foliation of $\psi$, then the foliation $\mathcal{F}'$ is also an invariant foliation of the pseudo-Anosov automorphism $\psi^m$. Thus, by the uniqueness, $\mathcal{F}' = \mathcal{F}_\phi$. We have just proved the following lemma.

**Lemma 2.1.7** *If $[\phi]$ is the set of all pseudo-Anosov automorphisms $\psi$ of $X$, such that $\psi^m = \phi^n$ for some positive integers $m$ and $n$, then the pseudo-Anosov foliation $\mathcal{F}_\phi$ is an invariant foliation for every pseudo-Anosov automorphism $\psi \in [\phi]$.*

In view of lemma 4.1.7, one gets the following identities for the AF-algebras

$$\mathbb{A}_\phi = \mathbb{A}_{\phi^2} = \ldots = \mathbb{A}_{\phi^n} = \mathbb{A}_{\psi^m} = \ldots = \mathbb{A}_{\psi^2} = \mathbb{A}_\psi.$$

Thus, functor $F$ is not an injective functor: the preimage, $Ker\ F$, of algbera $\mathbb{A}_\phi$ consists of a countable set of the pseudo-Anosov automorphisms $\psi \in [\phi]$, commensurable with the automorphism $\phi$. Theorem 4.1.2 is proved. □

### Proof of corollary 4.1.1

(i) Theorem 4.1.1 says that $\mathbb{A}_\phi$ is a stationary AF-algebra. An arithmetic invariant of the stable isomorphism classes of the stationary AF-algebras has been found by D. Handelman in [Handelman 1981] [33]. Summing up his results, the invariant is as follows. Let $A \in GL(n, \mathbb{Z})$ be a matrix with the strictly positive entries, such that $A$ is equal to the minimal period of the Bratteli diagram of the stationary AF-algebra. (In case the matrix $A$ has zero entries, it is necessary to take a proper minimal power of the matrix $A$.) By the Perron-Frobenius theory, matrix $A$ has a real eigenvalue $\lambda_A > 1$, which exceeds the absolute values of other roots of the characteristic polynomial of $A$. Note that $\lambda_A$ is an invertible algebraic integer (the unit). Consider the real algebraic number field $K = \mathbb{Q}(\lambda_A)$ obtained as an extension of the field of the rational numbers by the algebraic number $\lambda_A$. Let $(v_A^{(1)}, \dots, v_A^{(n)})$ be the eigenvector corresponding to the eigenvalue $\lambda_A$. One can normalize the eigenvector so that $v_A^{(i)} \in K$. The departure point of Handelman's invariant is the $\mathbb{Z}$-module $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \dots + \mathbb{Z}v_A^{(n)}$. The module $\mathfrak{m}$ brings in two new arithmetic objects: (i) the ring $\Lambda$ of the endomorphisms of $\mathfrak{m}$ and (ii) an ideal $I$ in the ring $\Lambda$, such that $I = \mathfrak{m}$ after a scaling, see e.g. [Borevich & Shafarevich 1966] [11], Lemma 1, p. 88. The ring $\Lambda$ is an order in the algebraic number field $K$ and therefore one can talk about the ideal classes in $\Lambda$. The ideal class of $I$ is denoted by $[I]$. Omitting the embedding question for the field $K$, the triple $(\Lambda, [I], K)$ is an invariant of the stable isomorphism class of the stationary AF-algebra $\mathbb{A}_\phi$, see [Handelman 1981] [33], §5. Item (i) follows.

(ii) Numerical invariants of the stable isomorphism classes of the stationary AF-algebras can be derived from the triple $(\Lambda, [I], K)$. These invariants are the rational integers – called the determinant and signature – can be obtained as follows. Let $\mathfrak{m}, \mathfrak{m}'$ be the full $\mathbb{Z}$-modules in an algebraic number field $K$. It follows from (i), that if $\mathfrak{m} \neq \mathfrak{m}'$ are distinct as the $\mathbb{Z}$-modules, then the corresponding AF-algebras cannot be stably isomorphic. We wish to find the numerical invariants, which discern the case $\mathfrak{m} \neq \mathfrak{m}'$. It is assumed that a $\mathbb{Z}$-module is given by the set of generators $\{\lambda_1, \dots, \lambda_n\}$. Therefore, the problem can be formulated as follows: find a number attached to the set of generators $\{\lambda_1, \dots, \lambda_n\}$, which does not change on the set of generators $\{\lambda_1', \dots, \lambda_n'\}$ of the same $\mathbb{Z}$-module. One such invariant is associated with the trace function on the algebraic number field $K$. Recall that

$$Tr : K \to \mathbb{Q}$$

is a linear function on $K$ such that $Tr\ (\alpha + \beta) = \mathbf{Tr}\ (\alpha) + \mathbf{Tr}\ (\beta)$ and $\mathbf{Tr}\ (a\alpha) = a\ \mathbf{Tr}\ (\alpha)$ for $\forall \alpha, \beta \in K$ and $\forall a \in \mathbb{Q}$. Let $\mathfrak{m}$ be a full $\mathbb{Z}$-module in the field $K$. The trace function defines a symmetric bilinear form $q(x, y)$ : $\mathfrak{m} \times \mathfrak{m} \to \mathbb{Q}$ by the formula

$$(x, y) \longmapsto \mathbf{Tr}\ (xy), \qquad \forall x, y \in \mathfrak{m}.$$

The form $q(x,y)$ depends on the basis $\{\lambda_1, \dots, \lambda_n\}$ in the module $\mathfrak{m}$

$$q(x,y) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_iy_j, \qquad \text{where} \quad a_{ij} = \mathbf{Tr}\ (\lambda_i\lambda_j).$$

However, the general theory of the bilinear forms (over the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ or the ring of rational integers $\mathbb{Z}$) tells us that certain numerical quantities will not depend on the choice of such a basis.

**Definition 2.1.5** *By a determinant of the bilinear form $q(x,y)$ one understands the rational integer number*

$$\Delta = \mathbf{det}\ (\mathbf{Tr}\ (\lambda_i\lambda_j)).$$

**Lemma 2.1.8** *The determinant $\Delta(\mathfrak{m})$ is independent of the choice of the basis $\{\lambda_1, \dots, \lambda_n\}$ in the module $\mathfrak{m}$.*

*Proof.* Consider a symmetric matrix $A$ corresponding to the bilinear form $q(x,y)$, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}.$$

It is known that the matrix $A$, written in a new basis, will take the form $A' = U^TAU$, where $U \in GL(n,\mathbb{Z})$. Then $\mathbf{det}\ (A') = \mathbf{det}\ (U^TAU) = \mathbf{det}\ (U^T)\mathbf{det}\ (A)\mathbf{det}\ (U) = \mathbf{det}\ (A)$. Therefore, the rational integer number

$$\Delta = \mathbf{det}\ (\mathbf{Tr}\ (\lambda_i\lambda_j)),$$

does not depend on the choice of the basis $\{\lambda_1, \dots, \lambda_n\}$ in the module $\mathfrak{m}$. Lemma 4.1.8 follows. □

**Remark 2.1.6 ($p$-adic invariants)** Roughly speaking, Lemma 4.1.8 says that determinant $\Delta(\mathfrak{m})$ discerns two distinct modules, i.e. $\mathfrak{m} \neq \mathfrak{m}'$. Note that if $\Delta(\mathfrak{m}) = \Delta(\mathfrak{m}')$ for the modules $\mathfrak{m}$ and $\mathfrak{m}'$, one cannot conclude that $\mathfrak{m} = \mathfrak{m}'$. The problem of equivalence of the symmetric bilinear forms over $\mathbb{Q}$ (i.e. the existence of a linear substitution over $\mathbb{Q}$, which transforms one form to the other), is a fundamental question of number theory. The Minkowski-Hasse theorem says that two such forms are equivalent if and only if they are equivalent over the $p$-adic field $\mathbf{Q}_p$ for every prime number $p$ and over the field $\mathbb{R}$. Clearly, the resulting $p$-adic quantities will give new invariants of the stable isomorphism classes of the AF-algebras. The question is much similar to the Minkowski units attached to knots, see e.g. [Reidemeister 1932] [87].

**Definition 2.1.6** *By a signature of the bilinear form $q(x, y)$ one understands the rational integer $\Sigma = (\#a_i^+) - (\#a_i^-)$, where $a_i^+$ are the positive and $a_i^-$ the negative entries in the diagonal form*

$$a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2$$

*of $q(x, y)$; recall that each $q(x, y)$ can be brought by an integer linear transformation to the diagonal form.*

**Lemma 2.1.9** *The signature $\Sigma(\mathfrak{m})$ is independent of the choice of basis in the module $\mathfrak{m}$ and, therefore, $\Sigma(\mathfrak{m}) \neq \Sigma(\mathfrak{m}')$ implies $\mathfrak{m} \neq \mathfrak{m}'$.*

*Proof.* The claim follows from the Law of Inertia for the signature of the bilinear form $q(x, y)$. $\square$

Corollary 4.1.1 follows from Lemmas 4.1.8 and 4.1.9. $\square$

### 2.1.6 Anosov maps of the torus

We shall calculate the noncommutative invariants $\Delta(\mathfrak{m})$ and $\Sigma(\mathfrak{m})$ for the Anosov automorphisms of the two-dimensional torus; we construct concrete examples of Anosov automorphisms with the same Alexander polynomial $\Delta(t)$ but different invariant $\Delta(\mathfrak{m})$, i.e. showing that $\Sigma(\mathfrak{m})$ is *finer* than $\Delta(t)$. Recall that isotopy classes of the orientation-preserving diffeomorphisms of the torus $T^2$ are bijective with the $2 \times 2$ matrices with integer entries and determinant $+1$, i.e. $Mod\ (T^2) \cong SL(2, \mathbb{Z})$. Under the identification, the non-periodic automorphisms correspond to the matrices $A \in SL(2, \mathbb{Z})$ with $|\mathbf{Tr}\ A| > 2$. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension of the field of rational numbers $\mathbb{Q}$. Further we suppose that $d$ is a positive square free integer. Let

$$\omega = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1\ mod\ 4, \\ \sqrt{d} & \text{if } d \equiv 2, 3\ mod\ 4. \end{cases}$$

**Remark 2.1.7** Recall that if $f$ is a positive integer then every order in $K$ has the form $\Lambda_f = \mathbb{Z} + (f\omega)\mathbb{Z}$, where $f$ is the conductor of $\Lambda_f$, see e.g. [Borevich & Shafarevich 1966] [11], pp. 130-132. This formula allows to classify the similarity classes of the full modules in the field $K$. Indeed, there exists a finite number of $\mathfrak{m}_f^{(1)}, \dots, \mathfrak{m}_f^{(s)}$ of the non-similar full modules in the field $K$ whose coefficient ring is the order $\Lambda_f$, see [Borevich & Shafarevich 1966] [11], Chapter 2.7, Theorem 3. Thus one gets a finite-to-one classification of the similarity classes of full modules in the field $K$.

### Numerical invariants of the Anosov maps

Let $\Lambda_f$ be an order in $K$ with the conductor $f$. Under the addition operation, the order $\Lambda_f$ is a full module, which we denote by $\mathfrak{m}_f$. Let us evaluate the invariants $q(x,y)$, $\Delta$ and $\Sigma$ on the module $\mathfrak{m}_f$. To calculate $(a_{ij}) = \mathbf{Tr}\,(\lambda_i\lambda_j)$, we let $\lambda_1 = 1, \lambda_2 = f\omega$. Then:

$$\begin{aligned} a_{11} &= 2, \quad a_{12} = a_{21} = f, \quad a_{22} = \frac{1}{2}f^2(d+1) \quad \text{if} \quad d \equiv 1 \ mod\ 4 \\ a_{11} &= 2, \quad a_{12} = a_{21} = 0, \quad a_{22} = 2f^2 d \quad \text{if} \quad d \equiv 2,3 \ mod\ 4, \end{aligned}$$

and

$$\begin{aligned} q(x,y) &= 2x^2 + 2fxy + \frac{1}{2}f^2(d+1)y^2 \quad \text{if} \quad d \equiv 1 \ mod\ 4 \\ q(x,y) &= 2x^2 + 2f^2 dy^2 \quad \text{if} \quad d \equiv 2,3 \ mod\ 4. \end{aligned}$$

Therefore

$$\Delta(\mathfrak{m}_f) = \begin{cases} f^2 d & \text{if } d \equiv 1 \ mod\ 4, \\ 4f^2 d & \text{if } d \equiv 2,3 \ mod\ 4 \end{cases}$$

and

$$\Sigma(\mathfrak{m}_f) = +2.$$

**Example 2.1.2** Consider the Anosov maps $\phi_A, \phi_B : T^2 \to T^2$ given by matrices

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix},$$

respectively. The reader can verify that the Alexander polynomials of $\phi_A$ and $\phi_B$ are identical and equal to $\Delta_A(t) = \Delta_B(t) = t^2 - 6t + 1$; yet $\phi_A$ and $\phi_B$ are *not* conjugate. Indeed, the Perron-Frobenius eigenvector of matrix $A$ is $v_A = (1, \sqrt{2} - 1)$ while of the matrix $B$ is $v_B = (1, 2\sqrt{2} - 2)$. The bilinear forms for the modules $\mathfrak{m}_A = \mathbb{Z} + (\sqrt{2}-1)\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (2\sqrt{2}-2)\mathbb{Z}$ can be written as

$$q_A(x,y) = 2x^2 - 4xy + 6y^2, \qquad q_B(x,y) = 2x^2 - 8xy + 24y^2,$$

respectively. The modules $\mathfrak{m}_A, \mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{2})$, since their determinants $\Delta(\mathfrak{m}_A) = 8$ and $\Delta(\mathfrak{m}_B) = 32$ are not equal. Therefore, matrices $A$ and $B$ are not similar in the group $SL(2,\mathbb{Z})$. Note that the class number $h_K = 1$ for the field $K$.

**Remark 2.1.8 (Gauss method)** The reader can verify that $A$ and $B$ are non-similar by using the *method of periods*, which dates back to C. -F. Gauss. According to the algorithm, we have to find the fixed points $Ax = x$ and $Bx = x$, which gives us $x_A = 1 + \sqrt{2}$ and $x_B = \frac{1+\sqrt{2}}{2}$, respectively. Then one unfolds the fixed points into a periodic continued fraction, which gives us $x_A = [2,2,2,\dots]$ and $x_B = [1,4,1,4,\dots]$. Since the period $(\overline{2})$ of $x_A$ differs from the period $(\overline{1,4})$ of $B$, the matrices $A$ and $B$ belong to different similarity classes in $SL(2,\mathbb{Z})$.

**Example 2.1.3** Consider the Anosov maps $\phi_A, \phi_B : T^2 \to T^2$ given by matrices

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix},$$

respectively. The Alexander polynomials of $\phi_A$ and $\phi_B$ are identical $\Delta_A(t) = \Delta_B(t) = t^2 - 8t + 1$; yet the automorphisms $\phi_A$ and $\phi_B$ are not conjugate. Indeed, the Perron-Frobenius eigenvector of matrix $A$ is $v_A = (1, \frac{1}{3}\sqrt{15})$ while of the matrix $B$ is $v_B = (1, \frac{1}{15}\sqrt{15})$. The corresponding modules are $\mathfrak{m}_A = \mathbb{Z} + (\frac{1}{3}\sqrt{15})\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (\frac{1}{15}\sqrt{15})\mathbb{Z}$; therefore

$$q_A(x,y) = 2x^2 + 18y^2, \qquad q_B(x,y) = 2x^2 + 450y^2,$$

respectively. The modules $\mathfrak{m}_A, \mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{15})$, since the module determinants $\Delta(\mathfrak{m}_A) = 36$ and $\Delta(\mathfrak{m}_B) = 900$ are not equal. Therefore, matrices $A$ and $B$ are not similar in the group $SL(2,\mathbb{Z})$.

**Example 2.1.4 ([Handelman 2009] [34], p.12***)* Let $a, b$ be a pair of positive integers satisfying the Pell equation $a^2 - 8b^2 = 1$; the latter has infinitely many solutions, e.g. $a = 3, b = 1$, *etc.* Denote by $\phi_A, \phi_B : T^2 \to T^2$ the Anosov maps given by matrices

$$A = \begin{pmatrix} a & 4b \\ 2b & a \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} a & 8b \\ b & a \end{pmatrix},$$

respectively. The Alexander polynomials of $\phi_A$ and $\phi_B$ are identical $\Delta_A(t) = \Delta_B(t) = t^2 - 2at + 1$; yet maps $\phi_A$ and $\phi_B$ are *not* conjugate. Indeed, the Perron-Frobenius eigenvector of matrix $A$ is $v_A = (1, \frac{1}{4b}\sqrt{a^2-1})$ while of the matrix $B$ is $v_B = (1, \frac{1}{8b}\sqrt{a^2-1})$. The corresponding modules are $\mathfrak{m}_A = \mathbb{Z} + (\frac{1}{4b}\sqrt{a^2-1})\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (\frac{1}{8b}\sqrt{a^2-1})\mathbb{Z}$. It is easy to see that the discriminant $d = a^2 - 1 \equiv 3 \mod 4$ for all $a \geq 2$. Indeed, $d = (a-1)(a+1)$ and, therefore, integer $a \not\equiv 1; 3 \mod 4$; hence $a \equiv 2 \mod 4$ so that $a - 1 \equiv 1 \mod 4$ and $a + 1 \equiv 3 \mod 4$ and, thus, $d = a^2 - 1 \equiv 3 \mod 4$. Therefore the corresponding conductors are $f_A = 4b$ and $f_B = 8b$, and

$$q_A(x,y) = 2x^2 + 32b^2(a^2-1)y^2, \quad q_B(x,y) = 2x^2 + 128b^2(a^2-1)y^2,$$

respectively. The modules $\mathfrak{m}_A, \mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{a^2-1})$, since their determinants $\Delta(\mathfrak{m}_A) = 64\ b^2(a^2-1)$ and $\Delta(\mathfrak{m}_B) = 256\ b^2(a^2-1)$ are not equal. Therefore, matrices $A$ and $B$ are not similar in the group $SL(2,\mathbb{Z})$.

**Guide to the literature.** The topology of surface automorphisms is the oldest part of geometric topology; it dates back to the works of J. Nielsen [Nielsen 1927; 1929; 1932] [61] and M. Dehn [Dehn 1938] [19]. W. Thurston proved that that there are only three types of such automorphisms: they are either of finite order, or pseudo-Anosov or else a mixture of the two, see [Thurston 1988] [105]; the topological classification of pseudo-Anosov automorphisms is the next biggest problem after the *Geometrization Conjecture* proved by G. Perelman, see [Thurston 1982] [104]. An excellent introduction to the subject are the books [Fathi, Laudenbach & Poénaru 1979] [24] and [Casson & Bleiler 1988] [15]. The measured foliations on compact surfaces were introduced in 1970's by W. Thurston [Thurston 1988] [105] and covered in [Hubbard & Masur 1979] [41]. The Jacobi-Perron algorithm can be found in [Perron 1907] [81] and [Bernstein 1971] [7]. The noncommutative invariants of pseudo-Anosov automorphisms were constructed in [69].

## 2.2 Torsion in the torus bundles

We assume that $M_\alpha$ is a *torus bundle*, i.e. an $(n+1)$-dimensional manifold fibering over the circle with monodromy $\alpha : T^n \to T^n$, where $T^n$ is the

$n$-dimensional torus. Roughly speaking, we want to construct a covariant functor on such bundles with the values in a category of the Cuntz-Krieger algebras, see Section 3.7; such a functor must map homeomorphic bundles $M_\alpha$ to the stably isomorphic (Morita equivalent) Cuntz-Krieger algebras. The functor (called a *Cuntz-Krieger functor*) is constructed below and it is proved that the $K$-theory of the Cuntz-Krieger algebra is linked to the *torsion subgroup* of the first homology group of $M_\alpha$. The Cuntz-Krieger functor can be regarded as an "abelianized" version of functor $\mathbf{F} : \alpha \mapsto \mathbb{A}_\alpha$ constructed in Section 4.1, see Remark 4.2.2.

### 2.2.1 Cuntz-Krieger functor

**Definition 2.2.1** *If $T^n$ is a torus of dimension $n \geq 1$, then by a torus bundle one understands an $(n+1)$-dimensional manifold*

$$M_\alpha = \{T^n \times [0,1] \mid (T^n, 0) = (\alpha(T^n), 1)\},$$

*where $\alpha : T^n \to T^n$ is an automorphism of $T^n$.*

**Remark 2.2.1** The torus bundles $M_\alpha$ and $M_{\alpha'}$ are homeomorphic, if and only if the automorphisms $\alpha$ and $\alpha'$ are conjugate, i.e. $\alpha' = \beta \circ \alpha \circ \beta^{-1}$ for an automorphism $\beta : T^n \to T^n$.

Let $H_1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n$ be the first homology of torus; consider the group $Aut\ (T^n)$ of (homotopy classes of) automorphisms of $T^n$. Any $\alpha \in Aut\ (T^n)$ induces a linear transformation of $H_1(T^n; \mathbb{Z})$, given by an invertible $n \times n$ matrix $A$ with the integer entries; conversely, each $A \in GL(n, \mathbb{Z})$ defines an automorphism $\alpha : T^n \to T^n$. In this matrix representation, the conjugate automorphisms $\alpha$ and $\alpha'$ define similar matrices $A, A' \in GL(n, \mathbb{Z})$, i.e. such that $A' = BAB^{-1}$ for a matrix $B \in GL(n, \mathbb{Z})$. Each class of matrices, similar to a matrix $A \in GL(n, \mathbb{Z})$ and such that $tr\ (A) \geq 0$ ($tr\ (A) \leq 0$), contains a matrix with only the non-negative (non-positive) entries. We always assume, that our bundle $M_\alpha$ is given by a non-negative matrix $A$; the matrices with $tr\ (A) \leq 0$ can be reduced to this case by switching the sign (from negative to positive) in the respective non-positive representative.

**Definition 2.2.2** *Denote by $\mathcal{M}$ a category of torus bundles (of fixed dimension) endowed with homeomorphisms between the bundles; denote by $\mathcal{A}$ a category of the Cuntz-Krieger algebras $\mathcal{O}_A$ with det $(A) = \pm 1$, endowed* *with stable isomorphisms between the algebras. By a Cuntz-Krieger map* $F : \mathcal{M} \to \mathcal{A}$ *one understands the map given by the formula*

$$M_\alpha \mapsto \mathcal{O}_A.$$

**Theorem 2.2.1 (Functor on torus bundles)** *The map* $F$ *is a covariant functor, which induces an isomorphism between the abelian groups*

$$H_1(M_\alpha; \mathbb{Z}) \cong \mathbb{Z} \oplus K_0(F(M_\alpha)).$$

**Remark 2.2.2** The functor $F : M_\alpha \mapsto \mathcal{O}_A$ can be obtained from functor $\mathbf{F} : \alpha \mapsto \mathbb{A}_\alpha$ on automorphisms $\alpha : T^n \to T^n$ with values in the stationary AF-algebras $\mathbb{A}_\alpha$, see Section 4.1; the correspondence between $F$ and $\mathbf{F}$ comes from the canonical isomorphism

$$\mathcal{O}_A \otimes \mathcal{K} \cong \mathbb{A}_\alpha \rtimes_\sigma \mathbb{Z},$$

where $\sigma$ is the shift automorphism of $\mathbb{A}_\alpha$, see Section 3.7. Thus, one can interpret the invariant $\mathbb{Z} \oplus K_0(\mathcal{O}_A)$ as "abelianized" Handelman's invariant $(\Lambda, [I], K)$ of algebra $\mathbb{A}_\alpha$; here we assume that $(\Lambda, [I], K)$ is an analog of the fundamental group $\pi_1(M_\alpha)$.

### 2.2.2 Proof of theorem 4.2.1

The idea of proof consists in a reduction of the conjugacy problem for the automorphisms of $T^n$ to the Cuntz-Krieger theorem on the flow equivalence of the subshifts of finite type, see e.g. [Lind & Marcus 1995] [51]. For a different proof of Theorem 4.2.1, see [Rodrigues & Ramos 2005] [90].

(i) The main reference to the subshifts of finite type (SFT) is [Lind & Marcus 1995] [51]. Recall that a *full* Bernoulli $n$-shift is the set $X_n$ of bi-infinite sequences $x = \{x_k\}$, where $x_k$ is a symbol taken from a set $S$ of cardinality $n$. The set $X_n$ is endowed with the product topology, making $X_n$ a Cantor set. The shift homeomorphism $\sigma_n : X_n \to X_n$ is given by the formula $\sigma_n(\dots x_{k-1}x_k x_{k+1}\dots) = (\dots x_k x_{k+1} x_{k+2}\dots)$ The homeomorphism defines a (discrete) dynamical system $\{X_n, \sigma_n\}$ given by the iterations of $\sigma_n$. Let $A$ be an $n \times n$ matrix, whose entries $a_{ij} := a(i,j)$ are 0 or 1. Consider a subset $X_A$ of $X_n$ consisting of the bi-infinite sequences, which satisfy the restriction $a(x_k, x_{k+1}) = 1$ for all $-\infty < k < \infty$. (It takes a moment to verify that $X_A$ is indeed a subset of $X_n$ and $X_A = X_n$, if and only if, all the entries of $A$ are 1's.) By definition, $\sigma_A = \sigma_n \mid X_A$ and the pair $\{X_A, \sigma_A\}$ is called a *SFT*. A standard edge shift construction described in [Lind & Marcus 1995] [51] allows to extend the notion of SFT to any matrix $A$ with the non-negative entries. It is well known that the SFT's $\{X_A, \sigma_A\}$ and $\{X_B, \sigma_B\}$ are topologically conjugate (as the dynamical systems), if and only if, the matrices $A$ and $B$ are *strong shift equivalent* (SSE), see [Lind & Marcus 1995] [51] for the corresponding definition. The SSE of two matrices is a difficult algorithmic problem, which motivates the consideration of a weaker equivalence between the matrices called a *shift equivalence* (SE). Recall, that the matrices $A$ and $B$ are said to be shift equivalent (over $\mathbb{Z}^+$), when there exist non-negative matrices $R$ and $S$ and a positive integer $k$ (a lag), satisfying the equations $AR = RB, BS = SA, A^k = RS$ and $SR = B^k$. Finally, the SFT's $\{X_A, \sigma_A\}$ and $\{X_B, \sigma_B\}$ (and the matrices $A$ and $B$) are said to be *flow equivalent* (FE), if the suspension flows of the SFT's act on the topological spaces, which are homeomorphic under a homeomorphism that respects the orientation of the orbits of the suspension flow. We shall use the following implications

$$SSE \Rightarrow SE \Rightarrow FE.$$

**Remark 2.2.3** The first implication is rather classical, while for the second we refer the reader to [Lind & Marcus 1995] [51], p. 456.

We further restrict to the SFT's given by the matrices with determinant $\pm 1$. In view of [Wagoner 1999] [106], Corollary 2.13, the matrices $A$ and $B$ with *det* $(A) = \pm 1$ and *det* $(B) = \pm 1$ are SE (over $\mathbb{Z}^+$), if and only if, matrices $A$ and $B$ are similar in $GL_n(\mathbb{Z})$. Let now $\alpha$ and $\alpha'$ be a pair of conjugate automorphisms of $T^n$. Since the corresponding matrices $A$ and $A'$ are similar in $GL_n(\mathbb{Z})$, one concludes that the SFT's $\{X_A, \sigma_A\}$ and $\{X_{A'}, \sigma_{A'}\}$ are SE. In particular, the SFT's $\{X_A, \sigma_A\}$ and $\{X_{A'}, \sigma_{A'}\}$ are FE. One can now apply the known result due to Cuntz and Krieger; it says, that the $C^*$-algebra $\mathcal{O}_A \otimes \mathcal{K}$ is an invariant of the flow equivalence of the irreducible SFT's, see [Cuntz & Krieger 1980] [18], p. 252 and its proof in Section 4 of the same work. Thus, the map $F$ sends the conjugate automorphisms of $T^n$ into the stably isomorphic Cuntz-Krieger algebras, i.e. $F$ is a functor.

Let us show that $F$ is a covariant functor. Consider the commutative diagram in Fig. 4.4, where $A, B \in GL_n(\mathbb{Z})$ and $\mathcal{O}_A, \mathcal{O}_{BAB^{-1}} \in \mathcal{A}$. Let $g_1, g_2$ be the arrows (similarity of matrices) in the upper category and $F(g_1), F(g_2)$ the corresponding arrows (stable isomorphisms) in the lower category. In view

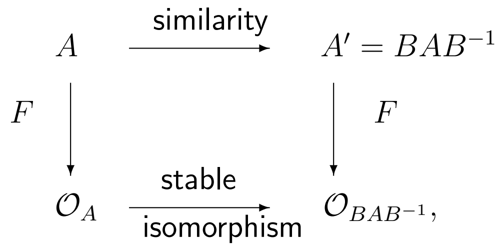



Figure 2.4: Cuntz-Krieger functor.

of the diagram, we have the following identities:

$$\begin{aligned} F(g_1 g_2) &= \mathcal{O}_{B_2 B_1 A B_1^{-1} B_2^{-1}} = \mathcal{O}_{B_2 (B_1 A B_1^{-1}) B_2^{-1}} \\ &= \mathcal{O}_{B_2 A' B_2^{-1}} = F(g_1) F(g_2), \end{aligned}$$

where $F(g_1)(\mathcal{O}_A) = \mathcal{O}_{A'}$ and $F(g_2)(\mathcal{O}_{A'}) = \mathcal{O}_{A''}$. Thus, $F$ does not reverse the arrows and is, therefore, a covariant functor. The first statement of Theorem 4.2.1 is proved.

(ii) Let $M_\alpha$ be a torus bundle with a monodromy, given by the matrix $A \in GL(n, \mathbb{Z})$. It can be calculated, e.g. using the Leray spectral sequence for the fiber bundles, that $H_1(M_\alpha; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}^n/(A - I)\mathbb{Z}^n$. Comparing this calculation with the $K$-theory of the Cuntz-Krieger algebra, one concludes that $H_1(M_\alpha; \mathbb{Z}) \cong \mathbb{Z} \oplus K_0(\mathcal{O}_A)$, where $\mathcal{O}_A = F(M_\alpha)$. The second statement of Theorem 4.2.1 follows. $\square$

### 2.2.3 Noncommutative invariants of torus bundles

To illustrate Theorem 4.2.1, we shall consider concrete examples of the torus bundles and calculate the noncommutative invariant $K_0(\mathcal{O}_A)$ for them. The reader can see, that in some cases $K_0(\mathcal{O}_A)$ is complete invariant of a family of the torus bundles. We compare $K_0(\mathcal{O}_A)$ with the corresponding Alexander polynomial $\Delta(t)$ of the bundle and prove that $K_0(\mathcal{O}_A)$ is *finer* than $\Delta(t)$ (in some cases).

**Example 2.2.1** Consider a three-dimensional torus bundle

$$A_1^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Using the reduction of matrix to its Smith normal form (see e.g. [Lind & Marcus 1995] [51]), one can easily calculate

$$K_0(\mathcal{O}_{A_1^n}) \cong \mathbb{Z} \oplus \mathbb{Z}_n.$$

**Remark 2.2.4** The Cuntz-Krieger invariant $K_0(\mathcal{O}_{A_1^n}) \cong \mathbb{Z} \oplus \mathbb{Z}_n$ is a complete topological invariant of the family of bundles $M_{\alpha_1^n}$; thus, such an invariant solves the classification problem for such bundles.

**Example 2.2.2** Consider a three-dimensional torus bundle

$$A_2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Using the reduction of matrix to its Smith normal form, one gets

$$K_0(\mathcal{O}_{A_2}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

**Example 2.2.3** Consider a three-dimensional torus bundle

$$A_3 = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}.$$

Using the reduction of matrix to its Smith normal form, one obtains

$$K_0(\mathcal{O}_{A_3}) \cong \mathbb{Z}_4.$$

**Remark 2.2.5 ($K_0(\mathcal{O}_A)$ versus the Alexander polynomial)** Note that for the bundles $M_{\alpha_2}$ and $M_{\alpha_3}$ the Alexander polynomial:

$$\Delta_{A_2}(t) = \Delta_{A_3}(t) = t^2 - 6t + 1. \tag{2.1}$$

Therefore, the Alexander polynomial cannot distinguish between the bundles $M_{\alpha_2}$ and $M_{\alpha_3}$; however, since $K_0(\mathcal{O}_{A_2}) \ncong K_0(\mathcal{O}_{A_3})$, Theorem 4.2.1 says that the torus bundles $M_{\alpha_2}$ and $M_{\alpha_3}$ are topologically distinct. Thus the noncommutative invariant $K_0(\mathcal{O}_A)$ is *finer* than the Alexander polynomial.

**Remark 2.2.6** According to the Thurston Geometrization Theorem, the torus bundle $M_{\alpha_1^n}$ is a *nilmanifold* for any $n$, while torus bundles $M_{\alpha_2}$ and $M_{\alpha_3}$ are *solvmanifolds*, see [Thurston 1982] [104].

**Guide to the literature.** For an excellent introduction to the subshifts of finite type we refer the reader to the book by [Lind & Marcus 1995] [51] and survey by [Wagoner 1999] [106]. The Cuntz-Krieger algebras $\mathcal{O}_A$, the abelian group $K_0(\mathcal{O}_A)$ and their connection to the subshifts of finite type were introduced in [Cuntz & Krieger 1980] [18]. Note that Theorem 4.2.1 follows from the results by [Rodrigues & Ramos 2005] [90]; however, our argument is different and the proof is more direct and shorter than in the above cited work. The Cuntz-Krieger functor was constructed in [67].

## 2.3 Obstruction theory for Anosov's bundles

We shall use functors ranging in the category of AF-algebras to study the *Anosov bundles* $M_\varphi$, i.e. mapping tori of the Anosov diffeomorphisms

$$\varphi : M \to M$$

of a smooth manifold $M$, see e.g. [Smale 1967] [98], p. 757. Namely, we construct a covariant functor $F$ form the category of Anosov's bundles to a category of stationary AF-algebras; the functor sends each continuous map between the bundles to a stable homomorphism between the corresponding AF-algebras. We develop an *obstruction theory* for continuous maps between Anosov's bundles; such a theory exploits noncommutative invariants derived from the triple $(\Lambda, [I], K)$ attached to stationary AF-algebras. We illustrate the obstruction theory by concrete examples of dimension 2, 3 and 4.

### 2.3.1 Fundamental AF-algebra

By a $q$-dimensional, class $C^r$ foliation of an $m$-dimensional manifold $M$ one understands a decomposition of $M$ into a union of disjoint connected subsets $\{\mathcal{L}_\alpha\}_{\alpha\in A}$, called the *leaves*, see e.g. [Lawson 1974] [50]. They must satisfy the following property: each point in $M$ has a neighborhood $U$ and a system of local class $C^r$ coordinates $x = (x^1, \dots, x^m) : U \to \mathbb{R}^m$ such that for each leaf $\mathcal{L}_\alpha$, the components of $U \cap \mathcal{L}_\alpha$ are described by the equations $x^{q+1} = Const, \dots, x^m = Const$. Such a foliation is denoted by $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha\in A}$. The number $k = m - q$ is called a codimension of the foliation. An example of a codimension $k$ foliation $\mathcal{F}$ is given by a closed $k$-form $\omega$ on $M$: the leaves of $\mathcal{F}$ are tangent to a plane defined by the normal vector $\omega(p) = 0$ at each point $p$ of $M$. The $C^r$-foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of codimension $k$ are said to be $C^s$-conjugate

$(0 \le s \le r)$, if there exists an (orientation-preserving) diffeomorphism of $M$, of class $C^s$, which maps the leaves of $\mathcal{F}_0$ onto the leaves of $\mathcal{F}_1$; when $s = 0$, $\mathcal{F}_0$ and $\mathcal{F}_1$ are *topologically conjugate*. Denote by $f : N \to M$ a map of class $C^s$ $(1 \le s \le r)$ of a manifold $N$ into $M$; the map $f$ is said to be *transverse* to $\mathcal{F}$, if for all $x \in N$ it holds $T_y(M) = \tau_y(\mathcal{F}) + f_* T_x(N)$, where $\tau_y(\mathcal{F})$ are the vectors of $T_y(M)$ tangent to $\mathcal{F}$ and $f_* : T_x(N) \to T_y(M)$ is the linear map on tangent vectors induced by $f$, where $y = f(x)$. If map $f : N \to M$ is transverse to a foliation $\mathcal{F}' = \{\mathcal{L}\}_{\alpha \in A}$ on $M$, then $f$ induces a class $C^s$ foliation $\mathcal{F}$ on $N$, where the leaves are defined as $f^{-1}(\mathcal{L}_\alpha)$ for all $\alpha \in A$; it is immediate, that *codim* $(\mathcal{F})$ = *codim* $(\mathcal{F}')$. We shall call $\mathcal{F}$ an *induced foliation*. When $f$ is a submersion, it is transverse to any foliation of $M$; in this case, the induced foliation $\mathcal{F}$ is correctly defined for all $\mathcal{F}'$ on $M$, see [Lawson 1974] [50], p.373. Notice, that for $M = N$ the above definition corresponds to topologically conjugate foliations $\mathcal{F}$ and $\mathcal{F}'$. To introduce measured foliations, denote by $P$ and $Q$ two $k$-dimensional submanifolds of $M$, which are everywhere transverse to a foliation $\mathcal{F}$ of codimension $k$. Consider a collection of $C^r$ homeomorphisms between subsets of $P$ and $Q$ induced by a return map along the leaves of $\mathcal{F}$. The collection of all such homeomorphisms between subsets of all possible pairs of transverse manifolds generates a *holonomy pseudogroup* of $\mathcal{F}$ under composition of the homeomorphisms, see [Plante 1975] [84], p.329. A foliation $\mathcal{F}$ is said to have measure preserving holonomy, if its holonomy pseudogroup has a non-trivial invariant measure, which is finite on compact sets; for brevity, we call $\mathcal{F}$ a *measured foliation*. An example of measured foliation is a foliation, determined by closed $k$-form $\omega$; the restriction of $\omega$ to a transverse $k$-dimensional manifold determines a volume element, which gives a positive invariant measure on open sets. Each measured foliation $\mathcal{F}$ defines an element of the cohomology group $H^k(M;\mathbb{R})$, see [Plante 1975] [84]; in the case of $\mathcal{F}$ given by a closed $k$-form $\omega$, such an element coincides with the de Rham cohomology class of $\omega$, *ibid.* In view of the isomorphism $H^k(M;\mathbb{R}) \cong Hom\ (H_k(M), \mathbb{R})$, foliation $\mathcal{F}$ defines a linear map $h$ from the $k$-th homology group $H_k(M)$ to $\mathbb{R}$.

**Definition 2.3.1** *By a Plante group $P(\mathcal{F})$ of measured foliation $\mathcal{F}$ one understand the finitely generated abelian subgroup $h(H_k(M)/Tors) \subset \mathbb{R}$.*

**Remark 2.3.1** If $\{\gamma_i\}$ is a basis of the homology group $H_k(M)$, then the periods $\lambda_i = \int_{\gamma_i} \omega$ are generators of the group $P(\mathcal{F})$, see [Plante 1975] [84].

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F})$ of a measured foliation $\mathcal{F}$, such that $\lambda_i > 0$. Take a vector $\theta = (\theta_1, \dots, \theta_{n-1})$ with $\theta_i = \lambda_{i+1}/\lambda_1$; the Jacobi-Perron continued fraction of vector $(1, \theta)$ (or, projective class of vector $\lambda$) is given by the formula

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ I & b_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} = \lim_{i\to\infty} B_i \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$ is a vector of the non-negative integers, $I$ the unit matrix and $\mathbb{I} = (0, \dots, 0, 1)^T$, see [Bernstein 1971] [7], p.13; the $b_i$ are obtained from $\theta$ by the Euclidean algorithm, *ibid.*, pp.2-3.

**Definition 2.3.2** *An AF-algebra given by the Bratteli diagram with the incidence matrices $B_i$ will be called associated to the measured foliation $\mathcal{F}$; we shall denote such an algebra by $\mathbb{A}_{\mathcal{F}}$.*

**Remark 2.3.2** Taking another basis of the Plante group $P(\mathcal{F})$ gives an AF-algebra which is stably isomorphic (Morita equivalent) to $\mathbb{A}_{\mathcal{F}}$; this is an algebraic recast of the main property of the Jacobi-Perron fractions.

If $\mathcal{F}'$ is a measured foliation on a manifold $M$ and $f : N \to M$ is a submersion, then induced foliation $\mathcal{F}$ on $N$ is a measured foliation. We shall denote by **MFol** the category of all manifolds with measured foliations (of fixed codimension), whose arrows are submersions of the manifolds; by $\mathbf{MFol}_0$ we understand a subcategory of **MFol**, consisting of manifolds, whose foliations have a unique transverse measure. Let **AF-alg** be a category of the (isomorphism classes of) AF-algebras given by *convergent* Jacobi-Perron fractions, so that the arrows of **AF-alg** are stable homomorphisms of the AF-algebras. By $F : \mathbf{MFol}_0 \to$ **AF-alg** we denote a map given by the formula $\mathcal{F} \mapsto \mathbb{A}_{\mathcal{F}}$. Notice, that $F$ is correctly defined, since foliations with the unique measure have the convergent Jacobi-Perron fractions; this assertion follows from [Bauer 1996] [6]. The following result will be proved in Section 4.3.2.

**Theorem 2.3.1** *The map $F : \mathbf{MFol}_0 \to$ **AF-alg** is a functor, which sends any pair of induced foliations to a pair of stably homomorphic AF-algebras.*

Let $M$ be an $m$-dimensional manifold and $\varphi : M \to M$ a diffeomorphisms of $M$; recall, that an orbit of point $x \in M$ is the subset $\{\varphi^n(x) \mid n \in \mathbb{Z}\}$ of $M$. The finite orbits $\varphi^m(x) = x$ are called periodic; when $m = 1$, $x$ is a *fixed point* of diffeomorphism $\varphi$. The fixed point $p$ is *hyperbolic* if the eigenvalues $\lambda_i$ of the linear map $D\varphi(p) : T_p(M) \to T_p(M)$ do not lie at the unit circle. If $p \in M$ is a hyperbolic fixed point of a diffeomorphism $\varphi : M \to M$, denote by $T_p(M) = V^s + V^u$ the corresponding decomposition of the tangent space under the linear map $D\varphi(p)$, where $V^s$ ($V^u$) is the eigenspace of $D\varphi(p)$ corresponding to $|\lambda_i| > 1$ ($|\lambda_i| < 1$). For a sub-manifold $W^s(p)$ there exists a contraction $g : W^s(p) \to W^s(p)$ with fixed point $p_0$ and an injective equivariant immersion $J : W^s(p) \to M$, such that $J(p_0) = p$ and $DJ(p_0) : T_{p_0}(W^s(p)) \to T_p(M)$ is an isomorphism; the image of $J$ defines an immersed submanifold $W^s(p) \subset M$ called a *stable manifold* of $\varphi$ at $p$. Clearly, $dim\ (W^s(p)) = dim\ (V^s)$.

**Definition 2.3.3 ([Anosov 1967] [2])** *A diffeomorphism $\varphi : M \to M$ is called Anosov if there exists a splitting of the tangent bundle $T(M)$ into a continuous Whitney sum $T(M) = E^s + E^u$ invariant under $D\varphi : T(M) \to T(M)$, so that $D\varphi : E^s \to E^s$ is contracting and $D\varphi : E^u \to E^u$ is expanding map.*

**Remark 2.3.3** The Anosov diffeomorphism imposes a restriction on topology of manifold $M$, in the sense that not each manifold can support such a diffeomorphism; however, if one Anosov diffeomorphism exists on $M$, there are infinitely many (conjugacy classes of) such diffeomorphisms on $M$. It is an open problem of S. Smale, which $M$ can carry an Anosov diffeomorphism; so far, it is proved that the hyperbolic diffeomorphisms of $m$-dimensional tori and certain automorphisms of the nilmanifolds are Anosov's, see e.g. [Smale 1967] [98].

Let $p$ be a fixed point of the Anosov diffeomorphism $\varphi : M \to M$ and $W^s(p)$ its stable manifold. Since $W^s(p)$ cannot have self-intersections or limit compacta, $W^s(p) \to M$ is a dense immersion, i.e. the closure of $W^s(p)$ is the entire $M$. Moreover, if $q$ is a periodic point of $\varphi$ of period $n$, then $W^s(q)$ is a translate of $W^s(p)$, i.e. locally they look like two parallel lines. Consider a foliation $\mathcal{F}$ of $M$, whose leaves are the translates of $W^s(p)$; the $\mathcal{F}$ is a continuous foliation, which is invariant under the action of diffeomorphism $\varphi$ on its leaves, i.e. $\varphi$ moves leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}$, see [Smale 1967] [98], p.760. The holonomy of $\mathcal{F}$ preserves the Lebesgue measure and, therefore, $\mathcal{F}$ is a measured foliation; we shall call it an *invariant measured foliation* and denote by $\mathcal{F}_\varphi$.

**Definition 2.3.4** *By a fundamental AF-algebra we shall understand the AF-algebra of foliation $\mathcal{F}_\varphi$, where $\varphi : M \to M$ is an Anosov diffeomorphism of a manifold $M$; the fundamental AF-algebra will be denoted by $\mathbb{A}_\varphi$.*

**Theorem 2.3.2** *The $\mathbb{A}_\varphi$ is a stationary AF-algebra.*

Consider the mapping torus of the Anosov diffeomorphism $\varphi$, i.e. a manifold

$$M_\varphi := M \times [0,1] \ / \sim, \text{ where } (x,0) \sim (\varphi(x),1), \ \forall x \in M.$$

Let **AnoBnd** be a category of the mapping tori of all Anosov's diffeomorphisms; the arrows of **AnoBnd** are continuous maps between the mapping tori. Likewise, let **Fund-AF** be a category of all fundamental AF-algebras; the arrows of **Fund-AF** are stable homomorphisms between the fundamental AF-algebras. By $F$ : **AnoBnd** $\to$ **Fund-AF** we understand a map given by the formula $M_\varphi \mapsto \mathbb{A}_\varphi$, where $M_\varphi \in$ **AnoBnd** and $\mathbb{A}_\varphi \in$ **Fund-AF**. The following theorem says that $F$ is a functor.

**Theorem 2.3.3 (Functor on Anosov's bundles)** *The map $F$ is a covariant functor, which sends each continuous map $N_\psi \to M_\varphi$ to a stable homomorphism $\mathbb{A}_\psi \to \mathbb{A}_\varphi$ of the corresponding fundamental AF-algebras.*

**Remark 2.3.4 (Obstruction theory)** Theorem 4.3.3 can be used e.g. in the obstruction theory, because stable homomorphisms of the fundamental AF-algebras are easier to detect, than continuous maps between manifolds $N_\psi$ and $M_\varphi$; such homomorphisms are bijective with the inclusions of certain $\mathbb{Z}$-modules belonging to a real algebraic number field. Often it is possible to prove, that no inclusion is possible and, thus, draw a topological conclusion about the maps, see Section 4.3.3.

## 2.3.2 Proofs

### Proof of Theorem 4.3.1

Let $\mathcal{F}'$ be measured foliation on $M$, given by a closed form $\omega' \in H^k(M;\mathbb{R})$; let $\mathcal{F}$ be measured foliation on $N$, induced by a submersion $f : N \to M$. Roughly speaking, we have to prove, that diagram in Fig. 4.5 is commutative; the proof amounts to the fact, that the periods of form $\omega'$ are contained among the periods of form $\omega \in H^k(N;\mathbb{R})$ corresponding to the foliation $\mathcal{F}$. The map $f$ defines a homomorphism $f_* : H_k(N) \to H_k(M)$ of the $k$-

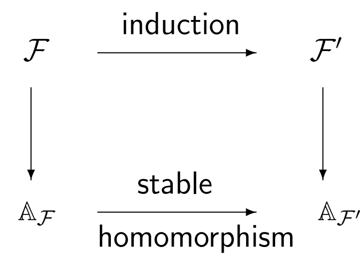



Figure 2.5: $F$ : **MFol**$_0 \to$ **AF-alg** .

th homology groups; let $\{e_i\}$ and $\{e'_i\}$ be a basis in $H_k(N)$ and $H_k(M)$, respectively. Since $H_k(M) = H_k(N) \ / \ ker \ (f_*)$, we shall denote by $[e_i] := e_i + ker \ (f_*)$ a coset representative of $e_i$; these can be identified with the elements $e_i \not\in ker \ (f_*)$. The integral $\int_{e_i} \omega$ defines a scalar product $H_k(N) \times H^k(N; \mathbb{R}) \to \mathbb{R}$, so that $f_*$ is a linear self-adjoint operator; thus, we can write:

$$\lambda'_i = \int_{e'_i} \omega' = \int_{e'_i} f^*(\omega) = \int_{f_*^{-1}(e'_i)} \omega = \int_{[e_i]} \omega \in P(\mathcal{F}),$$

where $P(\mathcal{F})$ is the Plante group (the group of periods) of foliation $\mathcal{F}$. Since $\lambda'_i$ are generators of $P(\mathcal{F}')$, we conclude that $P(\mathcal{F}') \subseteq P(\mathcal{F})$. Note, that $P(\mathcal{F}') = P(\mathcal{F})$ if and only if $f_*$ is an isomorphism.

One can apply a criterion of the stable homomorphism of AF-algebras; namely, $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are stably homomorphic, if and only if, there exists a positive homomorphism $h : G \to H$ between their dimension groups $G$ and $H$, see [Effros 1981] [21], p.15. But $G \cong P(\mathcal{F})$ and $H \cong P(\mathcal{F}')$, while $h = f_*$. Thus, $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are stably homomorphic.

The functor $F$ is compatible with the composition; indeed, let $f : N \to M$ and $f' : L \to N$ be submersions. If $\mathcal{F}$ is a measured foliation of $M$, one gets the induced foliations $\mathcal{F}'$ and $\mathcal{F}''$ on $N$ and $L$, respectively; these foliations fit the diagram $(L, \mathcal{F}'') \xrightarrow{f'} (N, \mathcal{F}') \xrightarrow{f} (M, \mathcal{F})$ and the corresponding Plante groups are included: $P(\mathcal{F}'') \supseteq P(\mathcal{F}') \supseteq P(\mathcal{F})$. Thus, $F(f' \circ f) = F(f') \circ F(f)$, since the inclusion of the Plante groups corresponds to the composition of homomorphisms; Theorem 4.3.1 is proved. $\square$

**Proof of Theorem 4.3.2**

Let $\varphi : M \to M$ be an Anosov diffeomorphism; we proceed by showing, that invariant foliation $\mathcal{F}_\varphi$ is given by form $\omega \in H^k(M;\mathbb{R})$, which is an eigenvector of the linear map $[\varphi] : H^k(M;\mathbb{R}) \to H^k(M;\mathbb{R})$ induced by $\varphi$. Indeed, let $0 < c < 1$ be contracting constant of the stable sub-bundle $E^s$ of diffeomorphism $\varphi$ and $\Omega$ the corresponding volume element; by definition, $\varphi(\Omega) = c\Omega$. Note, that $\Omega$ is given by restriction of form $\omega$ to a $k$-dimensional manifold, transverse to the leaves of $\mathcal{F}_\varphi$. The leaves of $\mathcal{F}_\varphi$ are fixed by $\varphi$ and, therefore, $\varphi(\Omega)$ is given by a multiple $c\omega$ of form $\omega$. Since $\omega \in H^k(M;\mathbb{R})$ is a vector, whose coordinates define $\mathcal{F}_\varphi$ up to a scalar, we conclude, that $[\varphi](\omega) = c\omega$, i.e. $\omega$ is an eigenvector of the linear map $[\varphi]$. Let $(\lambda_1, \dots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F}_\varphi)$, such that $\lambda_i > 0$. Notice, that $\varphi$ acts on $\lambda_i$ as multiplication by constant $c$; indeed, since $\lambda_i = \int_{\gamma_i} \omega$, we have:

$$\lambda_i' = \int_{\gamma_i} [\varphi](\omega) = \int_{\gamma_i} c\omega = c\int_{\gamma_i} \omega = c\lambda_i,$$

where $\{\gamma_i\}$ is a basis in $H_k(M)$. Since $\varphi$ preserves the leaves of $\mathcal{F}_\varphi$, one concludes that $\lambda_i' \in P(\mathcal{F}_\varphi)$; therefore, $\lambda_j' = \sum b_{ij}\lambda_i$ for a non-negative integer matrix $B = (b_{ij})$. According to [Bauer 1996] [6], matrix $B$ can be written as a finite product:

$$B = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ I & b_p \end{pmatrix} := B_1 \dots B_p,$$

where $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$ is a vector of non-negative integers and $I$ the unit matrix. Let $\lambda = (\lambda_1, \dots, \lambda_n)$. Consider a purely periodic Jacobi-Perron continued fraction:

$$\lim_{i\to\infty} \overline{B_1 \dots B_p} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $\mathbb{I} = (0, \dots, 0, 1)^T$; by a basic property of such fractions, it converges to an eigenvector $\lambda' = (\lambda_1', \dots, \lambda_n')$ of matrix $B_1 \dots B_p$, see [Bernstein 1971] [7], Chapter 3. But $B_1 \dots B_p = B$ and $\lambda$ is an eigenvector of matrix $B$; therefore, vectors $\lambda$ and $\lambda'$ are collinear. The collinear vectors are known to have the same continued fractions; thus, we have

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i\to\infty} \overline{B_1 \dots B_p} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $\theta = (\theta_1, \dots, \theta_{n-1})$ and $\theta_i = \lambda_{i+1}/\lambda_1$. Since vector $(1, \theta)$ unfolds into a periodic Jacobi-Perron continued fraction, we conclude, that the AF-algebra $\mathbb{A}_\varphi$ is stationary. Theorem 4.3.2 is proved. □

**Proof of Theorem 4.3.3**

Let $\psi : N \to N$ and $\varphi : M \to M$ be a pair of Anosov diffeomorphisms; denote by $(N, \mathcal{F}_\psi)$ and $(M, \mathcal{F}_\varphi)$ the corresponding invariant foliations of manifolds $N$ and $M$, respectively. In view of Theorem 4.3.1, it is sufficient to prove, that the diagram in Fig.4.6 is commutative. We shall split the proof in a series of lemmas.

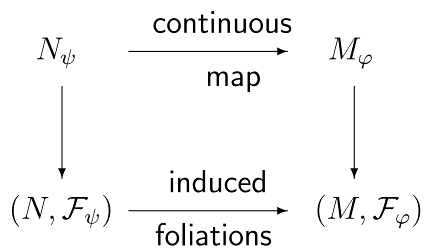



Figure 2.6: Mapping tori and invariant foliations.

**Lemma 2.3.1** *There exists a continuous map $N_\psi \to M_\varphi$, whenever $f \circ \varphi = \psi \circ f$ for a submersion $f : N \to M$.*

*Proof.* (i) Suppose, that $h : N_\psi \to M_\varphi$ is a continuous map; let us show, that there exists a submersion $f : N \to M$, such that $f \circ \varphi = \psi \circ f$. Both $N_\psi$ and $M_\varphi$ fiber over the circle $S^1$ with the projection map $p_\psi$ and $p_\varphi$, respectively; therefore, the diagram in Fig. 4.7 is commutative. Let $x \in S^1$; since $p_\psi^{-1} = N$ and $p_\varphi^{-1} = M$, the restriction of $h$ to $x$ defines a submersion $f : N \to M$, i.e. $f = h_x$. Moreover, since $\psi$ and $\varphi$ are monodromy maps of the bundle, it holds:

$$\begin{cases} p_\psi^{-1}(x + 2\pi) & = \psi(N), \\ p_\varphi^{-1}(x + 2\pi) & = \varphi(M). \end{cases}$$

From the diagram in Fig. 4.7, we get $\psi(N) = p_\psi^{-1}(x + 2\pi) = f^{-1}(p_\varphi^{-1}(x + 2\pi)) = f^{-1}(\varphi(M)) = f^{-1}(\varphi(f(N)))$; thus, $f \circ \psi = \varphi \circ f$. The necessary condition of Lemma 4.3.1 follows.

(ii) Suppose, that $f : N \to M$ is a submersion, such that $f \circ \varphi = \psi \circ f$; we have to construct a continuous map $h : N_\psi \to M_\varphi$. Recall, that

$$\begin{cases} N_\psi & = \{N \times [0, 1] \mid (x, 0) \sim (\psi(x), 1)\}, \\ M_\varphi & = \{M \times [0, 1] \mid (y, 0) \sim (\varphi(y), 1)\}. \end{cases}$$

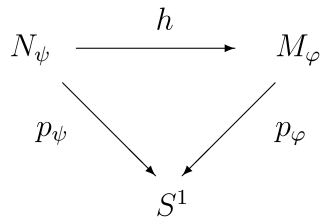



Figure 2.7: The fiber bundles $N_\psi$ and $M_\varphi$ over $S^1$.

We shall identify the points of $N_\psi$ and $M_\varphi$ using the substitution $y = f(x)$; it remains to verify, that such an identification will satisfy the gluing condition $y \sim \varphi(y)$. In view of condition $f \circ \varphi = \psi \circ f$, we have:

$$y = f(x) \sim f(\psi(x)) = \varphi(f(x)) = \varphi(y).$$

Thus, $y \sim \varphi(y)$ and, therefore, the map $h : N_\psi \to M_\varphi$ is continuous. The sufficient condition of lemma 4.3.1 is proved. □

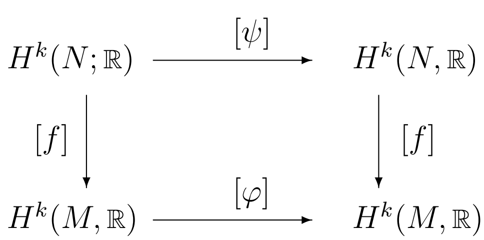



Figure 2.8: The linear maps $[\psi], [\varphi]$ and $[f]$.

**Lemma 2.3.2** *If a submersion $f : N \to M$ satisfies condition $f \circ \varphi = \psi \circ f$ for the Anosov diffeomorphisms $\psi : N \to N$ and $\varphi : M \to M$, then the invariant foliations $(N, \mathcal{F}_\psi)$ and $(M, \mathcal{F}_\varphi)$ are induced by $f$.*

*Proof.* The invariant foliations $\mathcal{F}_\psi$ and $\mathcal{F}_\varphi$ are measured; we shall denote by $\omega_\psi \in H^k(N;\mathbb{R})$ and $\omega_\varphi \in H^k(M;\mathbb{R})$ the corresponding cohomology class, respectively. The linear maps on $H^k(N;\mathbb{R})$ and $H^k(M;\mathbb{R})$ induced by $\psi$

and $\varphi$, we shall denote by $[\psi]$ and $[\varphi]$; the linear map between $H^k(N;\mathbb{R})$ and $H^k(M;\mathbb{R})$ induced by $f$, we write as $[f]$. Notice, that $[\psi]$ and $[\varphi]$ are isomorphisms, while $[f]$ is generally a homomorphism. It was shown earlier, that $\omega_\psi$ and $\omega_\varphi$ are eigenvectors of linear maps $[\psi]$ and $[\varphi]$, respectively; in other words, we have:

$$\begin{cases} [\psi]\omega_\psi & = c_1\omega_\psi, \\ [\varphi]\omega_\varphi & = c_2\omega_\varphi, \end{cases}$$

where $0 < c_1 < 1$ and $0 < c_2 < 1$. Consider a diagram in Fig. 4.8 , which involves the linear maps $[\psi], [\varphi]$ and $[f]$; the diagram is commutative, since condition $f \circ \varphi = \psi \circ f$ implies, that $[\varphi] \circ [f] = [f] \circ [\psi]$. Take the eigenvector $\omega_\psi$ and consider its image under the linear map $[\varphi] \circ [f]$:

$$[\varphi] \circ [f](\omega_\psi) = [f] \circ [\psi](\omega_\psi) = [f](c_1\omega_\psi) = c_1\left([f](\omega_\psi)\right).$$

Therefore, vector $[f](\omega_\psi)$ is an eigenvector of the linear map $[\varphi]$; let compare it with the eigenvector $\omega_\varphi$:

$$\begin{cases} [\varphi]\left([f](\omega_\psi)\right) & = c_1\left([f](\omega_\psi)\right), \\ [\varphi]\omega_\varphi & = c_2\omega_\varphi. \end{cases}$$

We conclude, therefore, that $\omega_\varphi$ and $[f](\omega_\psi)$ are collinear vectors, such that $c_1^m = c_2^n$ for some integers $m, n > 0$; a scaling gives us $[f](\omega_\psi) = \omega_\varphi$. The latter is an analytic formula, which says that the submersion $f : N \to M$ induces foliation $(N, \mathcal{F}_\psi)$ from the foliation $(M, \mathcal{F}_\varphi)$. Lemma 4.3.2 is proved. $\square$

To finish the proof of Theorem 4.3.3, let $N_\psi \to M_\varphi$ be a continuous map; by Lemma 4.3.1, there exists a submersion $f : N \to M$, such that $f \circ \varphi = \psi \circ f$. Lemma 4.3.2 says, that in this case the invariant measured foliations $(N, \mathcal{F}_\psi)$ and $(M, \mathcal{F}_\varphi)$ are induced. On the other hand, from Theorem 4.3.2 we know, that the Jacobi-Perron continued fraction connected to foliations $\mathcal{F}_\psi$ and $\mathcal{F}_\varphi$ are periodic and, hence, convergent, see e.g [Bernstein 1971] [7]; therefore, one can apply Theorem 4.3.1 which says that the AF-algebra $\mathbb{A}_\psi$ is stably homomorphic to the AF-algebra $\mathbb{A}_\varphi$. The latter are, by definition, the fundamental AF-algebras of the Anosov diffeomorphisms $\psi$ and $\varphi$, respectively. Theorem 4.3.3 is proved. $\square$

### 2.3.3 Obstruction theory

Let $\mathbb{A}_\psi$ be a fundamental AF-algebra and $B$ its primitive incidence matrix, i.e. $B$ is not a power of some positive integer matrix. Suppose that the characteristic polynomial of $B$ is irreducible and let $K_\psi$ be its splitting field; then $K_\psi$ is a Galois extension of $\mathbb{Q}$.

**Definition 2.3.5** *We call $Gal\ (\mathbb{A}_\psi) := Gal\ (K_\psi|\mathbb{Q})$ the Galois group of the fundamental AF-algebra $\mathbb{A}_\psi$; such a group is determined by the AF-algebra $\mathbb{A}_\psi$.*

The second algebraic field is connected to the Perron-Frobenius eigenvalue $\lambda_B$ of the matrix $B$; we shall denote this field $\mathbb{Q}(\lambda_B)$. Note, that $\mathbb{Q}(\lambda_B) \subseteq K_\psi$ and $\mathbb{Q}(\lambda_B)$ is not, in general, a Galois extension of $\mathbb{Q}$; the reason being complex roots the polynomial *char* $(B)$ may have and if there are no such roots $\mathbb{Q}(\lambda_B) = K_\psi$. There is still a group $Aut\ (\mathbb{Q}(\lambda_B))$ of automorphisms of $\mathbb{Q}(\lambda_B)$ fixing the field $\mathbb{Q}$ and $Aut\ (\mathbb{Q}(\lambda_B)) \subseteq Gal\ (K_\psi)$ is a subgroup inclusion.

**Lemma 2.3.3** *If $h : \mathbb{A}_\psi \to \mathbb{A}_\varphi$ is a stable homomorphism, then $\mathbb{Q}(\lambda_{B'}) \subseteq K_\psi$ is a field inclusion.*

*Proof.* Notice that the non-negative matrix $B$ becomes strictly positive, when a proper power of it is taken; we always assume $B$ positive. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F}_\psi)$. Following the proof of Theorem 4.3.2, one concludes that $\lambda_i \in K_\psi$; indeed, $\lambda_B \in K_\psi$ is the Perron-Frobenius eigenvalue of $B$ , while $\lambda$ the corresponding eigenvector. The latter can be scaled so, that $\lambda_i \in K_\psi$. Any stable homomorphism $h : \mathbb{A}_\psi \to \mathbb{A}_\varphi$ induces a positive homomorphism of their dimension groups $[h] : G \to H$; but $G \cong P(\mathcal{F}_\psi)$ and $H \cong P(\mathcal{F}_\varphi)$. From inclusion $P(\mathcal{F}_\varphi) \subseteq P(\mathcal{F}_\psi)$, one gets $\mathbb{Q}(\lambda_{B'}) \cong P(\mathcal{F}_\varphi) \otimes \mathbb{Q} \subseteq P(\mathcal{F}_\psi) \otimes \mathbb{Q} \cong \mathbb{Q}(\lambda_B) \subseteq K_f$ and, therefore, $\mathbb{Q}(\lambda_{B'}) \subseteq K_\psi$. Lemma 4.3.3 follows. $\square$

**Corollary 2.3.1** *If $h : \mathbb{A}_\psi \to \mathbb{A}_\varphi$ is a stable homomorphism, then $Aut\ (\mathbb{Q}(\lambda_{B'}))$ (or, $Gal\ (\mathbb{A}_\varphi)$) is a subgroup (or, a normal subgroup) of $Gal\ (\mathbb{A}_\psi)$.*

*Proof.* The (Galois) subfields of the Galois field $K_\psi$ are bijective with the (normal) subgroups of the group $Gal\ (K_\psi)$, see e.g. [Morandi 1996] [55]. $\square$

Let $T^m \cong \mathbb{R}^m/\mathbb{Z}^m$ be an $m$-dimensional torus; let $\psi_0$ be a $m \times m$ integer matrix with $det\ (\psi_0) = 1$, such that it is similar to a positive matrix. The matrix

$\psi_0$ defines a linear transformation of $\mathbb{R}^m$, which preserves the lattice $L \cong \mathbb{Z}^m$ of points with integer coordinates. There is an induced diffeomorphism $\psi$ of the quotient $T^m \cong \mathbb{R}^m/\mathbb{Z}^m$ onto itself; this diffeomorphism $\psi : T^m \to T^m$ has a fixed point $p$ corresponding to the origin of $\mathbb{R}^m$. Suppose that $\psi_0$ is hyperbolic, i.e. there are no eigenvalues of $\psi_0$ at the unit circle; then $p$ is a hyperbolic fixed point of $\psi$ and the stable manifold $W^s(p)$ is the image of the corresponding eigenspace of $\psi_0$ under the projection $\mathbb{R}^m \to T^m$. If *codim* $W^s(p) = 1$, the hyperbolic linear transformation $\psi_0$ (and the diffeomorphism $\psi$) will be called *tight*.

**Lemma 2.3.4** *The tight hyperbolic matrix $\psi_0$ is similar to the matrix $B$ of the fundamental AF-algebra $\mathbb{A}_\psi$.*

*Proof.* Since $H_k(T^m;\mathbb{R}) \cong \mathbb{R}^{\frac{m!}{k!(m-k)!}}$, one gets $H_{m-1}(T^m;\mathbb{R}) \cong \mathbb{R}^m$; in view of the Poincaré duality, $H^1(T^m;\mathbb{R}) = H_{m-1}(T^m;\mathbb{R}) \cong \mathbb{R}^m$. Since *codim* $W^s(p) = 1$, measured foliation $\mathcal{F}_\psi$ is given by a closed form $\omega_\psi \in H^1(T^m;\mathbb{R})$, such that $[\psi]\omega_\psi = \lambda_\psi\omega_\psi$, where $\lambda_\psi$ is the eigenvalue of the linear transformation $[\psi] : H^1(T^m;\mathbb{R}) \to H^1(T^m;\mathbb{R})$. It is easy to see that $[\psi] = \psi_0$, because $H^1(T^m;\mathbb{R}) \cong \mathbb{R}^m$ is the universal cover for $T^m$, where the eigenspace $W^u(p)$ of $\psi_0$ is the span of the eigenform $\omega_\psi$. On the other hand, from the proof of Theorem 4.3.2 we know that the Plante group $P(\mathcal{F}_\psi)$ is generated by the coordinates of vector $\omega_\psi$; the matrix $B$ is nothing but the matrix $\psi_0$ written in a new basis of $P(\mathcal{F}_\psi)$. Each change of basis in the $\mathbb{Z}$-module $P(\mathcal{F}_\psi)$ is given by an integer invertible matrix $S$; therefore, $B = S^{-1}\psi_0 S$. Lemma 4.3.4 follows. $\square$

Let $\psi : T^m \to T^m$ be a hyperbolic diffeomorphism; the mapping torus $T^m_\psi$ will be called a (hyperbolic) *torus bundle* of dimension $m$. Let $k = |Gal\ (\mathbb{A}_\psi)|$; it follows from the Galois theory, that $1 < k \le m!$. Denote $t_i$ the cardinality of a subgroup $G_i \subseteq Gal\ (\mathbb{A}_\psi)$.

**Corollary 2.3.2** *There are no (non-trivial) continuous map $T^m_\psi \to T^{m'}_\varphi$, whenever $t'_i \nmid k$ for all $G'_i \subseteq Gal\ (\mathbb{A}_\varphi)$.*

*Proof.* If $h : T^m_\psi \to T^{m'}_\varphi$ was a continuous map to a torus bundle of dimension $m' < m$, then, by Theorem 4.3.3 and Corollary 4.3.1, the $Aut\ (\mathbb{Q}(\lambda_{B'}))$ (or, $Gal\ (\mathbb{A}_\varphi)$) were a non-trivial subgroup (or, normal subgroup) of the group $Gal\ (\mathbb{A}_\psi)$; since $k = |Gal\ (\mathbb{A}_\psi)|$, one concludes that one of $t'_i$ divides $k$. This contradicts our assumption. $\square$

**Definition 2.3.6** *The torus bundle $T_\psi^m$ is called robust, if there exists $m' < m$, such that no continuous map $T_\psi^m \to T_\varphi^{m'}$ is possible.*

**Remark 2.3.5** Are there robust bundles? It is shown in this section, that for $m = 2, 3$ and 4 there are infinitely many robust bundles.

**Case** $m = 2$

This case is trivial; $\psi_0$ is a hyperbolic matrix and always tight. The *char* $(\psi_0) =$ *char* $(B)$ is an irreducible quadratic polynomial with two real roots; *Gal* $(\mathbb{A}_\psi) \cong \mathbb{Z}_2$ and, therefore, $|Gal\ (\mathbb{A}_\psi)| = 2$. Formally, $T_\psi^2$ is robust, since no torus bundle of a smaller dimension is defined.

**Case** $m = 3$

The $\psi_0$ is hyperbolic; it is always tight, since one root of *char* $(\psi_0)$ is real and isolated inside or outside the unit circle.

**Corollary 2.3.3** *Let*

$$\psi_0(b,c) = \begin{pmatrix} -b & 1 & 0 \\ -c & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

*be such, that char* $(\psi_0(b,c)) = x^3+bx^2+cx+1$ *is irreducible and* $-4b^3+b^2c^2+18bc-4c^3-27$ *is the square of an integer; then* $T_\psi^3$ *admits no continuous map to any* $T_\varphi^2$.

*Proof.* The *char* $(\psi_0(b,c)) = x^3 + bx^2 + cx + 1$ and the discriminant $D = -4b^3 + b^2c^2 + 18bc - 4c^3 - 27$. By [Morandi 1996] [55], Theorem 13.1, we have *Gal* $(\mathbb{A}_\psi) \cong \mathbb{Z}_3$ and, therefore, $k = |Gal\ (\mathbb{A}_\psi)| = 3$. For $m' = 2$, it was shown that *Gal* $(\mathbb{A}_\varphi) \cong \mathbb{Z}_2$ and, therefore, $t'_1 = 2$. Since $2 \nmid 3$, Corollary 4.3.2 says that no continuous map $T_\psi^3 \to T_\varphi^2$ can be constructed. Corollary 4.3.3 follows. □

**Example 2.3.1** There are infinitely many matrices $\psi_0(b,c)$ satisfying the assumptions of Corollary 4.3.3; below are a few numerical examples of robust bundles:

$$\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

**Remark 2.3.6** Notice that the above matrices are not pairwise similar; it can be gleaned from their traces; thus they represent topologically distinct torus bundles.

**Case** $m = 4$

Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a quartic. Consider the associated cubic polynomial $r(x) = x^3 - bx^2 + (ac - 4d)x + 4bd - a^2d - c^2$; denote by $L$ the splitting field of $r(x)$.

**Corollary 2.3.4** *Let*

$$\psi_0(a,b,c) = \begin{pmatrix} -a & 1 & 0 & 0 \\ -b & 0 & 1 & 0 \\ -c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

*be tight and such, that char* $(\psi_0(a,b,c)) = x^4+ax^3+bx^2+cx+1$ *is irreducible and one of the following holds: (i)* $L = \mathbb{Q}$*; (ii)* $r(x)$ *has a unique root* $t \in \mathbb{Q}$ *and* $h(x) = (x^2-tx+1)[x^2+ax+(b-t)]$ *splits over* $L$*; (iii)* $r(x)$ *has a unique root* $t \in \mathbb{Q}$ *and* $h(x)$ *does not split over* $L$*. Then* $T^4_\psi$ *admits no continuous map to any* $T^3_\varphi$ *with* $D > 0$*.*

*Proof.* According to [Morandi 1996] [55], Theorem 13.4, $Gal\ (\mathbb{A}_\psi) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in case (i); $Gal\ (\mathbb{A}_\psi) \cong \mathbb{Z}_4$ in case (ii); and $Gal\ (\mathbb{A}_\psi) \cong D_4$ (the dihedral group) in case (iii). Therefore, $k = |\mathbb{Z}_2 \oplus \mathbb{Z}_2| = |\mathbb{Z}_4| = 4$ or $k = |D_4| = 8$. On the other hand, for $m' = 3$ with $D > 0$ (all roots are real), we have $t'_1 = |\mathbb{Z}_3| = 3$ and $t'_2 = |S_3| = 6$. Since $3; 6 \nmid 4; 8$, corollary 4.3.2 says that continuous map $T^4_\psi \to T^3_\varphi$ is impossible. Corollary 4.3.4 follows. □

**Example 2.3.2** There are infinitely many matrices $\psi_0$, which satisfy the assumption of corollary 4.3.4; indeed, consider a family

$$\psi_0(a,c) = \begin{pmatrix} -2a & 1 & 0 & 0 \\ -a^2 - c^2 & 0 & 1 & 0 \\ -2c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

where $a, c \in \mathbb{Z}$. The associated cubic becomes $r(x) = x[x^2-(a^2+c^2)x+4(ac-1)]$, so that $t = 0$ is a rational root; then $h(x) = (x^2+1)[x^2+2ax+a^2+c^2]$.

The matrix $\psi_0(a,c)$ satisfies one of the conditions (i)-(iii) of corollary 4.3.4 for each $a, c \in \mathbb{Z}$; it remains to eliminate the (non-generic) matrices, which are not tight or irreducible. Thus, $\psi_0(a,c)$ defines a family of topologically distinct robust bundles.

**Guide to the literature.** The Anosov diffeomorphisms were introduced and studied in [Anosov 1967] [2]; for a classical account of the differentiable dynamical systems see [Smale 1967] [98]. An excellent survey of foliations has been compiled by [Lawson 1974] [50]. The Galois theory is covered in the textbook by [Morandi 1996] [55]. The original proof of Theorem 4.3.3 and obstruction theory for Anosov's bundles can be found in [73].

# Exercises, problems and conjectures

1. Verify that $F : \phi \mapsto \mathbb{A}_\phi$ is a well-defined function on the set of all Anosov automorphisms given by the hyperbolic matrices with the non-negative entries.

2. Verify that the definition of the AF-algebra $\mathbb{A}_\phi$ for the pseudo-Anosov maps coincides with the one for the Anosov maps. (Hint: the Jacobi-Perron fractions of dimension $n = 2$ coincide with the regular continued fractions.)

3. **$p$-adic invariants of pseudo-Anosov maps.** Let $\phi \in Mod\ (X)$ be pseudo-Anosov automorphism of a surface $X$. If $\lambda_\phi$ is the dilatation of $\phi$, then one can consider a $\mathbb{Z}$-module $\mathfrak{m} = \mathbb{Z}v^{(1)} + \ldots + \mathbb{Z}v^{(n)}$ in the number field $K = \mathbb{Q}(\lambda_\phi)$ generated by the normalized eigenvector $(v^{(1)}, \ldots, v^{(n)})$ corresponding to the eigenvalue $\lambda_\phi$. The trace function on the number field $K$ gives rise to a symmetric bilinear form $q(x,y)$ on the module $\mathfrak{m}$. The form is defined over the field $\mathbb{Q}$. It has been shown that a pseudo-Anosov automorphism $\phi'$, conjugate to $\phi$, yields a form $q'(x,y)$, equivalent to $q(x,y)$, i.e. $q(x,y)$ can be transformed to $q'(x,y)$ by an invertible linear substitution with the coefficients in $\mathbb{Z}$. It is well known that two rational bilinear forms $q(x,y)$ and $q'(x,y)$ are equivalent whenever the following conditions are satisfied:

   (i) $\Delta = \Delta'$, where $\Delta$ is the determinant of the form;

(ii) for each prime number $p$ (including $p = \infty$) certain $p$-adic equation between the coefficients of forms $q, q'$ must be satisfied, see e.g. [Borevich & Shafarevich 1966] [11], Chapter 1, §7.5. (In fact, only a *finite* number of such equations have to be verified.)

Condition (i) has been already used to discern between the conjugacy classes of the pseudo-Anosov automorphisms. One can use condition (ii) to discern between the pseudo-Anosov automorphisms with $\Delta = \Delta'$; in other words, one gets a problem:

*To define p-adic invariants of the pseudo-Anosov maps.*

4. **The signature of a pseudo-Anosov map.** The signature is an important and well-known invariant connected to the chirality and knotting number of knots and links, see e.g. [Reidemeister 1932] [87]. It will be interesting to find a geometric interpretation of the signature $\Sigma$ for the pseudo-Anosov automorphisms; one can ask the following question:

   *To find geometric interpretation of the invariant $\Sigma$.*

5. **The number of conjugacy classes of pseudo-Anosov maps with the same dilatation.** The dilatation $\lambda_\phi$ is an invariant of the conjugacy class of the pseudo-Anosov automorphism $\phi \in Mod\ (X)$. On the other hand, it is known that there exist non-conjugate pseudo-Anosov's with the same dilatation and the number of such classes is finite, see [Thurston 1988] [105], p.428. It is natural to expect that the invariants of operator algebras can be used to evaluate the number; we have the following

   **Conjecture 2.3.1** *Let $(\Lambda, [I], K)$ be the triple corresponding to a pseudo-Anosov map $\phi \in Mod\ (X)$. Then the number of the conjugacy classes of the pseudo-Anosov automorphisms with the dilatation $\lambda_\phi$ is equal to the class number $h_\Lambda = |\Lambda/[I]|$ of the integral order $\Lambda$.*

# Chapter 3

# Algebraic Geometry

The NCG-valued functors arise in algebraic geometry; what is going on conceptually? Remember the covariant functor $GL_n$ : **CRng** $\to$ **Grp** from the category of commutative rings $R$ to the category of groups; functor $GL_n$ produces a multiplicative group of all $n \times n$ invertible matrices with entries in $R$ and preserves homomorphisms between the objects in the respective categories, see Example 2.3.4. The NCG-valued functors take one step further: they deal with the natural embedding **Grp** $\hookrightarrow$ **Grp-Rng**, where **Grp-Rng** is the category of associative group rings; thus we have

$$\mathbf{CRng} \overset{\mathrm{GL_n}}{\longrightarrow} \mathbf{Grp} \hookrightarrow \mathbf{Grp\text{-}Rng},$$

where **Grp-Rng** is an associative ring, i.e. the NCG. (Of course this simple observation would be of little use if the objects in **Grp-Rng** was fuzzy and nothing concrete can be said about them; note also that the abelianization **Grp-Rng**$/[\bullet, \bullet]$ of **Grp-Rng** is naturally isomorphic to **CRng**.) For $n = 2$ and **CRng** being the coordinate ring of elliptic curves, the category **Grp-Rng** consists of the noncommutative tori (with scaled units); this fact will be proved in Section 5.1 in two independent ways. For the higher genus algebraic curves the category **Grp-Rng** consists of the so-called *toric* AF-algebras, see Section 5.2. The general case of complex projective varieties is considered in Section 5.3 and **Grp-Rng** consists of the *Serre $C^*$-algebras*. In Section 5.4 we use the stable isomorphism group of toric AF-algebras to prove Harvey's conjecture on the linearity of the mapping class groups.

## 3.1 Elliptic curves

Let us repeat some known facts. We will be working with the ground field of complex numbers $\mathbb{C}$; by an *elliptic curve* we shall understand the subset of the complex projective plane of the form

$$\mathcal{E}(\mathbb{C}) = \{(x, y, z) \in \mathbb{C}P^2 \mid y^2 z = 4x^3 + axz^2 + bz^3\},$$

where $a$ and $b$ are some constant complex numbers. One can visualize the real points of $\mathcal{E}(\mathbb{C})$ as it is shown in Figure 5.1.

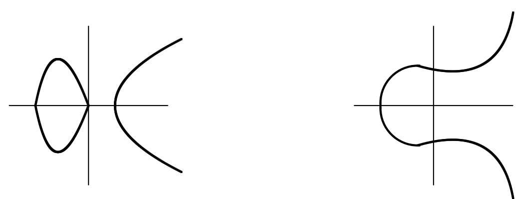



Figure 3.1: The real points of an affine elliptic curve $y^2 = 4x^3 + ax$.

**Remark 3.1.1** It is known that each elliptic curve $\mathcal{E}(\mathbb{C})$ is isomorphic to the set of points of intersection of two *quadric surfaces* in the complex projective space $\mathbb{C}P^3$ given by the system of homogeneous equations

$$\begin{cases} u^2 + v^2 + w^2 + z^2 &= 0, \\ Av^2 + Bw^2 + z^2 &= 0, \end{cases}$$

where $A$ and $B$ are some constant complex numbers and $(u, v, w, z) \in \mathbb{C}P^3$; the system is called the *Jacobi form* of elliptic curve $\mathcal{E}(\mathbb{C})$.

**Definition 3.1.1** *By a complex torus one understands the space $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$, where $\omega_1$ and $\omega_2$ are linearly independent vectors in the complex plane $\mathbb{C}$, see Fig. 5.2; the ratio $\tau = \omega_2/\omega_1$ is called a complex modulus.*

**Remark 3.1.2** Two complex tori $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ and $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau')$ are isomorphic if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

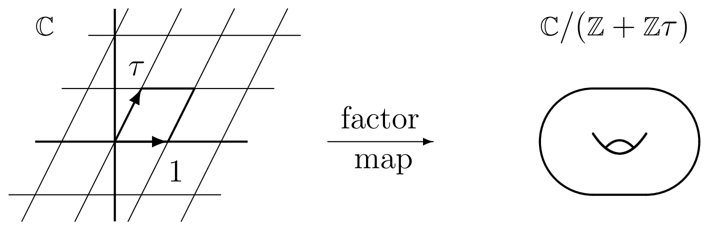



Figure 3.2: Complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$.

The complex analytic manifold $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ can be embedded into an $n$-dimensional complex projective space as an algebraic variety. For $n=2$ we have the following classical result, which relates complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with an elliptic curve $\mathcal{E}(\mathbb{C})$ in the projective plane $\mathbb{C}P^2$.

**Theorem 3.1.1 (Weierstrass)** *There exists a holomorphic embedding*

$$\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \hookrightarrow \mathbb{C}P^2$$

*given by the formula*

$$z \mapsto \begin{cases} (\wp(z), \wp'(z), 1) & \text{for } z \notin L_\tau := \mathbb{Z}+\mathbb{Z}\tau, \\ (0,1,0) & \text{for } z \in L_\tau \end{cases},$$

*which is an isomorphism between complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ and elliptic curve*

$$\mathcal{E}(\mathbb{C}) = \{(x,y,z) \in \mathbb{C}P^2 \mid y^2 z = 4x^3 + axz^2 + bz^3\},$$

*where $\wp(z)$ is the Weierstrass function defined by the convergent series*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L_\tau - \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

*and*

$$\begin{cases} a = -60 \sum_{\omega \in L_\tau - \{0\}} \frac{1}{\omega^4}, \\ b = -140 \sum_{\omega \in L_\tau - \{0\}} \frac{1}{\omega^6}. \end{cases}$$

**Remark 3.1.3** The Weierstrass Theorem identifies elliptic curves $\mathcal{E}(\mathbb{C})$ and complex tori $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$; we shall write $\mathcal{E}_\tau$ to denote elliptic curve corresponding to the complex torus of modulus $\tau = \omega_2/\omega_1$.

**Definition 3.1.2** *By* **Ell** *we shall mean the category of all elliptic curves* $\mathcal{E}_\tau$*; the arrows of* **Ell** *are identified with the isomorphisms between elliptic curves* $\mathcal{E}_\tau$*. We shall write* **NC-Tor** *to denote the category of all noncommutative tori* $\mathcal{A}_\theta$*; the arrows of* **NC-Tor** *are identified with the stable isomorphisms (Morita equivalences) between noncommutative tori* $\mathcal{A}_\theta$*.*

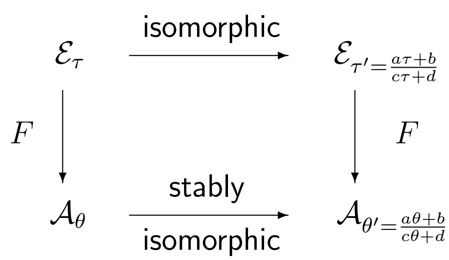



Figure 3.3: Functor on elliptic curves.

**Theorem 3.1.2 (Functor on elliptic curves)** *There exists a covariant functor*

$$F : \mathbf{Ell} \longrightarrow \mathbf{NC\text{-}Tor},$$

*which maps isomorphic elliptic curves* $\mathcal{E}_\tau$ *to the stably isomorphic (Morita equivalent) noncommutative tori* $\mathcal{A}_\theta$*, see Fig. 5.3; the functor* $F$ *is non-injective and* $Ker\ F \cong (0, \infty)$*.*

Theorem 5.1.2 will be proved in Section 5.1.1 using the Sklyanin algebras and in Section 5.1.2 using measured foliations and the Teichmüller theory.

### 3.1.1 Noncommutative tori via Sklyanin algebras

**Definition 3.1.3 ([Sklyanin 1982] [96])** *By the Sklyanin algebra* $S(\alpha, \beta, \gamma)$ *one understands a free* $\mathbb{C}$*-algebra on four generators* $x_1, \dots, x_4$ *and six quadratic relations*

$$\left\{\begin{array}{ccc} x_1x_2 - x_2x_1 &=& \alpha(x_3x_4 + x_4x_3),\\ x_1x_2 + x_2x_1 &=& x_3x_4 - x_4x_3,\\ x_1x_3 - x_3x_1 &=& \beta(x_4x_2 + x_2x_4),\\ x_1x_3 + x_3x_1 &=& x_4x_2 - x_2x_4,\\ x_1x_4 - x_4x_1 &=& \gamma(x_2x_3 + x_3x_2),\\ x_1x_4 + x_4x_1 &=& x_2x_3 - x_3x_2, \end{array}\right.$$

*where* $\alpha+\beta+\gamma+\alpha\beta\gamma=0$.

**Remark 3.1.4 ([Smith & Stafford 1992] [99], p. 260)** The algebra $S(\alpha,\beta,\gamma)$ is isomorphic to a *twisted homogeneous coordinate ring* of elliptic curve $\mathcal{E}_\tau\subset\mathbb{C}P^3$ given in its Jacobi form

$$\left\{\begin{array}{lll} u^2+v^2+w^2+z^2 &=& 0,\\ \frac{1-\alpha}{1+\beta}v^2+\frac{1+\alpha}{1-\gamma}w^2+z^2 &=& 0,\end{array}\right.$$

i.e. $S(\alpha,\beta,\gamma)$ satisfies an isomorphism

$$\textbf{Mod }(S(\alpha,\beta,\gamma))/\textbf{Tors}\cong\textbf{Coh }(\mathcal{E}_\tau),$$

where **Coh** is the category of quasi-coherent sheaves on $\mathcal{E}_\tau$, **Mod** the category of graded left modules over the graded ring $S(\alpha,\beta,\gamma)$ and **Tors** the full subcategory of **Mod** consisting of the torsion modules, see [Serre 1955] [91]. The algebra $S(\alpha,\beta,\gamma)$ defines a natural *automorphism* $\sigma:\mathcal{E}_\tau\to\mathcal{E}_\tau$ of the elliptic curve $\mathcal{E}_\tau$, see e.g. [Stafford & van den Bergh 2001] [100], p. 173.

**Lemma 3.1.1** *If* $\sigma^4=Id$, *then algebra* $S(\alpha,\beta,\gamma)$ *is isomorphic to a free algebra* $\mathbb{C}\langle x_1,x_2,x_3,x_4\rangle$ *modulo an ideal generated by six skew-symmetric quadratic relations*

$$\left\{\begin{array}{ll} x_3x_1 &= \mu e^{2\pi i\theta}x_1x_3,\\ x_4x_2 &= \frac{1}{\mu}e^{2\pi i\theta}x_2x_4,\\ x_4x_1 &= \mu e^{-2\pi i\theta}x_1x_4,\\ x_3x_2 &= \frac{1}{\mu}e^{-2\pi i\theta}x_2x_3,\\ x_2x_1 &= x_1x_2,\\ x_4x_3 &= x_3x_4,\end{array}\right.$$

*where* $\theta\in S^1$ *and* $\mu\in(0,\infty)$.

*Proof.* (i) Since $\sigma^4=Id$, the Sklyanin algebra $S(\alpha,\beta,\gamma)$ is isomorphic to a free algebra $\mathbb{C}\langle x_1,x_2,x_3,x_4\rangle$ modulo an ideal generated by the skew-symmetric relations

$$\left\{\begin{array}{lll} x_3x_1 &=& q_{13}x_1x_3,\\ x_4x_2 &=& q_{24}x_2x_4,\\ x_4x_1 &=& q_{14}x_1x_4,\\ x_3x_2 &=& q_{23}x_2x_3,\\ x_2x_1 &=& q_{12}x_1x_2,\\ x_4x_3 &=& q_{34}x_3x_4,\end{array}\right.$$

where $q_{ij} \in \mathbb{C} \setminus \{0\}$, see [Feigin & Odesskii 1989] [26], Remark 1 and [Feigin & Odesskii 1993] [27], §2 for the proof.

(ii) It is verified directly, that above relations are invariant of the involution $x_1^* = x_2, x_3^* = x_4$, if and only if, the following restrictions on the constants $q_{ij}$ hold

$$\begin{cases} q_{13} &= (\bar{q}_{24})^{-1}, \\ q_{24} &= (\bar{q}_{13})^{-1}, \\ q_{14} &= (\bar{q}_{23})^{-1}, \\ q_{23} &= (\bar{q}_{14})^{-1}, \\ q_{12} &= \bar{q}_{12}, \\ q_{34} &= \bar{q}_{34}, \end{cases}$$

where $\bar{q}_{ij}$ means the complex conjugate of $q_{ij} \in \mathbb{C} \setminus \{0\}$.

**Remark 3.1.5** The skew-symmetric relations invariant of the involution $x_1^* = x_2, x_3^* = x_4$ define an involution on the Sklyanin algebra; we shall call such an algebra a *Sklyanin $*$-algebra*.

(iii) Consider a one-parameter family $S(q_{13})$ of the Sklyanin $*$-algebras defined by the following additional constraints

$$\begin{cases} q_{13} &= \bar{q}_{14}, \\ q_{12} &= q_{34} = 1. \end{cases}$$

It is not hard to see, that the $*$-algebras $S(q_{13})$ are pairwise non-isomorphic for different values of complex parameter $q_{13}$; therefore the family $S(q_{13})$ is a normal form of the Sklyanin $*$-algebra $S(\alpha, \beta, \gamma)$ with $\sigma^4 = Id$. It remains to notice, that one can write complex parameter $q_{13}$ in the polar form $q_{13} = \mu e^{2\pi i\theta}$, where $\theta = Arg\ (q_{13})$ and $\mu = |q_{13}|$. Lemma 5.1.1 follows. □

**Lemma 3.1.2** *The system of relations*

$$\begin{cases} x_3x_1 &= e^{2\pi i\theta}x_1x_3, \\ x_1x_2 &= x_2x_1 = e, \\ x_3x_4 &= x_4x_3 = e \end{cases}$$

*defining the noncommutative torus $\mathcal{A}_\theta$ is equivalent to the following system of quadratic relations*

$$\begin{cases} x_3x_1 &= e^{2\pi i\theta}x_1x_3,\\ x_4x_2 &= e^{2\pi i\theta}x_2x_4,\\ x_4x_1 &= e^{-2\pi i\theta}x_1x_4,\\ x_3x_2 &= e^{-2\pi i\theta}x_2x_3,\\ x_2x_1 &= x_1x_2 = e,\\ x_4x_3 &= x_3x_4 = e. \end{cases}$$

*Proof.* Indeed, the first and the two last equations of both systems coincide; we shall proceed stepwise for the rest of the equations.

(i) Let us prove that equations for $\mathcal{A}_\theta$ imply $x_1x_4 = e^{2\pi i\theta}x_4x_1$. It follows from $x_1x_2 = e$ and $x_3x_4 = e$ that $x_1x_2x_3x_4 = e$. Since $x_1x_2 = x_2x_1$ we can bring the last equation to the form $x_2x_1x_3x_4 = e$ and multiply the both sides by the constant $e^{2\pi i\theta}$; thus one gets the equation $x_2(e^{2\pi i\theta}x_1x_3)x_4 = e^{2\pi i\theta}$. But $e^{2\pi i\theta}x_1x_3 = x_3x_1$ and our main equation takes the form $x_2x_3x_1x_4 = e^{2\pi i\theta}$.

We can multiply on the left the both sides of the equation by the element $x_1$ and thus get the equation $x_1x_2x_3x_1x_4 = e^{2\pi i\theta}x_1$; since $x_1x_2 = e$ one arrives at the equation $x_3x_1x_4 = e^{2\pi i\theta}x_1$.

Again one can multiply on the left the both sides by the element $x_4$ and thus get the equation $x_4x_3x_1x_4 = e^{2\pi i\theta}x_4x_1$; since $x_4x_3 = e$ one gets the required identity $x_1x_4 = e^{2\pi i\theta}x_4x_1$.

(ii) Let us prove that equations for $\mathcal{A}_\theta$ imply $x_2x_3 = e^{2\pi i\theta}x_3x_2$. As in the case (i), it follows from the equations $x_1x_2 = e$ and $x_3x_4 = e$ that $x_3x_4x_1x_2 = e$. Since $x_3x_4 = x_4x_3$ we can bring the last equation to the form $x_4x_3x_1x_2 = e$ and multiply the both sides by the constant $e^{-2\pi i\theta}$; thus one gets the equation $x_4(e^{-2\pi i\theta}x_3x_1)x_2 = e^{-2\pi i\theta}$. But $e^{-2\pi i\theta}x_3x_1 = x_1x_3$ and our main equation takes the form $x_4x_1x_3x_2 = e^{-2\pi i\theta}$.

We can multiply on the left the both sides of the equation by the element $x_3$ and thus get the equation $x_3x_4x_1x_3x_2 = e^{-2\pi i\theta}x_3$; since $x_3x_4 = e$ one arrives at the equation $x_1x_3x_2 = e^{-2\pi i\theta}x_3$.

Again one can multiply on the left the both sides by the element $x_2$ and thus get the equation $x_2x_1x_3x_2 = e^{-2\pi i\theta}x_2x_3$; since $x_2x_1 = e$ one gets the equation $x_3x_2 = e^{-2\pi i\theta}x_2x_3$. Multiplying both sides by constant $e^{2\pi i\theta}$ we obtain the required identity $x_2x_3 = e^{2\pi i\theta}x_3x_2$.

(iii) Let us prove that equations for $\mathcal{A}_\theta$ imply $x_4x_2 = e^{2\pi i\theta}x_2x_4$. Indeed, it was proved in (i) that $x_1x_4 = e^{2\pi i\theta}x_4x_1$; we shall multiply this equation on the right by the equation $x_2x_1 = e$. Thus one arrives at the equation $x_1x_4x_2x_1 = e^{2\pi i\theta}x_4x_1$.

Notice that in the last equation one can cancel $x_1$ on the right thus bringing it to the simpler form $x_1x_4x_2 = e^{2\pi i\theta}x_4$.

We shall multiply on the left both sides of the above equation by the element $x_2$; one gets therefore $x_2x_1x_4x_2 = e^{2\pi i\theta}x_2x_4$. But $x_2x_1 = e$ and the left hand side simplifies giving the required identity $x_4x_2 = e^{2\pi i\theta}x_2x_4$.

Lemma 5.1.2 follows. □

**Lemma 3.1.3 (Basic isomorphism)** *The system of relations for noncommutative torus $\mathcal{A}_\theta$*

$$\left\{\begin{array}{ll} x_3x_1 & = e^{2\pi i\theta}x_1x_3, \\ x_4x_2 & = e^{2\pi i\theta}x_2x_4, \\ x_4x_1 & = e^{-2\pi i\theta}x_1x_4, \\ x_3x_2 & = e^{-2\pi i\theta}x_2x_3, \\ x_2x_1 & = x_1x_2 = e, \\ x_4x_3 & = x_3x_4 = e, \end{array}\right.$$

*is equivalent to the system of relations for the Sklyanin $*$-algebra*

$$\left\{\begin{array}{ll} x_3x_1 & = \mu e^{2\pi i\theta}x_1x_3, \\ x_4x_2 & = \frac{1}{\mu}e^{2\pi i\theta}x_2x_4, \\ x_4x_1 & = \mu e^{-2\pi i\theta}x_1x_4, \\ x_3x_2 & = \frac{1}{\mu}e^{-2\pi i\theta}x_2x_3, \\ x_2x_1 & = x_1x_2, \\ x_4x_3 & = x_3x_4, \end{array}\right.$$

*modulo the following "scaled unit relation"*

$$x_1x_2 = x_3x_4 = \frac{1}{\mu}e.$$

*Proof.* (i) Using the last two relations, one can bring the noncommutative torus relations to the form

$$\left\{\begin{array}{ccc} x_3x_1x_4 & = & e^{2\pi i\theta}x_1, \\ x_4 & = & e^{2\pi i\theta}x_2x_4x_1, \\ x_4x_1x_3 & = & e^{-2\pi i\theta}x_1, \\ x_2 & = & e^{-2\pi i\theta}x_4x_2x_3, \\ x_1x_2 & = & x_2x_1 = e, \\ x_3x_4 & = & x_4x_3 = e. \end{array}\right.$$

(ii) The system of relations for the Sklyanin $*$-algebra complemented by the scaled unit relation, i.e.

$$\begin{cases} x_3x_1 &= \mu e^{2\pi i\theta}x_1x_3,\\ x_4x_2 &= \frac{1}{\mu}e^{2\pi i\theta}x_2x_4,\\ x_4x_1 &= \mu e^{-2\pi i\theta}x_1x_4,\\ x_3x_2 &= \frac{1}{\mu}e^{-2\pi i\theta}x_2x_3,\\ x_2x_1 &= x_1x_2 = \frac{1}{\mu}e,\\ x_4x_3 &= x_3x_4 = \frac{1}{\mu}e \end{cases}$$

is equivalent to the system

$$\begin{cases} x_3x_1x_4 &= e^{2\pi i\theta}x_1,\\ x_4 &= e^{2\pi i\theta}x_2x_4x_1,\\ x_4x_1x_3 &= e^{-2\pi i\theta}x_1,\\ x_2 &= e^{-2\pi i\theta}x_4x_2x_3,\\ x_2x_1 &= x_1x_2 = \frac{1}{\mu}e,\\ x_4x_3 &= x_3x_4 = \frac{1}{\mu}e \end{cases}$$

by using multiplication and cancellation involving the last two equations.

(iii) For each $\mu \in (0,\infty)$ consider a *scaled unit* $e' := \frac{1}{\mu}e$ of the Sklyanin $*$-algebra $S(q_{13})$ and the two-sided ideal $I_\mu \subset S(q_{13})$ generated by the relations $x_1x_2 = x_3x_4 = e'$. Comparing the defining relations for $S(q_{13})$ with such for the noncommutative torus $\mathcal{A}_\theta$, one gets an isomorphism

$$S(q_{13})\ /\ I_\mu \cong \mathcal{A}_\theta,$$

see items (i) and (ii). The isomorphism maps generators $x_1,\dots,x_4$ of $*$-algebra $S(q_{13})$ to such of the $C^*$-algebra $\mathcal{A}_\theta$ and the scaled unit $e' \in S(q_{13})$ to the *ordinary* unit of algebra $\mathcal{A}_\theta$. Lemma 5.1.3 follows. $\square$

To finish the proof of Theorem 5.1.2, recall that the Sklyanin $*$-algebra $S(q_{13})$ satisfies the fundamental isomorphism $\textbf{Mod}\ (S(q_{13}))/\ \textbf{Tors} \cong \textbf{Coh}\ (\mathcal{E}_\tau)$. Using the isomorphism $S(q_{13})/I_\mu \cong \mathcal{A}_\theta$ established in Lemma 5.1.3, we conclude that

$$I_\mu\backslash\textbf{Coh}\ (\mathcal{E}_\tau) \cong \textbf{Mod}\ (I_\mu\backslash S(q_{13}))/\textbf{Tors} \cong \textbf{Mod}\ (\mathcal{A}_\theta)/\textbf{Tors}.$$

Thus one gets an isomorphism $\textbf{Coh}\ (\mathcal{E}_\tau)/I_\mu \cong \textbf{Mod}\ (\mathcal{A}_\theta)/\ \textbf{Tors}$, which defines a map $F : \textbf{Ell} \to \textbf{NC-Tor}$. Moreover, map $F$ is a functor because isomorphisms in the category **Mod** ($\mathcal{A}_\theta$) give rise to the stable isomorphisms (Morita equivalences) in the category **NC-Tor**. The second part of Theorem 5.1.2 is due to the fact that $F$ forgets scaling of the unit, i.e. for each $\mu \in (0,\infty)$ we have a constant map

$$S(q_{13}) \ni e' := \frac{1}{\mu} e \longmapsto e \in \mathcal{A}_\theta.$$

Thus $Ker\ F \cong (0,\infty)$. Theorem 5.1.2 is proved. $\square$

### 3.1.2 Noncommutative tori via measured foliations

**Definition 3.1.4 ([Thurston 1988] [105])** *By a measured foliation $\mathcal{F}$ on a surface $X$ one understands partition of $X$ into the singular points $x_1,\dots,x_n$ of order $k_1,\dots,k_n$ and the regular leaves, i.e. 1-dimensional submanifolds of $X$; on each open cover $U_i$ of $X\backslash\{x_1,\dots,x_n\}$ there exists a non-vanishing real-valued closed 1-form $\phi_i$ such that:*

*(i) $\phi_i = \pm\phi_j$ on $U_i \cap U_j$;*

*(ii) at each $x_i$ there exists a local chart $(u,v) : V \to \mathbb{R}^2$ such that for $z = u + iv$, it holds $\phi_i = Im\ (z^{\frac{k_i}{2}} dz)$ on $V \cap U_i$ for some branch of $z^{\frac{k_i}{2}}$.*

*The pair $(U_i,\phi_i)$ is called an atlas for measured foliation $\mathcal{F}$. A measure $\mu$ is assigned to each segment $(t_0,t) \in U_i$; the measure is transverse to the leaves of $\mathcal{F}$ and is defined by the integral $\mu(t_0,t) = \int_{t_0}^{t}\phi_i$. Such a measure is invariant along the leaves of $\mathcal{F}$, hence the name.*

**Remark 3.1.6** In case $X \cong T^2$ (a torus) each measured foliation is given by a family of parallel lines of a slope $\theta > 0$ as shown in Fig. 5.4.

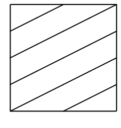


Figure 3.4: A measured foliation on the torus $\mathbb{R}^2/\mathbb{Z}^2$.

Let $T(g)$ be the Teichmüller space of surface $X$ of genus $g \geq 1$, i.e. the space of the complex structures on $X$. Consider the vector bundle $p : Q \to T(g)$

over $T(g)$, whose fiber above a point $S \in T_g$ is the vector space $H^0(S, \Omega^{\otimes 2})$. Given a non-zero $q \in Q$ above $S$, we can consider the horizontal measured foliation $\mathcal{F}_q \in \Phi_X$ of $q$, where $\Phi_X$ denotes the space of the equivalence classes of the measured foliations on $X$. If $\{0\}$ is the zero section of $Q$, the above construction defines a map $Q - \{0\} \longrightarrow \Phi_X$. For any $\mathcal{F} \in \Phi_X$, let $E_{\mathcal{F}} \subset Q - \{0\}$ be the fiber above $\mathcal{F}$. In other words, $E_{\mathcal{F}}$ is a subspace of the holomorphic quadratic forms, whose horizontal trajectory structure coincides with the measured foliation $\mathcal{F}$.

**Remark 3.1.7** If $\mathcal{F}$ is a measured foliation with the simple zeroes (a generic case), then $E_{\mathcal{F}} \cong \mathbb{R}^n - 0$ and $T(g) \cong \mathbb{R}^n$, where $n = 6g-6$ if $g \geq 2$ and $n = 2$ if $g = 1$.

**Theorem 3.1.3 ([Hubbard & Masur 1979] [41])** *The restriction of $p$ to $E_{\mathcal{F}}$ defines a homeomorphism (an embedding) $h_{\mathcal{F}} : E_{\mathcal{F}} \to T(g)$.*

**Corollary 3.1.1** *There exists a canonical homeomorphism $h : \Phi_X \to T(g) - \{pt\}$, where $pt = h_{\mathcal{F}}(0)$ and $\Phi_X \cong \mathbb{R}^n - 0$ is the space of equivalence classes of measured foliations $\mathcal{F}'$ on $X$.*

*Proof.* Denote by $\mathcal{F}'$ a vertical trajectory structure of $q$. Since $\mathcal{F}$ and $\mathcal{F}'$ define $q$, and $\mathcal{F} = Const$ for all $q \in E_{\mathcal{F}}$, one gets a homeomorphism between $T(g) - \{pt\}$ and $\Phi_X$. Corollary 5.1.1 follows. □

**Remark 3.1.8** The homeomorphism $h : \Phi_X \to T(g) - \{pt\}$ depends on a foliation $\mathcal{F}$; yet there exists a canonical homeomorphism $h = h_{\mathcal{F}}$ as follows. Let $Sp\ (S)$ be the length spectrum of the Riemann surface $S$ and $Sp\ (\mathcal{F}')$ be the set positive reals $\inf \mu(\gamma_i)$, where $\gamma_i$ runs over all simple closed curves, which are transverse to the foliation $\mathcal{F}'$. A canonical homeomorphism $h = h_{\mathcal{F}} : \Phi_X \to T(g) - \{pt\}$ is defined by the formula $Sp\ (\mathcal{F}') = Sp\ (h_{\mathcal{F}}(\mathcal{F}'))$ for $\forall \mathcal{F}' \in \Phi_X$.

Let $X \cong T^2$; then $T(1) \cong \mathbb{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}$. Since $q \neq 0$ there are no singular points and each $q \in H^0(S, \Omega^{\otimes 2})$ has the form $q = \omega^2$, where $\omega$ is a nowhere zero holomorphic differential on the complex torus $S$. Note that $\omega$ is just a constant times $dz$, and hence its vertical trajectory structure is just a family of the parallel lines of a slope $\theta$, see e.g. [Strebel 1984] [101], pp. 54–55. Therefore, $\Phi_{T^2}$ consists of the equivalence classes of the non-singular measured foliations on the two-dimensional torus. It is well known (the Denjoy theory), that every such foliation is measure equivalent to the foliation of a slope $\theta$ and a transverse measure $\mu > 0$, which is invariant along the leaves of the foliation Thus one obtains a canonical bijection

$$h : \Phi_{T^2} \longrightarrow \mathbb{H} - \{pt\}.$$

**Definition 3.1.5 (Category of lattices)** *By a lattice in the complex plane $\mathbb{C}$ one understands a triple $(\Lambda, \mathbb{C}, j)$, where $\Lambda \cong \mathbb{Z}^2$ and $j : \Lambda \to \mathbb{C}$ is an injective homomorphism with the discrete image. A morphism of lattices $(\Lambda, \mathbb{C}, j) \to (\Lambda', \mathbb{C}, j')$ is the identity $j \circ \psi = \varphi \circ j'$ where $\varphi$ is a group homomorphism and $\psi$ is a $\mathbb{C}$-linear map. It is not hard to see, that any isomorphism class of a lattice contains a representative given by $j : \mathbb{Z}^2 \to \mathbb{C}$ such that $j(1,0) = 1, j(0,1) = \tau \in \mathbb{H}$. The category of lattices $\mathcal{L}$ consists of Ob $(\mathcal{L})$, which are lattices $(\Lambda, \mathbb{C}, j)$ and morphisms $H(L, L')$ between $L, L' \in$ Ob $(\mathcal{L})$ which coincide with the morphisms of lattices specified above. For any $L, L', L'' \in$ Ob $(\mathcal{L})$ and any morphisms $\varphi' : L \to L'$, $\varphi'' : L' \to L''$ a morphism $\phi : L \to L''$ is the composite of $\varphi'$ and $\varphi''$, which we write as $\phi = \varphi''\varphi'$. The identity morphism, $1_L$, is a morphism $H(L, L)$.*

**Remark 3.1.9** The lattices are bijective with the complex tori (and elliptic curves) via the formula $(\Lambda, \mathbb{C}, j) \mapsto \mathbb{C}/j(\Lambda)$; thus $\mathcal{L} \cong$ **Ell**.

**Definition 3.1.6 (Category of pseudo-lattices)** *By a pseudo- lattice (of rank 2) in the real line $\mathbb{R}$ one understands a triple $(\Lambda, \mathbb{R}, j)$, where $\Lambda \cong \mathbb{Z}^2$ and $j : \Lambda \to \mathbb{R}$ is a homomorphism. A morphism of the pseudo-lattices $(\Lambda, \mathbb{R}, j) \to (\Lambda', \mathbb{R}, j')$ is the identity $j \circ \psi = \varphi \circ j'$, where $\varphi$ is a group homomorphism and $\psi$ is an inclusion map (i.e. $j'(\Lambda') \subseteq j(\Lambda)$). Any isomorphism class of a pseudo-lattice contains a representative given by $j : \mathbb{Z}^2 \to \mathbb{R}$, such that $j(1,0) = \lambda_1, j(0,1) = \lambda_2$, where $\lambda_1, \lambda_2$ are the positive reals. The pseudo-lattices make up a category, which we denote by $\mathcal{PL}$.*

**Lemma 3.1.4** *The pseudo-lattices are bijective with the measured foliations on torus via the formula $(\Lambda, \mathbb{R}, j) \mapsto \mathcal{F}^{\lambda_1}_{\lambda_2/\lambda_1}$, where $\mathcal{F}^{\lambda_1}_{\lambda_2/\lambda_1}$ is a foliation of the slope $\theta = \lambda_2/\lambda_1$ and measure $\mu = \lambda_1$.*

*Proof.* Define a pairing by the formula $(\gamma, Re\ \omega) \mapsto \int_\gamma Re\ \omega$, where $\gamma \in H_1(T^2, \mathbb{Z})$ and $\omega \in H^0(S; \Omega)$. The trajectories of the closed differential $\phi := Re\ \omega$ define a measured foliation on $T^2$. Thus, in view of the pairing, the linear spaces $\Phi_{T^2}$ and $Hom\ (H_1(T^2, \mathbb{Z}); \mathbb{R})$ are isomorphic. Notice that the latter space coincides with the space of the pseudo-lattices. To obtain an explicit bijection formula, let us evaluate the integral:

$$\int_{\mathbb{Z}\gamma_1+\mathbb{Z}\gamma_2}\phi=\mathbb{Z}\int_{\gamma_1}\phi+\mathbb{Z}\int_{\gamma_2}\phi=\mathbb{Z}\int_0^1\mu dx+\mathbb{Z}\int_0^1\mu dy,$$

where $\{\gamma_1,\gamma_2\}$ is a basis in $H_1(T^2,\mathbb{Z})$. Since $\frac{dy}{dx}=\theta$, one gets:

$$\begin{cases}\int_0^1\mu dx & =\mu & =\lambda_1\\ \int_0^1\mu dy & =\int_0^1\mu\theta dx & =\mu\theta & =\lambda_2.\end{cases}$$

Thus, $\mu=\lambda_1$ and $\theta=\frac{\lambda_2}{\lambda_1}$. Lemma 5.1.4 follows. $\square$

**Remark 3.1.10** It follows from Lemma 5.1.4 and the canonical bijection $h:\Phi_{T^2}\to\mathbb{H}-\{pt\}$, that $\mathcal{L}\cong\mathcal{PL}$ are the equivalent categories.

**Definition 3.1.7 (Category of projective pseudo-lattices)** *By a projective pseudo- lattice (of rank 2) one understands a triple $(\Lambda,\mathbb{R},j)$, where $\Lambda\cong\mathbb{Z}^2$ and $j:\Lambda\to\mathbb{R}$ is a homomorphism. A morphism of the projective pseudo-lattices $(\Lambda,\mathbb{C},j)\to(\Lambda',\mathbb{R},j')$ is the identity $j\circ\psi=\varphi\circ j'$, where $\varphi$ is a group homomorphism and $\psi$ is an $\mathbb{R}$-linear map. (Notice, that unlike the case of the pseudo-lattices, $\psi$ is a scaling map as opposite to an inclusion map. Thus, the two pseudo-lattices can be projectively equivalent, while being distinct in the category $\mathcal{PL}$.) It is not hard to see that any isomorphism class of a projective pseudo-lattice contains a representative given by $j:\mathbb{Z}^2\to\mathbb{R}$ such that $j(1,0)=1, j(0,1)=\theta$, where $\theta$ is a positive real. The projective pseudo-lattices make up a category, which we shall denote by $\mathcal{PPL}$.*

**Lemma 3.1.5** $\mathcal{PPL}\cong$ **NC-Tor**, *i.e projective pseudo-lattices and noncommutative tori are equivalent categories.*

*Proof.* Notice that projective pseudo-lattices are bijective with the noncommutative tori, via the formula $(\Lambda,\mathbb{R},j)\mapsto\mathcal{A}_\theta$. An isomorphism $\varphi:\Lambda\to\Lambda'$ acts by the formula $1\mapsto a+b\theta$, $\theta\mapsto c+d\theta$, where $ad-bc=1$ and $a,b,c,d\in\mathbb{Z}$. Therefore, $\theta'=\frac{c+d\theta}{a+b\theta}$. Thus, isomorphic projective pseudo-lattices map to the stably isomorphic (Morita equivalent) noncommutative tori. Lemma 5.1.5 follows. $\square$

To define a map $F:$ **Ell** $\to$ **NC-Tor**, we shall consider a composition of the following morphisms

$$\mathbf{Ell}\xrightarrow{\sim}\mathcal{L}\xrightarrow{\sim}\mathcal{PL}\xrightarrow{\text{F}}\mathcal{PPL}\xrightarrow{\sim}\mathbf{NC\text{-}Tor},$$

where all the arrows, but $F$, have been defined. To define $F$, let $PL \in \mathcal{PL}$ be a pseudo-lattice, such that $PL = PL(\lambda_1, \lambda_2)$, where $\lambda_1 = j(1,0), \lambda_2 = j(0,1)$ are positive reals. Let $PPL \in \mathcal{PPL}$ be a projective pseudo-lattice, such that $PPL = PPL(\theta)$, where $j(1,0) = 1$ and $j(0,1) = \theta$ is a positive real. Then $F : \mathcal{PL} \to \mathcal{PPL}$ is given by the formula $PL(\lambda_1, \lambda_2) \longmapsto PPL\left(\frac{\lambda_2}{\lambda_1}\right)$. It is easy to see, that $Ker\ F \cong (0, \infty)$ and $F$ is not an injective map. Since all the arrows, but $F$, are the isomorphisms between the categories, one gets a map

$$F : \mathbf{Ell} \longrightarrow \mathbf{NC\text{-}Tor}.$$

**Lemma 3.1.6 (Basic lemma)** *The map* $F : \mathbf{Ell} \to \mathbf{NC\text{-}Tor}$ *is a covariant functor which maps isomorphic complex tori to the stably isomorphic (Morita equivalent) noncommutative tori; the functor is non-injective functor and* $Ker\ F \cong (0, \infty)$.

*Proof.* (i) Let us show that $F$ maps isomorphic complex tori to the stably isomorphic noncommutative tori. Let $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ be a complex torus. Recall that the periods $\omega_1 = \int_{\gamma_1} \omega_E$ and $\omega_2 = \int_{\gamma_2} \omega_E$, where $\omega_E = dz$ is an invariant (Néron) differential on the complex torus and $\{\gamma_1, \gamma_2\}$ is a basis in $H_1(T^2, \mathbb{Z})$. The map $F$ can be written as

$$\mathbb{C}/L_{(\int_{\gamma_2} \omega_E)/(\int_{\gamma_1} \omega_E)} \overset{\text{F}}{\longmapsto} \mathcal{A}_{(\int_{\gamma_2} \phi)/(\int_{\gamma_1} \phi)},$$

where $L_{\omega_2/\omega_1}$ is a lattice and $\phi = Re\ \omega$ is a closed differential defined earlier. Note that every isomorphism in the category **Ell** is induced by an orientation preserving automorphism, $\varphi$, of the torus $T^2$. The action of $\varphi$ on the homology basis $\{\gamma_1, \gamma_2\}$ of $T^2$ is given by the formula:

$$\left\{\begin{array}{ll} \gamma_1' & = a\gamma_1 + b\gamma_2 \\ \gamma_2' & = c\gamma_1 + d\gamma_2 \end{array}\right. , \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The functor $F$ acts by the formula:

$$\tau = \frac{\int_{\gamma_2} \omega_E}{\int_{\gamma_1} \omega_E} \longmapsto \theta = \frac{\int_{\gamma_2} \phi}{\int_{\gamma_1} \phi}.$$

(a) From the left-hand side of the above equation, one obtains

$$\left\{\begin{array}{lllll} \omega_1' & = \int_{\gamma_1'} \omega_E & = \int_{a\gamma_1 + b\gamma_2} \omega_E & = a\int_{\gamma_1} \omega_E + b\int_{\gamma_2} \omega_E & = a\omega_1 + b\omega_2 \\ \omega_2' & = \int_{\gamma_2'} \omega_E & = \int_{c\gamma_1 + d\gamma_2} \omega_E & = c\int_{\gamma_1} \omega_E + d\int_{\gamma_2} \omega_E & = c\omega_1 + d\omega_2, \end{array}\right.$$

and therefore $\tau' = \frac{\int_{\gamma_2'} \omega_E}{\int_{\gamma_1'} \omega_E} = \frac{c+d\tau}{a+b\tau}$.

(b) From the right-hand side, one obtains

$$\begin{cases} \lambda_1' & = \int_{\gamma_1'} \phi & = \int_{a\gamma_1 + b\gamma_2} \phi & = a\int_{\gamma_1} \phi + b\int_{\gamma_2} \phi & = a\lambda_1 + b\lambda_2 \\ \lambda_2' & = \int_{\gamma_2'} \phi & = \int_{c\gamma_1 + d\gamma_2} \phi & = c\int_{\gamma_1} \phi + d\int_{\gamma_2} \phi & = c\lambda_1 + d\lambda_2, \end{cases}$$

and therefore $\theta' = \frac{\int_{\gamma_2'} \phi}{\int_{\gamma_1'} \phi} = \frac{c+d\theta}{a+b\theta}$. Comparing (a) and (b), one concludes that $F$ maps isomorphic complex tori to the stably isomorphic (Morita equivalent) noncommutative tori.

(ii) Let us show that $F$ is a covariant functor, i.e. $F$ does not reverse the arrows. Indeed, it can be verified directly using the above formulas, that $F(\varphi_1\varphi_2) = \varphi_1\varphi_2 = F(\varphi_1)F(\varphi_2)$ for any pair of the isomorphisms $\varphi_1, \varphi_2 \in Aut\ (T^2)$.

(iii) Since $F : \mathcal{PL} \to \mathcal{PPL}$ is given by the formula $PL(\lambda_1, \lambda_2) \longmapsto PPL\left(\frac{\lambda_2}{\lambda_1}\right)$, one gets $Ker\ F \cong (0, \infty)$ and $F$ is not an injective map. Lemma 5.1.6 is proved. $\square$

Theorem 5.1.2 follows from Lemma 5.1.6. $\square$

**Guide to the literature.** The basics of elliptic curves are covered by [Husemöller 1986] [42], [Knapp 1992] [44], [Koblitz 1984] [46], [Silverman 1985] [93], [Silverman 1994] [94], [Silverman & Tate 1992] [95] and others. More advanced topics are discussed in the survey papers [Cassels 1966] [14], [Mazur 1986] [53] and [Tate 1974] [103]. The Sklyanin algebras were introduced and studied in [Sklyanin 1982] [96] and [Sklyanin 1983] [97]; for a detailed account, see [Feigin & Odesskii 1989] [26] and [Feigin & Odesskii 1993] [27]. The general theory is covered by [Stafford & van den Bergh 2001] [100]. The basics of measured foliations and the Teichmüller theory can be found in [Thurston 1988] [105] and [Hubbard & Masur 1979] [41]. The functor from elliptic curves to noncommutative tori was constructed in [64] and [66] using measured foliations and in [74] using Sklyanin's algebras. The idea of infinite-dimensional representations of Sklyanin's algebras by the linear operators on a Hilbert space $\mathcal{H}$ belongs to [Sklyanin 1982] [96], the end of Section 3.

## 3.2 Higher genus algebraic curves

By a *complex algebraic curve* one understands a subset of the complex projective plane of the form

$$C = \{(x,y,z) \in \mathbb{C}P^2 \mid P(x,y,z) = 0\},$$

where $P(x,y,z)$ is a homogeneous polynomial with complex coefficients; such curves are isomorphic to the complex 2-dimensional manifolds, i.e. the *Riemann surfaces* $S$. We shall construct a functor $F$ on a generic set of complex algebraic curves with values in the category of *toric* AF-algebras; the functor maps isomorphic algebraic curves to the stably isomorphic (Morita equivalent) toric AF-algebras. For genus $g = 1$ algebraic (i.e. elliptic) curves, the toric AF-algebras are isomorphic to the Effros-Shen algebras $\mathbb{A}_\theta$; such AF-algebras are known to contain the noncommutative torus $\mathcal{A}_\theta$, see Theorem 3.5.3. The functor $F$ will be used to prove Harvey's conjecture by construction of a faithful representation of the mapping class group of genus $g \geq 2$ in the matrix group $GL(6g-6,\mathbb{Z})$, see Section 5.4.

### 3.2.1 Toric AF-algebras

We repeat some facts of the Teichmüller theory, see e.g. [Hubbard & Masur 1979] [41]. Denote by $T_S(g)$ the Teichmüller space of genus $g \geq 1$ (i.e. the space of all complex 2-dimensional manifolds of genus $g$) endowed with a distinguished point $S$. Let $q \in H^0(S, \Omega^{\otimes 2})$ be a holomorphic quadratic differential on the Riemann surface $S$, such that all zeroes of $q$ (if any) are simple. By $\widetilde{S}$ we understand a double cover of $S$ ramified over the zeroes of $q$ and by $H_1^{odd}(\widetilde{S})$ the odd part of the integral homology of $\widetilde{S}$ relatively the zeroes. Note that $H_1^{odd}(\widetilde{S}) \cong \mathbb{Z}^n$, where $n = 6g-6$ if $g \geq 2$ and $n = 2$ if $g = 1$. It is known that

$$T_S(g) \cong Hom\ (H_1^{odd}(\widetilde{S}); \mathbb{R}) - \{0\},$$

where 0 is the zero homomorphism [Hubbard & Masur 1979] [41]. Denote by $\lambda = (\lambda_1, \dots, \lambda_n)$ the image of a basis of $H_1^{odd}(\widetilde{S})$ in the real line $\mathbb{R}$, such that $\lambda_1 \neq 0$.

**Remark 3.2.1** The claim $\lambda_1 \neq 0$ is not restrictive, because the zero homomorphism is excluded.

We let $\theta = (\theta_1, \ldots, \theta_{n-1})$, where $\theta_i = \lambda_{i-1}/\lambda_1$. Recall that, up to a scalar multiple, vector $(1, \theta) \in \mathbb{R}^n$ is the limit of a generically convergent Jacobi-Perron continued fraction [Bernstein 1971] [7]

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $b_i = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^T$ is a vector of the non-negative integers, $I$ the unit matrix and $\mathbb{I} = (0, \ldots, 0, 1)^T$.

**Definition 3.2.1** *By a toric AF-algebra $\mathbb{A}_\theta$ one understands the AF-algebra given by the Bratteli diagram in Fig. 5.5, where numbers $b_j^{(i)}$ indicate the multiplicity of edges of the graph.*

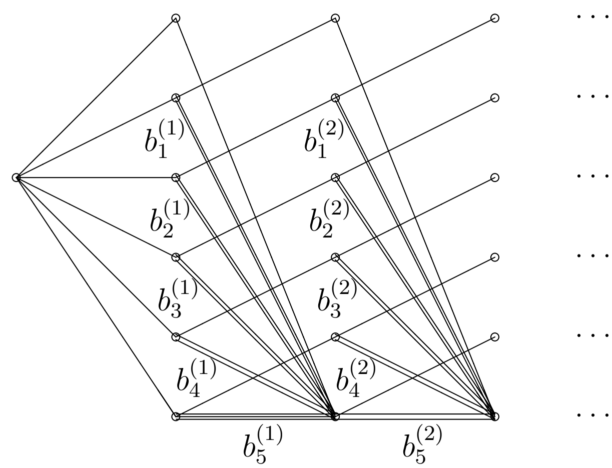



Figure 3.5: Toric AF-algebra $\mathbb{A}_\theta$ (case $g = 2$).

**Remark 3.2.2** Note that in the case $g = 1$, the Jacobi-Perron fraction coincides with the regular continued fraction and $\mathbb{A}_\theta$ becomes the Effros-Shen algebra, see Example 3.5.2.

**Definition 3.2.2** *By* **Alg-Gen** *we shall mean the maximal subset of $T_S(g)$ such that for each complex algebraic curve $C \in$* **Alg-Gen** *the corresponding Jacobi-Perron continued fraction is convergent; the arrows of* **Alg-Gen** *are* *isomorphisms between complex algebraic curves $C$. We shall write* **AF-Toric** *to denote the category of all toric AF-algebras $\mathbb{A}_\theta$; the arrows of* **AF-Toric** *are stable isomorphisms (Morita equivalences) between toric AF-algebras $\mathbb{A}_\theta$. By $F$ we understand a map given by the formula $C \mapsto \mathbb{A}_\theta$; in other words, we have a map*

$$F : \textbf{Alg-Gen} \longrightarrow \textbf{AF-Toric}.$$

**Theorem 3.2.1 (Functor on algebraic curves)** *The set* **Alg-Gen** *is a generic subset of $T_S(g)$ and the map $F$ has the following properties:*

*(i)* **Alg-Gen** $\cong$ **AF-Toric** $\times$ $(0, \infty)$ *is a trivial fiber bundle, whose projection map $p$ :* **Alg-Gen** $\to$ **AF-Toric** *coincides with $F$;*

*(ii) $F$ :* **Alg-Gen** $\to$ **AF-Toric** *is a covariant functor, which maps isomorphic complex algebraic curves $C, C' \in$* **Alg-Gen** *to the stably isomorphic (Morita equivalent) toric AF-algebras $\mathbb{A}_\theta, \mathbb{A}_{\theta'} \in$* **AF-Toric**.

### 3.2.2 Proof of Theorem 5.2.1

We shall repeat some known facts and notation. Let $S$ be a Riemann surface, and $q \in H^0(S, \Omega^{\otimes 2})$ a holomorphic quadratic differential on $S$. The lines $Re\ q = 0$ and $Im\ q = 0$ define a pair of measured foliations on $R$, which are transversal to each other outside the set of singular points. The set of singular points is common to both foliations and coincides with the zeroes of $q$. The above measured foliations are said to represent the vertical and horizontal trajectory structure of $q$, respectively. Denote by $T(g)$ the Teichmüller space of the topological surface $X$ of genus $g \ge 1$, i.e. the space of the complex structures on $X$. Consider the vector bundle $p : Q \to T(g)$ over $T(g)$ whose fiber above a point $S \in T_g$ is the vector space $H^0(S, \Omega^{\otimes 2})$. Given non-zero $q \in Q$ above $S$, we can consider horizontal measured foliation $\mathcal{F}_q \in \Phi_X$ of $q$, where $\Phi_X$ denotes the space of equivalence classes of measured foliations on $X$. If $\{0\}$ is the zero section of $Q$, the above construction defines a map $Q - \{0\} \longrightarrow \Phi_X$. For any $\mathcal{F} \in \Phi_X$, let $E_{\mathcal{F}} \subset Q - \{0\}$ be the fiber above $\mathcal{F}$. In other words, $E_{\mathcal{F}}$ is a subspace of the holomorphic quadratic forms whose horizontal trajectory structure coincides with the measured foliation $\mathcal{F}$. If $\mathcal{F}$ is a measured foliation with the simple zeroes (a generic case), then $E_{\mathcal{F}} \cong \mathbb{R}^n - 0$, while $T(g) \cong \mathbb{R}^n$, where $n = 6g - 6$ if $g \ge 2$ and $n = 2$ if $g = 1$. The restriction of $p$ to $E_{\mathcal{F}}$ defines a homeomorphism (an embedding)

$$h_{\mathcal{F}} : E_{\mathcal{F}} \to T(g).$$

The above result implies that the measured foliations parametrize the space $T(g) - \{pt\}$, where $pt = h_{\mathcal{F}}(0)$. Indeed, denote by $\mathcal{F}'$ a vertical trajectory structure of $q$. Since $\mathcal{F}$ and $\mathcal{F}'$ define $q$, and $\mathcal{F} = Const$ for all $q \in E_{\mathcal{F}}$, one gets a homeomorphism between $T(g) - \{pt\}$ and $\Phi_X$, where $\Phi_X \cong \mathbb{R}^n - 0$ is the space of equivalence classes of the measured foliations $\mathcal{F}'$ on $X$. Note that the above parametrization depends on a foliation $\mathcal{F}$. However, there exists a unique canonical homeomorphism $h = h_{\mathcal{F}}$ as follows. Let $Sp\ (S)$ be the length spectrum of the Riemann surface $S$ and $Sp\ (\mathcal{F}')$ be the set positive reals $\inf \mu(\gamma_i)$, where $\gamma_i$ runs over all simple closed curves, which are transverse to the foliation $\mathcal{F}'$. A canonical homeomorphism $h = h_{\mathcal{F}} : \Phi_X \to T(g) - \{pt\}$ is defined by the formula $Sp\ (\mathcal{F}') = Sp\ (h_{\mathcal{F}}(\mathcal{F}'))$ for $\forall \mathcal{F}' \in \Phi_X$. Thus, there exists a canonical homeomorphism

$$h : \Phi_X \to T(g) - \{pt\}.$$

Recall that $\Phi_X$ is the space of equivalence classes of measured foliations on the topological surface $X$. Following [Douady & Hubbard 1975] [20], we consider the following coordinate system on $\Phi_X$. For clarity, let us make a generic assumption that $q \in H^0(S, \Omega^{\otimes 2})$ is a non-trivial holomorphic quadratic differential with only simple zeroes. We wish to construct a Riemann surface of $\sqrt{q}$, which is a double cover of $S$ with ramification over the zeroes of $q$. Such a surface, denoted by $\widetilde{S}$, is unique and has an advantage of carrying a holomorphic differential $\omega$, such that $\omega^2 = q$. We further denote by $\pi : \widetilde{S} \to S$ the covering projection. The vector space $H^0(\widetilde{S}, \Omega)$ splits into the direct sum $H^0_{even}(\widetilde{S}, \Omega) \oplus H^0_{odd}(\widetilde{S}, \Omega)$ in view of the involution $\pi^{-1}$ of $\widetilde{S}$, and the vector space $H^0(S, \Omega^{\otimes 2}) \cong H^0_{odd}(\widetilde{S}, \Omega)$. Let $H_1^{odd}(\widetilde{S})$ be the odd part of the homology of $\widetilde{S}$ relatively the zeroes of $q$. Consider the pairing $H_1^{odd}(\widetilde{S}) \times H^0(S, \Omega^{\otimes 2}) \to \mathbb{C}$, defined by the integration $(\gamma, q) \mapsto \int_\gamma \omega$. We shall take the associated map $\psi_q : H^0(S, \Omega^{\otimes 2}) \to Hom\ (H_1^{odd}(\widetilde{S}); \mathbb{C})$ and let $h_q = Re\ \psi_q$.

**Lemma 3.2.1 ([Douady & Hubbard 1975] [20])** *The map*

$$h_q : H^0(S, \Omega^{\otimes 2}) \longrightarrow Hom\ (H_1^{odd}(\widetilde{S}); \mathbb{R})$$

*is an $\mathbb{R}$-isomorphism.*

**Remark 3.2.3** Since each $\mathcal{F} \in \Phi_X$ is the vertical foliation $Re\ q = 0$ for a $q \in H^0(S, \Omega^{\otimes 2})$, Lemma 5.2.1 implies that $\Phi_X \cong Hom\ (H_1^{odd}(\widetilde{S}); \mathbb{R})$. By formulas for the relative homology, one finds that $H_1^{odd}(\widetilde{S}) \cong \mathbb{Z}^n$, where $n = 6g - 6$ if $g \geq 2$ and $n = 2$ if $g = 1$. Each $h \in Hom\ (\mathbb{Z}^n; \mathbb{R})$ is given by the reals $\lambda_1 = h(e_1), \dots, \lambda_n = h(e_n)$, where $(e_1, \dots, e_n)$ is a basis in $\mathbb{Z}^n$. The numbers $(\lambda_1, \dots, \lambda_n)$ are the coordinates in the space $\Phi_X$ and, therefore, in the Teichmüller space $T(g)$.

To prove Theorem 5.2.1, we shall consider the following categories: (i) generic complex algebraic curves **Alg-Gen**; (ii) pseudo-lattices $\mathcal{PL}$; (iii) projective pseudo-lattices $\mathcal{PPL}$ and (iv) category **AF-Toric** of the toric AF-algebras. First, we show that **Alg-Gen** $\cong \mathcal{PL}$ are equivalent categories, such that isomorphic complex algebraic curves $C, C' \in$ **Alg-Gen** map to isomorphic pseudo-lattices $PL, PL' \in \mathcal{PL}$. Next, a non-injective functor $F : \mathcal{PL} \to \mathcal{PPL}$ is constructed. The $F$ maps isomorphic pseudo-lattices to isomorphic projective pseudo-lattices and $Ker\ F \cong (0, \infty)$. Finally, it is shown that a subcategory $U \subseteq \mathcal{PPL}$ and **AF-Toric** are equivalent categories. In other words, we have the following diagram

$$\textbf{Alg-Gen} \stackrel{\alpha}{\longrightarrow} \mathcal{PL} \stackrel{\mathrm{F}}{\longrightarrow} U \stackrel{\beta}{\longrightarrow} \textbf{AF-Toric},$$

where $\alpha$ is an injective map, $\beta$ is a bijection and $Ker\ F \cong (0, \infty)$.

**Definition 3.2.3** *Let $Mod\ X$ be the mapping class group of the surface $X$. A complex algebraic curve is a triple $(X, C, j)$, where $X$ is a topological surface of genus $g \geq 1$, $j : X \to C$ is a complex (conformal) parametrization of $X$ and $C$ is a Riemann surface. A morphism of complex algebraic curves $(X, C, j) \to (X, C', j')$ is the identity $j \circ \psi = \varphi \circ j'$, where $\varphi \in Mod\ X$ is a diffeomorphism of $X$ and $\psi$ is an isomorphism of Riemann surfaces. A category of generic complex algebraic curves,* **Alg-Gen**, *consists of Ob* (**Alg-Gen**) *which are complex algebraic curves $C \in T_S(g)$ and morphisms $H(C, C')$ between $C, C' \in Ob$* (**Alg-Gen**) *which coincide with the morphisms specified above. For any $C, C', C'' \in Ob$* (**Alg-Gen**) *and any morphisms $\varphi' : C \to C'$, $\varphi'' : C' \to C''$ a morphism $\phi : C \to C''$ is the composite of $\varphi'$ and $\varphi''$, which we write as $\phi = \varphi''\varphi'$. The identity morphism, $1_C$, is a morphism $H(C, C)$.*

**Definition 3.2.4** *By a pseudo-lattice (of rank n) one undestands the triple $(\Lambda, \mathbb{R}, j)$, where $\Lambda \cong \mathbb{Z}^n$ and $j : \Lambda \to \mathbb{R}$ is a homomorphism. A morphism of pseudo-lattices $(\Lambda, \mathbb{R}, j) \to (\Lambda, \mathbb{R}, j')$ is the identity $j \circ \psi = \varphi \circ j'$, where $\varphi$ is a group homomorphism and $\psi$ is an inclusion map, i.e. $j'(\Lambda') \subseteq j(\Lambda)$. Any isomorphism class of a pseudo-lattice contains a representative given by $j : \mathbb{Z}^n \to$*

$\mathbb{R}$ *such that* $j(1,0,\dots,0)=\lambda_1,\quad j(0,1,\dots,0)=\lambda_2,\quad \dots,\quad j(0,0,\dots,1)=\lambda_n$, *where* $\lambda_1,\lambda_2,\dots,\lambda_n$ *are positive reals. The pseudo-lattices of rank n make up a category, which we denote by* $\mathcal{PL}_n$.

**Lemma 3.2.2 (Basic lemma)** *Let* $g\ge 2$ $(g=1)$ *and* $n=6g-6$ $(n=2)$. *There exists an injective covariant functor* $\alpha:\mathbf{Alg\text{-}Gen}\to\mathcal{PL}_n$, *which maps isomorphic complex algebraic curves* $C,C'\in$ **Alg-Gen** *to the isomorphic pseudo-lattices* $PL,PL'\in\mathcal{PL}_n$.

*Proof.* Recall that we have a map $\alpha: T(g)-\{pt\}\to Hom\ (H_1^{odd}(\widetilde{S});\mathbb{R})-\{0\}$; $\alpha$ is a homeomorphism and, therefore, $\alpha$ is injective. Let us find the image $\alpha(\varphi)\in Mor\ (\mathcal{PL})$ of $\varphi\in Mor$ (**Alg-Gen**). Let $\varphi\in Mod\ X$ be a diffeomorphism of $X$, and let $\widetilde{X}\to X$ be the ramified double cover of $X$. We denote by $\widetilde{\varphi}$ the induced mapping on $\widetilde{X}$. Note that $\widetilde{\varphi}$ is a diffeomorphism of $\widetilde{X}$ modulo the covering involution $\mathbb{Z}_2$. Denote by $\widetilde{\varphi}^*$ the action of $\widetilde{\varphi}$ on $H_1^{odd}(\widetilde{X})\cong\mathbb{Z}^n$. Since $\widetilde{\varphi}\ mod\ \mathbb{Z}_2$ is a diffeomorphism of $\widetilde{X}$, $\widetilde{\varphi}^*\in GL_n(\mathbb{Z})$. Thus, $\alpha(\varphi)=\widetilde{\varphi}^*\in Mor\ (\mathcal{PL})$. Let us show that $\alpha$ is a functor. Indeed, let $C,C'\in$ **Alg-Gen** be isomorphic complex algebraic curves, such that $C'=\varphi(C)$ for a $\varphi\in Mod\ X$. Let $a_{ij}$ be the elements of matrix $\widetilde{\varphi}^*\in GL_n(\mathbb{Z})$. Recall that $\lambda_i=\int_{\gamma_i}\phi$ for a closed 1-form $\phi=Re\ \omega$ and $\gamma_i\in H_1^{odd}(\widetilde{X})$. Then $\gamma_j=\sum_{i=1}^n a_{ij}\gamma_i,\quad j=1,\dots,n$ are the elements of a new basis in $H_1^{odd}(\widetilde{X})$. By integration rules we have

$$\lambda'_j=\int_{\gamma_j}\phi=\int_{\sum a_{ij}\gamma_i}\phi=\sum_{i=1}^n a_{ij}\lambda_i.$$

Let $j(\Lambda)=\mathbb{Z}\lambda_1+\dots+\mathbb{Z}\lambda_n$ and $j'(\Lambda)=\mathbb{Z}\lambda'_1+\dots+\mathbb{Z}\lambda'_n$. Since $\lambda'_j=\sum_{i=1}^n a_{ij}\lambda_i$ and $(a_{ij})\in GL_n(\mathbb{Z})$, we conclude that $j(\Lambda)=j'(\Lambda)\subset\mathbb{R}$. In other words, the pseudo-lattices $(\Lambda,\mathbb{R},j)$ and $(\Lambda,\mathbb{R},j')$ are isomorphic. Hence, $\alpha:\mathbf{Alg\text{-}Gen}\to\mathcal{PL}$ maps isomorphic complex algebraic curves to the isomorphic pseudo-lattices, i.e. $\alpha$ is a functor. Let us show that $\alpha$ is a covariant functor. Indeed, let $\varphi_1,\varphi_2\in Mor(\mathbf{Alg\text{-}Gen})$. Then $\alpha(\varphi_1\varphi_2)=(\widetilde{\varphi_1\varphi_2})^*=\widetilde{\varphi}_1^*\widetilde{\varphi}_2^*=\alpha(\varphi_1)\alpha(\varphi_2)$. Lemma 5.2.2 follows. □

**Definition 3.2.5** *By a projective pseudo-lattice (of rank n) one understands a triple* $(\Lambda,\mathbb{R},j)$, *where* $\Lambda\cong\mathbb{Z}^n$ *and* $j:\Lambda\to\mathbb{R}$ *is a homomorphism. A morphism of projective pseudo-lattices* $(\Lambda,\mathbb{C},j)\to(\Lambda,\mathbb{R},j')$ *is the identity* $j\circ\psi=\varphi\circ j'$, *where* $\varphi$ *is a group homomorphism and* $\psi$ *is an* $\mathbb{R}$*-linear map. It is not hard to see that any isomorphism class of a projective pseudo-lattice* *contains a representative given by* $j : \mathbb{Z}^n \to \mathbb{R}$ *such that* $j(1,0,\dots,0) = 1, \quad j(0,1,\dots,0) = \theta_1, \quad \dots, \quad j(0,0,\dots,1) = \theta_{n-1}$*, where* $\theta_i$ *are positive reals. The projective pseudo-lattices of rank n make up a category, which we denote by* $\mathcal{PPL}_n$.

**Remark 3.2.4** Notice that unlike the case of pseudo-lattices, $\psi$ is a scaling map as opposite to an inclusion map. This allows to the two pseudo-lattices to be projectively equivalent, while being distinct in the category $\mathcal{PL}_n$.

**Definition 3.2.6** *Finally, the toric AF-algebras* $\mathbb{A}_\theta$*, modulo stable isomorphism (Morita equivalences), make up a category which we shall denote by* **AF-Toric**.

**Lemma 3.2.3** *Let* $U_n \subseteq \mathcal{PPL}_n$ *be a subcategory consisting of the projective pseudo-lattices* $PPL = PPL(1,\theta_1,\dots,\theta_{n-1})$ *for which the Jacobi-Perron fraction of the vector* $(1,\theta_1,\dots,\theta_{n-1})$ *converges to the vector. Define a map* $\beta : U_n \to$ **AF-Toric** *by the formula* $PPL(1,\theta_1,\dots,\theta_{n-1}) \mapsto \mathbb{A}_\theta$*, where* $\theta = (\theta_1,\dots,\theta_{n-1})$*. Then* $\beta$ *is a bijective functor, which maps isomorphic projective pseudo-lattices to the stably isomorphic toric AF-algebras.*

*Proof.* It is evident that $\beta$ is injective and surjective. Let us show that $\beta$ is a functor. Indeed, according to [Effros 1981] [21], Corollary 4.7, every totally ordered abelian group of rank $n$ has form $\mathbb{Z} + \theta_1\mathbb{Z} + \dots + \mathbb{Z}\theta_{n-1}$. The latter is a projective pseudo-lattice $PPL$ from the category $U_n$. On the other hand, by Theorem 3.5.2 the $PPL$ defines a stable isomorphism class of the toric AF-algebra $\mathbb{A}_\theta \in$ **AF-Toric**. Therefore, $\beta$ maps isomorphic projective pseudo-lattices (from the set $U_n$) to the stably isomorphic toric AF-algebras, and *vice versa*. Lemma 5.2.3 follows. □

**Lemma 3.2.4** *Let* $F : \mathcal{PL}_n \to \mathcal{PPL}_n$ *be a map given by formula*

$$PL(\lambda_1,\lambda_2,\dots,\lambda_n) \mapsto PPL\left(1, \frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_n}{\lambda_1}\right),$$

*where* $PL(\lambda_1,\lambda_2,\dots,\lambda_n) \in \mathcal{PL}_n$ *and* $PPL(1,\theta_1,\dots,\theta_{n-1}) \in \mathcal{PPL}_n$*. Then* $Ker\ F = (0,\infty)$ *and* $F$ *is a functor which maps isomorphic pseudo-lattices to isomorphic projective pseudo-lattices.*

*Proof.* Indeed, $F$ can be thought as a map from $\mathbb{R}^n$ to $\mathbb{R}P^n$. Hence $Ker\ F = \{\lambda_1 : \lambda_1 > 0\} \cong (0, \infty)$. The second part of lemma is evident. Lemma 5.2.4 is proved. $\square$

Theorem 5.2.1 follows from Lemmas 5.2.2 - 5.2.4 with $n = 6g - 6$ ($n = 2$) for $g \geq 2$ ($g = 1$). $\square$

**Guide to the literature.** An excellent introduction to complex algebraic curves is the book by [Kirwan 1992] [43]. For measured foliations and their relation to the Teichmüller theory the reader is referred to [Hubbard & Masur 1979] [41]. Functor $F$ : **Alg-Gen** $\rightarrow$ **AF-Toric** was constructed in [65]; the term *toric AF-algebras* was coined by Yu. Manin (private communication).

## 3.3 Complex projective varieties

We shall generalize functors constructed in Sections 5.1 and 5.2 to arbitrary complex projective varieties $X$. Namely, for the category **Proj-Alg** of all such varieties (of fixed dimension $n$) we construct a covariant functor

$$F : \textbf{Proj-Alg} \longrightarrow \textbf{C*-Serre},$$

where **C*-Serre** is a category of the *Serre $C^*$-algebras*, $\mathcal{A}_X$, attached to variety $X$. In particular, if $X \cong \mathcal{E}_\tau$ is an elliptic curve, then $\mathcal{A}_X \cong \mathcal{A}_\theta$ is a noncommutative torus and if $X \cong C$ is a complex algebraic curve, then $\mathcal{A}_X \cong \mathbb{A}_\theta$ is a toric AF-algebra. For $n \geq 2$ the description of $\mathcal{A}_X$ in terms of its semigroup $K_0^+(\mathcal{A}_X)$ is less satisfactory (so far?) but using the Takai duality for crossed product $C^*$-algebras, it is possible to prove the following general result. If $B$ is the commutative coordinate ring of variety $X$, then it is well known that $X \cong$ **Spec** $(B)$, where **Spec** is the space of prime ideals of $B$; an analog of this important formula for $\mathcal{A}_X$ is proved to be $X \cong$ **Irred** $(\mathcal{A}_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}})$, where $\hat{\alpha}$ is an automorphism of $\mathcal{A}_X$ and **Irred** the space of all irreducible representations of the crossed product $C^*$-algebra. We illustrate the formula in an important special case $\mathcal{A}_X \cong \mathcal{A}_{RM}$, i.e the case of noncommutative torus with real multiplication.

### 3.3.1 Serre $C^*$-algebras

Let $X$ be a projective scheme over a field $k$, and let $\mathcal{L}$ be the invertible sheaf $\mathcal{O}_X(1)$ of linear forms on $X$. Recall that the homogeneous coordinate ring of $X$ is a graded $k$-algebra, which is isomorphic to the algebra

$$B(X, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X,\ \mathcal{L}^{\otimes n}).$$

Denote by **Coh** the category of quasi-coherent sheaves on a scheme $X$ and by **Mod** the category of graded left modules over a graded ring $B$. If $M = \oplus M_n$ and $M_n = 0$ for $n >> 0$, then the graded module $M$ is called *right bounded*. The direct limit $M = \lim M_\alpha$ is called a *torsion*, if each $M_\alpha$ is a right bounded graded module. Denote by **Tors** the full subcategory of **Mod** of the torsion modules. The following result is the fundamental fact about the graded ring $B = B(X, \mathcal{L})$.

**Theorem 3.3.1 ([Serre 1955] [91])**

$$\mathbf{Mod}\ (B(X, \mathcal{L}))\ /\ \mathbf{Tors} \cong \mathbf{Coh}\ (X).$$

**Definition 3.3.1** *Let $\alpha$ be an automorphism of a projective scheme $X$; the pullback of sheaf $\mathcal{L}$ along $\alpha$ will be denoted by $\mathcal{L}^\alpha$, i.e. $\mathcal{L}^\alpha(U) := \mathcal{L}(\alpha U)$ for every $U \subset X$. We shall set*

$$B(X, \mathcal{L}, \alpha) = \bigoplus_{n \geq 0} H^0(X,\ \mathcal{L} \otimes \mathcal{L}^\alpha \otimes \ldots \otimes \mathcal{L}^{\alpha^n}).$$

*The multiplication of sections is defined by the rule*

$$ab = a \otimes b^{\alpha^m},$$

*whenever $a \in B_m$ and $b \in B_n$. Given a pair $(X, \alpha)$ consisting of a Noetherian scheme $X$ and an automorphism $\alpha$ of $X$, an invertible sheaf $\mathcal{L}$ on $X$ is called $\alpha$-ample, if for every coherent sheaf $\mathcal{F}$ on $X$, the cohomology group $H^q(X,\ \mathcal{L} \otimes \mathcal{L}^\alpha \otimes \ldots \otimes \mathcal{L}^{\alpha^{n-1}} \otimes \mathcal{F})$ vanishes for $q > 0$ and $n >> 0$. (Notice, that if $\alpha$ is trivial, this definition is equivalent to the usual definition of ample invertible sheaf, see [Serre 1955] [91].) If $\alpha : X \to X$ is an automorphism of a projective scheme $X$ over $k$ and $\mathcal{L}$ is an $\alpha$-ample invertible sheaf on $X$, then $B(X, \mathcal{L}, \alpha)$ is called a twisted homogeneous coordinate ring of $X$.*

**Theorem 3.3.2 ([Artin & van den Bergh 1990] [4])**

$$\textbf{Mod}\ (B(X,\mathcal{L},\alpha))\ /\ \textbf{Tors} \cong \textbf{Coh}\ (X).$$

**Remark 3.3.1** Theorem 5.3.2 extends Theorem 5.3.1 to the non-commutative rings; hence the name for ring $B(X,\mathcal{L},\alpha)$. The question of which invertible sheaves are $\alpha$-ample is fairly subtle, and there is no characterization of the automorphisms $\alpha$ for which such an invertible sheaf exists. However, in many important special cases this problem is solvable, see [Artin & van den Bergh 1990] [4], Corollary 1.6.

**Remark 3.3.2** In practice, any twisted homogeneous coordinate ring $B(X,\mathcal{L},\alpha)$ of a projective scheme $X$ can be constructed as follows. Let $R$ be a commutative graded ring, such that $X = Spec\ (R)$. Consider the ring $B(X,\mathcal{L},\alpha) := R[t,t^{-1};\alpha]$, where $R[t,t^{-1};\alpha]$ is the ring of skew Laurent polynomials defined by the commutation relation $b^{\alpha}t = tb$, for all $b \in R$; here $b^{\alpha} \in R$ is the image of $b$ under automorphism $\alpha$. The ring $B(X,\mathcal{L},\alpha)$ satisfies the isomorphism $\textbf{Mod}\ (B(X,\mathcal{L},\alpha))/\textbf{Tors} \cong \textbf{Coh}\ (X)$, i.e. is the twisted homogeneous coordinate ring of projective scheme $X$, see Lemma 5.3.1.

**Example 3.3.1** Let $k$ be a field and $U_{\infty}(k)$ the algebra of polynomials over $k$ in two non-commuting variables $x_1$ and $x_2$, and a quadratic relation $x_1x_2 - x_2x_1 - x_1^2 = 0$; let $\mathbb{P}^1(k)$ be the projective line over $k$. Then $B(X,\mathcal{L},\alpha) = U_{\infty}(k)$ and $X = \mathbb{P}^1(k)$ satisfy equation $\textbf{Mod}\ (B(X,\mathcal{L},\alpha))/\textbf{Tors} \cong \textbf{Coh}\ (X)$. The ring $U_{\infty}(k)$ corresponds to the automorphism $\alpha(u) = u+1$ of the projective line $\mathbb{P}^1(k)$. Indeed, $u = x_2x_1^{-1} = x_1^{-1}x_2$ and, therefore, $\alpha$ maps $x_2$ to $x_1+x_2$; if one substitutes $t = x_1, b = x_2$ and $b^{\alpha} = x_1+x_2$ in equation $b^{\alpha}t = tb$ (see Remark 5.3.2), then one gets the defining relation $x_1x_2 - x_2x_1 - x_1^2 = 0$ for the algebra $U_{\infty}(k)$.

To get a $C^*$-algebra from the ring $B(X,\mathcal{L},\alpha)$, we shall consider infinite-dimensional representations of $B(X,\mathcal{L},\alpha)$ by bounded linear operators on a Hilbert space $\mathcal{H}$; as usual, let $\mathcal{B}(\mathcal{H})$ stay for the algebra of all bounded linear operators on $\mathcal{H}$. For a ring of skew Laurent polynomials $R[t,t^{-1};\alpha]$ described in Remark 5.3.2, we shall consider a homomorphism

$$\rho : R[t,t^{-1};\alpha] \longrightarrow \mathcal{B}(\mathcal{H}).$$

Recall that algebra $\mathcal{B}(\mathcal{H})$ is endowed with a $*$-involution; such an involution is the adjoint with respect to the scalar product on the Hilbert space $\mathcal{H}$.

**Definition 3.3.2** *The representation $\rho$ will be called $*$-coherent if:*

*(i) $\rho(t)$ and $\rho(t^{-1})$ are unitary operators, such that $\rho^*(t) = \rho(t^{-1})$;*

*(ii) for all $b \in R$ it holds $(\rho^*(b))^{\alpha(\rho)} = \rho^*(b^\alpha)$, where $\alpha(\rho)$ is an automorphism of $\rho(R)$ induced by $\alpha$.*

**Example 3.3.2** The ring $U_\infty(k)$ has no $*$-coherent representations. Indeed, involution acts on the generators of $U_\infty(k)$ by formula $x_1^* = x_2$; the latter does not preserve the defining relation $x_1x_2 - x_2x_1 - x_1^2 = 0$.

**Definition 3.3.3** *By a Serre $C^*$-algebra $\mathcal{A}_X$ of the projective scheme $X$ one understands the norm-closure of an $*$-coherent representation $\rho(B(X, \mathcal{L}, \alpha))$ of the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \alpha) \cong R[t, t^{-1}; \alpha]$ of scheme $X$.*

**Example 3.3.3** For $X \cong \mathcal{E}_\tau$ is an elliptic curve, the ring $R[t, t^{-1}; \alpha]$ is isomorphic to the Sklyanin algebra, see Section 5.1.1. For such algebras there exists a $*$-coherent representation *ibid.*; the resulting Serre $C^*$-algebra $\mathcal{A}_X \cong \mathcal{A}_\theta$, where $\mathcal{A}_\theta$ is the noncommutative torus.

**Remark 3.3.3 (Functor on complex projective varieties)** If **Proj-Alg** is the category of all complex projective varieties $X$ (of dimension $n$) and **C*-Serre** the category of all Serre $C^*$-algebras $\mathcal{A}_X$, then the formula $X \mapsto \mathcal{A}_X$ gives rise to a map

$$F : \textbf{Proj-Alg} \longrightarrow \textbf{C*-Serre}.$$

The map $F$ is actually a functor which takes isomorphisms between projective varieties to the stable isomorphisms (Morita equivalences) between the corresponding Serre $C^*$-algebras; the proof repeats the argument for elliptic curves given in Section 5.1.1 and is left to the reader.

Let **Spec** $(B(X, \mathcal{L}))$ be the space of all prime ideals of the commutative homogeneous coordinate ring $B(X, \mathcal{L})$ of a complex projective variety $X$, see Theorem 5.3.1. To get an analog of the classical formula

$$X \cong \textbf{Spec}\ (B(X, \mathcal{L}))$$

for the Serre $C^*$-algebras $\mathcal{A}_X$, we shall recall that for each continuous homomorphism $\alpha : G \to Aut\ (\mathcal{A})$ of a locally compact group $G$ into the group of automorphisms of a $C^*$-algebra $\mathcal{A}$, there exists a crossed product $C^*$-algebra $\mathcal{A} \rtimes_\alpha G$, see e.g. Section 3.2. Let $G = \mathbb{Z}$ and let $\hat{\mathbb{Z}} \cong S^1$ be its Pontryagin dual. We shall write **Irred** for the set of all irreducible representations of given $C^*$-algebra.

**Theorem 3.3.3** *For each Serre $C^*$-algebra $\mathcal{A}_X$ there exists $\hat{\alpha} \in Aut\ (\mathcal{A}_X)$, such that:*

$$X \cong \textbf{Irred}\ (\mathcal{A}_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}).$$

**Remark 3.3.4** Note that a naive generalization $X \cong \textbf{Spec}\ (\mathcal{A}_X)$ is wrong, because most of the Serre $C^*$-algebras are simple, i.e. have no ideals whatsoever.

### 3.3.2 Proof of theorem 5.3.3

**Lemma 3.3.1** $B(X, \mathcal{L}, \alpha) \cong R[t, t^{-1}; \alpha]$, *where* $X = \textbf{Spec}\ (R)$.

*Proof.* Let us write the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \alpha)$ of projective variety $X$ in the following form:

$$B(X, \mathcal{L}, \alpha) = \bigoplus_{n \geq 0} H^0(X, \mathfrak{B}_n),$$

where $\mathfrak{B}_n = \mathcal{L} \otimes \mathcal{L}^\alpha \otimes \ldots \otimes \mathcal{L}^{\alpha^n}$ and $H^0(X, \mathfrak{B}_n)$ is the zero sheaf cohomology of $X$, i.e. the space of sections $\Gamma(X, \mathfrak{B}_n)$; compare with [Artin & van den Bergh 1990] [4], formula (3.5). If one denotes by $\mathcal{O}$ the structure sheaf of $X$, then

$$\mathfrak{B}_n = \mathcal{O} t^n$$

can be interpreted as a free left $\mathcal{O}$-module of rank one with basis $\{t^n\}$, see [Artin & van den Bergh 1990] [4], p. 252. Recall, that spaces $B_i = H^0(X, \mathfrak{B}_i)$ have been endowed with the multiplication rule between the sections $a \in B_m$ and $b \in B_n$, see Definition 5.3.1; such a rule translates into the formula

$$a t^m b t^n = a b^{\alpha^m} t^{m+n}.$$

One can eliminate $a$ and $t^n$ on the both sides of the above equation; this operation gives us the following equation

$$t^m b = b^{\alpha^m} t^m.$$

First notice, that our ring $B(X,\mathcal{L},\alpha)$ contains a commutative subring $R$, such that **Spec** $(R) = X$. Indeed, let $m = 0$ in formula $t^m b = b^{\alpha^m} t^m$; then $b = b^{Id}$ and, thus, $\alpha = Id$. We conclude therefore, that $R = B_0$ is a commutative subring of $B(X,\mathcal{L},\alpha)$, and **Spec** $(R) = X$.

Let us show that equations $b^\alpha t = tb$ of Remark 5.3.2 and $t^m b = b^{\alpha^m} t^m$ are equivalent. First, let us show that $b^\alpha t = tb$ implies $t^m b = b^{\alpha^m} t^m$. Indeed, equation $b^\alpha t = tb$ can be written as $b^\alpha = tbt^{-1}$. Then:

$$\begin{cases} b^{\alpha^2} &= \quad tb^\alpha t^{-1} = t^2 b t^{-2}, \\ b^{\alpha^3} &= \quad tb^{\alpha^2} t^{-1} = t^3 b t^{-3}, \\ &\vdots \\ b^{\alpha^m} &= \quad tb^{\alpha^{m-1}} t^{-1} = t^m b t^{-m}. \end{cases}$$

The last equation of the above system is equivalent to equation $t^m b = b^{\alpha^m} t^m$. The converse is evident; one sets $m = 1$ in $t^m b = b^{\alpha^m} t^m$ and obtains equation $b^\alpha t = tb$. Thus, $b^\alpha t = tb$ and $t^m b = b^{\alpha^m} t^m$ are equivalent equations. It is easy now to establish an isomorphism $B(X,\mathcal{L},\alpha) \cong R[t,t^{-1};\alpha]$. For that, take $b \in R \subset B(X,\mathcal{L},\alpha)$; then $B(X,\mathcal{L},\alpha)$ coincides with the ring of the skew Laurent polynomials $R[t,t^{-1};\alpha]$, since the commutation relation $b^\alpha t = tb$ is equivalent to equation $t^m b = b^{\alpha^m} t^m$. Lemma 5.3.1 follows. $\square$

**Lemma 3.3.2** *$\mathcal{A}_X \cong C(X) \rtimes_\alpha \mathbb{Z}$, where $C(X)$ is the $C^*$-algebra of all continuous complex-valued functions on $X$ and $\alpha$ is a $*$-coherent automorphism of $X$.*

*Proof.* By definition of the Serre algebra $\mathcal{A}_X$, the ring of skew Laurent polynomials $R[t,t^{-1};\alpha]$ is dense in $\mathcal{A}_X$; roughly speaking, one has to show that this property defines a crossed product structure on $\mathcal{A}_X$. We shall proceed in the following steps.

(i) Recall that $R[t,t^{-1};\alpha]$ consists of the finite sums

$$\sum b_k t^k, \qquad b_k \in R,$$

subject to the commutation relation

$$b_k^\alpha t = t b_k.$$

Because of the $*$-coherent representation, there is also an involution on $R[t,t^{-1};\alpha]$, subject to the following rules

$$\begin{cases} (i) & t^* &= & t^{-1}, \\ (ii) & (b_k^*)^\alpha &= & (b_k^\alpha)^*. \end{cases}$$

(ii) Following [Williams 2007] [110], p.47, we shall consider the set $C_c(\mathbb{Z}, R)$ of continuous functions from $\mathbb{Z}$ to $R$ having a compact support; then the finite sums can be viewed as elements of $C_c(\mathbb{Z}, R)$ via the identification

$$k \longmapsto b_k.$$

It can be verified, that multiplication operation of the finite sums translates into a convolution product of functions $f, g \in C_c(\mathbb{Z}, R)$ given by the formula

$$(fg)(k) = \sum_{l \in \mathbb{Z}} f(l) t^l g(k-l) t^{-l},$$

while involution translates into an involution on $C_c(\mathbb{Z}, R)$ given by the formula

$$f^*(k) = t^k f^*(-k) t^{-k}.$$

It is easy to see, that the multiplication given by the convolution product and involution turn $C_c(\mathbb{Z}, R)$ into an $*$-algebra, which is isomorphic to the algebra $R[t, t^{-1}; \alpha]$.

(iii) There exists the standard construction of a norm on $C_c(\mathbb{Z}, R)$; we omit it here referring the reader to [Williams 2007] [110], Section 2.3. The completion of $C_c(\mathbb{Z}, R)$ in that norm defines a crossed product $C^*$-algebra $R \rtimes_\alpha \mathbb{Z}$ [Williams 2007] [110], Lemma 2.27.

(iv) Since $R$ is a commutative $C^*$-algebra and $X = \mathbf{Spec}\ (R)$, one concludes that $R \cong C(X)$. Thus, one obtains $\mathcal{A}_X = C(X) \rtimes_\alpha \mathbb{Z}$. Lemma 5.3.2 follows. $\square$

**Remark 3.3.5** It is easy to prove, that equations $b_k^\alpha t = t b_k$ and $t^* = t^{-1}$ imply equation $(b_k^*)^\alpha = (b_k^\alpha)^*$; in other words, if involution does not commute with automorphism $\alpha$, representation $\rho$ cannot be unitary, i.e. $\rho^*(t) \neq \rho(t^{-1})$.

**Lemma 3.3.3** *There exists $\hat{\alpha} \in Aut\ (\mathcal{A}_X)$, such that:*

$$X \cong \mathbf{Irred}\ (\mathcal{A}_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}).$$

*Proof.* The above formula is an implication of the Takai duality for the crossed products, see e.g. [Williams 2007] [110], Section 7.1; for the sake of clarity, we shall repeat this construction. Let $(A, G, \alpha)$ be a $C^*$-dynamical system with $G$ locally compact abelian group; let $\hat{G}$ be the dual of $G$. For each $\gamma \in \hat{G}$, one can define a map $\hat{a}_\gamma : C_c(G, A) \to C_c(G, A)$ given by the formula:

$$\hat{a}_\gamma(f)(s) = \bar{\gamma}(s) f(s), \qquad \forall s \in G.$$

In fact, $\hat{a}_\gamma$ is a $*$-homomorphism, since it respects the convolution product and involution on $C_c(G, A)$ [Williams 2007] [110]. Because the crossed product $A \rtimes_\alpha G$ is the closure of $C_c(G, A)$, one gets an extension of $\hat{a}_\gamma$ to an element of $Aut\ (A \rtimes_\alpha G)$ and, therefore, a homomorphism:

$$\hat{\alpha} : \hat{G} \to Aut\ (A \rtimes_\alpha G).$$

The *Takai duality* asserts, that

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes \mathcal{K}(L^2(G)),$$

where $\mathcal{K}(L^2(G))$ is the algebra of compact operators on the Hilbert space $L^2(G)$. Let us substitute $A = C_0(X)$ and $G = \mathbb{Z}$ in the above equation; one gets the following isomorphism

$$(C_0(X) \rtimes_\alpha \mathbb{Z}) \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}} \cong C_0(X) \otimes \mathcal{K}(L^2(\mathbb{Z})).$$

Lemma 5.3.2 asserts that $C_0(X) \rtimes_\alpha \mathbb{Z} \cong \mathcal{A}_X$; therefore one arrives at the following isomorphism

$$\mathcal{A}_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}} \cong C_0(X) \otimes \mathcal{K}(L^2(\mathbb{Z})).$$

Consider the set of all irreducible representations of the $C^*$-algebras in the above equation; then one gets the following equality of representations

$$\textbf{Irred}\ (\mathcal{A}_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) = \textbf{Irred}\ (C_0(X) \otimes \mathcal{K}(L^2(\mathbb{Z}))).$$

Let $\pi$ be a representation of the tensor product $C_0(X) \otimes \mathcal{K}(L^2(\mathbb{Z}))$ on the Hilbert space $\mathcal{H} \otimes L^2(\mathbb{Z})$; then $\pi = \varphi \otimes \psi$, where $\varphi : C_0(X) \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{K} \to \mathcal{B}(L^2(\mathbb{Z}))$. It is known, that the only irreducible representation of the algebra of compact operators is the identity representation. Thus, one gets:

$$\begin{aligned} \textbf{Irred}\ (C_0(X) \otimes \mathcal{K}(L^2(\mathbb{Z}))) &= \textbf{Irred}\ (C_0(X)) \otimes \{pt\} = \\ &= \textbf{Irred}(C_0(X)). \end{aligned}$$

Further, the $C^*$-algebra $C_0(X)$ is commutative, hence the following equations are true

$$\textbf{Irred}\ (C_0(X)) = \textbf{Spec}\ (C_0(X)) = X.$$

Putting together the last three equations, one obtains:

$$\mathbf{Irred}\ (\mathcal{A}_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) \cong X.$$

The conclusion of lemma 5.3.3 follows from the above equation. □

Theorem 5.3.3 follows from Lemma 5.3.3. □

### 3.3.3 Real multiplication revisited

We shall test Theorem 5.3.3 for $\mathcal{A}_X \cong \mathcal{A}_{RM}$, i.e a noncommutative torus with real multiplication; notice that $\mathcal{A}_{RM}$ is the Serre $C^*$-algebra, see Example 5.3.3.

**Theorem 3.3.4**

$$\mathbf{Irred}\ (\mathcal{A}_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) \cong \mathcal{E}(K),$$

*where $\mathcal{E}(K)$ is non-singular elliptic curve defined over a field of algebraic numbers $K$.*

*Proof.* We shall view the crossed product $\mathcal{A}_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}$ as a $C^*$-dynamical system $(\mathcal{A}_{RM}, \hat{\mathbb{Z}}, \hat{\alpha})$, see [Williams 2007] [110] for the details. Recall that the irreducible representations of $C^*$-dynamical system $(\mathcal{A}_{RM}, \hat{\mathbb{Z}}, \hat{\alpha})$ are in the one-to-one correspondence with the minimal sets of the dynamical system (i.e. closed $\hat{\alpha}$-invariant sub-$C^*$-algebras of $\mathcal{A}_{RM}$ not containing a smaller object with the same property). To calculate the minimal sets of $(\mathcal{A}_{RM}, \hat{\mathbb{Z}}, \hat{\alpha})$, let $\theta$ be quadratic irrationality such that $\mathcal{A}_{RM} \cong \mathcal{A}_\theta$. It is known that every non-trivial sub-$C^*$-algebra of $\mathcal{A}_\theta$ has the form $\mathcal{A}_{n\theta}$ for some positive integer $n$, see [Rieffel 1981] [88], p. 419. It is easy to deduce that the *maximal* proper sub-$C^*$-algebra of $\mathcal{A}_\theta$ has the form $\mathcal{A}_{p\theta}$, where $p$ is a prime number. (Indeed, each composite $n = n_1 n_2$ cannot be maximal since $\mathcal{A}_{n_1 n_2 \theta} \subset \mathcal{A}_{n_1\theta} \subset \mathcal{A}_\theta$ or $\mathcal{A}_{n_1 n_2 \theta} \subset \mathcal{A}_{n_2\theta} \subset \mathcal{A}_\theta$, where all inclusions are strict.) We claim that $(\mathcal{A}_{p\theta}, \hat{\mathbb{Z}}, \hat{\alpha}^{\pi(p)})$ is the minimal $C^*$-dynamical system, where $\pi(p)$ is certain power of the automorphism $\hat{\alpha}$. Indeed, the automorphism $\hat{\alpha}$ of $\mathcal{A}_\theta$ corresponds to multiplication by the fundamental unit, $\varepsilon$, of pseudo-lattice $\Lambda = \mathbb{Z} + \theta\mathbb{Z}$. It is known that certain power, $\pi(p)$, of $\varepsilon$ coincides with the fundamental unit of pseudo-lattice $\mathbb{Z} + (p\theta)\mathbb{Z}$, see e.g. [Hasse 1950] [37], p. 298. Thus one gets the minimal $C^*$-dynamical system $(\mathcal{A}_{p\theta}, \hat{\mathbb{Z}}, \hat{\alpha}^{\pi(p)})$, which is defined on the sub-$C^*$-algebra $\mathcal{A}_{p\theta}$ of $\mathcal{A}_\theta$. Therefore we have an isomorphism

$$\mathbf{Irred}\ (\mathcal{A}_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) \cong \bigcup_{p \in \mathcal{P}} \mathbf{Irred}\ (\mathcal{A}_{p\theta} \rtimes_{\hat{\alpha}^{\pi(p)}} \hat{\mathbb{Z}}),$$

where $\mathcal{P}$ is the set of all (but a finite number) of primes. To simplify the RHS of the above equation, let us introduce some notation. Recall that matrix form of the fundamental unit $\varepsilon$ of pseudo-lattice $\Lambda$ coincides with the matrix $A$, see above. For each prime $p \in \mathcal{P}$ consider the matrix

$$L_p = \begin{pmatrix} tr\ (A^{\pi(p)}) - p & p \\ tr\ (A^{\pi(p)}) - p - 1 & p \end{pmatrix},$$

where $tr$ is the trace of matrix. Let us show, that

$$\mathcal{A}_{p\theta} \rtimes_{\hat{\alpha}^{\pi(p)}} \hat{\mathbb{Z}} \cong \mathcal{A}_\theta \rtimes_{L_p} \hat{\mathbb{Z}},$$

where $L_p$ is an endomorphism of $\mathcal{A}_\theta$ (of degree $p$) induced by matrix $L_p$. Indeed, because $deg\ (L_p) = p$ the endomorphism $L_p$ maps pseudo-lattice $\Lambda = \mathbb{Z} + \theta\mathbb{Z}$ to a sub-lattice of index $p$; any such can be written in the form $\Lambda_p = \mathbb{Z} + (p\theta)\mathbb{Z}$, see e.g. [Borevich & Shafarevich 1966] [11], p.131. Notice that pseudo-lattice $\Lambda_p$ corresponds to the sub-$C^*$-algebra $\mathcal{A}_{p\theta}$ of algebra $\mathcal{A}_\theta$ and $L_p$ induces a shift automorphism of $\mathcal{A}_{p\theta}$, see e.g. [Cuntz 1977] [17] beginning of Section 2.1 for terminology and details of this construction. It is not hard to see, that the shift automorphism coincides with $\hat{\alpha}^{\pi(p)}$. Indeed, it is verified directly that $tr\ (\hat{\alpha}^{\pi(p)}) = tr\ (A^{\pi(p)}) = tr\ (L_p)$; thus one gets a bijection between powers of $\hat{\alpha}^{\pi(p)}$ and such of $L_p$. But $\hat{\alpha}^{\pi(p)}$ corresponds to the fundamental unit of pseudo-lattice $\Lambda_p$; therefore the shift automorphism induced by $L_p$ must coincide with $\hat{\alpha}^{\pi(p)}$. The required isomorphism is proved and, therefore, our last formula can be written in the form

$$\mathbf{Irred}\ (\mathcal{A}_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) \cong \bigcup_{p \in \mathcal{P}} \mathbf{Irred}\ (\mathcal{A}_{RM} \rtimes_{L_p} \hat{\mathbb{Z}}).$$

To calculate irreducible representations of the crossed product $C^*$-algebra $\mathcal{A}_{RM} \rtimes_{L_p} \hat{\mathbb{Z}}$ at the RHS of the above equation, recall that such are in a one-to-one correspondence with the set of invariant measures on a subshift of finite type given by the positive integer matrix $L_p$, see [Bowen & Franks 1977] [12] and [Cuntz 1977] [17]; the measures make an abelian group under the addition operation. Such a group is isomorphic to $\mathbb{Z}^2\ /\ (I - L_p)\mathbb{Z}^2$, where $I$ is the identity matrix, see [Bowen & Franks 1977] [12], Theorem 2.2. Therefore our last equation can be written in the form

$$\mathbf{Irred}\ (\mathcal{A}_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) \cong \bigcup_{p \in \mathcal{P}} \frac{\mathbb{Z}^2}{(I - L_p)\mathbb{Z}^2}.$$

Let $\mathcal{E}(K)$ be a non-singular elliptic curve defined over the algebraic number field $K$; let $\mathcal{E}(\mathbb{F}_p)$ be the reduction of $\mathcal{E}(K)$ modulo prime ideal over a "good" prime number $p$. Recall that $|\mathcal{E}(\mathbb{F}_p)| = det\ (I - Fr_p)$, where $Fr_p$ is an integer two-by-two matrix corresponding to the action of Frobenius endomorphism on the $\ell$-adic cohomology of $\mathcal{E}(K)$, see e.g. [Tate 1974] [103], p. 187. Since $|\mathbb{Z}^2/(I - L_p)\mathbb{Z}^2| = det\ (I - L_p)$, one can identify $Fr_p$ and $L_p$ and, therefore, one obtains an isomorphism $\mathcal{E}(\mathbb{F}_p) \cong \mathbb{Z}^2/(I - L_p)\mathbb{Z}^2$. Thus our equation can be written in the form

$$\mathbf{Irred}\ (\mathcal{A}_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) \cong \bigcup_{p \in \mathcal{P}} \mathcal{E}(\mathbb{F}_p).$$

Finally, consider an arithmetic scheme, $X$, corresponding to $\mathcal{E}(K)$; the latter fibers over $\mathbb{Z}$, see [Silverman 1994] [101], Example 4.2.2 for the details. It can be immediately seen, that the RHS of our last equation coincides with the scheme $X$, where the regular fiber over $p$ corresponds to $\mathcal{E}(\mathbb{F}_p)$ *ibid.* The argument finishes the proof of Theorem 5.3.4. $\square$

**Guide to the literature.** The standard reference to complex projective varieties is the monograph [Hartshorne 1977] [35]. Twisted homogeneous coordinate rings of projective varieties are covered in the excellent survey by [Stafford & van den Bergh 2001] [100]. The Serre $C^*$-algebras were introduced and studied in [75].

## 3.4 Application: Mapping class groups

In the foreword it was asked: *Why does NCG matter?* We shall answer this question by solving a problem of classical geometry (Harvey's conjecture) using invariants attached to the functor $F$ : **Alg-Gen** $\to$ **AF-Toric**, see Theorem 5.2.1; the author is unaware of a "classical" proof of this result. The invariant in question is the stable isomorphism group of toric AF-algebra $\mathbb{A}_\theta$.

### 3.4.1 Harvey's conjecture

The mapping class group has been introduced in the 1920-ies by M. Dehn [Dehn 1938] [19]. Such a group, $Mod\ (X)$, is defined as the group of isotopy classes of the orientation-preserving diffeomorphisms of a two-sided closed surface $X$ of genus $g \geq 1$. The group is known to be prominent in algebraic geometry [Hain & Looijenga 1997] [31], topology [Thurston 1982] [104] and dynamics [Thurston 1988] [105]. When $X$ is a torus, the $Mod\ (X)$ is isomorphic to the group $SL(2,\mathbb{Z})$. (The $SL(2,\mathbb{Z})$ is called a modular group, hence our notation for the mapping class group.) A little is known about the representations of $Mod\ (X)$ beyond the case $g=1$. Recall, that the group is called *linear*, if there exists a faithful representation into the matrix group $GL(m,R)$, where $R$ is a commutative ring. The braid groups are known to be linear [Bigelow 2001] [8]. Using a modification of the argument for the braid groups, it is possible to prove, that $Mod\ (X)$ is linear in the case $g=2$ [Bigelow & Budney 2001] [9].

**Definition 3.4.1 ([Harvey 1979] [36], p.267)** *By Harvey's conjecture we understand the claim that the mapping class group is linear for $g \geq 3$.*

Recall that a covariant functor $F$ : **Alg-Gen** $\to$ **AF-Toric** from a category of generic Riemann surfaces (i.e. complex algebraic curves) to a category of toric AF-algebras was constructed in Section 5.2; the functor maps any pair of isomorphic Riemann surfaces to a pair of stably isomorphic (Morita equivalent) toric AF-algebras. Since each isomorphism of Riemann surfaces is given by an element of $Mod\ (X)$ [Hain & Looijenga 1997] [31], it is natural to ask about a representation of $Mod\ (X)$ by the stable isomorphisms of toric AF-algebras. Recall that the stable isomorphisms of toric AF-algebras are well understood and surprisingly simple; provided the automorphism group of the algebra is trivial (this is true for a generic algebra), its group of stable isomorphism admits a faithful representation into the matrix group $GL(m,\mathbb{Z})$, see e.g. [Effros 1981] [21]. This fact, combined with the properties of functor $F$, implies a positive solution to the Harvey conjecture.

**Theorem 3.4.1** *For every surface $X$ of genus $g \geq 2$, there exists a faithful representation $\rho : Mod\ (X) \to GL(6g-6,\mathbb{Z})$.*

## 3.4.2 Proof of Theorem 5.4.1

Let **AF-Toric** denote the set of all toric AF-algebras of genus $g \geq 2$. Let $G$ be a finitely presented group and

$$G \times \textbf{AF-Toric} \longrightarrow \textbf{AF-Toric}$$

be its action on **AF-Toric** by the stable isomorphisms (Morita equivalences) of toric AF-algebras; in other words, $\gamma(\mathbb{A}_\theta)\otimes\mathcal{K}\cong\mathbb{A}_\theta\otimes\mathcal{K}$ for all $\gamma\in G$ and all $\mathbb{A}_\theta\in$ **AF-Toric**. The following preparatory lemma will be important.

**Lemma 3.4.1** *For each* $\mathbb{A}_\theta\in$ **AF-Toric**, *there exists a representation*

$$\rho_{\mathbb{A}_\theta}: G\to GL(6g-6,\mathbb{Z}).$$

*Proof.* The proof of lemma is based on the following well known criterion of the stable isomorphism for the (toric) AF-algebras: a pair of such algebras $\mathbb{A}_\theta,\mathbb{A}_{\theta'}$ are stably isomorphic if and only if their Bratteli diagrams coincide, except (possibly) a finite part of the diagram, see e.g. [Effros 1981] [21], Theorem 2.3.

**Remark 3.4.1** Note that the order isomorphism between the dimension groups *ibid.*, translates to the language of the Bratteli diagrams as stated.

Let $G$ be a finitely presented group on the generators $\{\gamma_1,\dots,\gamma_m\}$ subject to relations $r_1,\dots,r_n$. Let $\mathbb{A}_\theta\in$ **AF-Toric**. Since $G$ acts on the toric AF-algebra $\mathbb{A}_\theta$ by stable isomorphisms, the toric AF-algebras $\mathbb{A}_{\theta_1}:=\gamma_1(\mathbb{A}_\theta),\dots,\mathbb{A}_{\theta_m}:=\gamma_m(\mathbb{A}_\theta)$ are stably isomorphic to $\mathbb{A}_\theta$; moreover, by transitivity, they are also pairwise stably isomorphic. Therefore, the Bratteli diagrams of $\mathbb{A}_{\theta_1},\dots,\mathbb{A}_{\theta_m}$ coincide everywhere except, possibly, some finite parts. We shall denote by $\mathbb{A}_{\theta_{\max}}\in$ **AF-Toric** a toric AF-algebra, whose Bratteli diagram is the maximal common part of the Bratteli diagrams of $\mathbb{A}_{\theta_i}$ for $1\le i\le m$; such a choice is unique and defined correctly because the set $\{\mathbb{A}_{\theta_i}\}$ is a finite set. By the Definition 5.2.1 of a toric AF-algebra, the vectors $\theta_i=(1,\theta_1^{(i)},\dots,\theta_{6g-7}^{(i)})$ are related to the vector $\theta_{\max}=(1,\theta_1^{(\max)},\dots,\theta_{6g-7}^{(\max)})$ by the formula

$$\begin{pmatrix}1\\ \theta_1^{(i)}\\ \vdots\\ \theta_{6g-7}^{(i)}\end{pmatrix}=\underbrace{\begin{pmatrix}0&0&\dots&0&1\\ 1&0&\dots&0&b_1^{(1)(i)}\\ \vdots&\vdots&&\vdots&\vdots\\ 0&0&\dots&1&b_{6g-7}^{(1)(i)}\end{pmatrix}\dots\begin{pmatrix}0&0&\dots&0&1\\ 1&0&\dots&0&b_1^{(k)(i)}\\ \vdots&\vdots&&\vdots&\vdots\\ 0&0&\dots&1&b_{6g-7}^{(k)(i)}\end{pmatrix}}_{A_i}\begin{pmatrix}1\\ \theta_1^{(\max)}\\ \vdots\\ \theta_{6g-7}^{(\max)}\end{pmatrix}$$

The above expression can be written in the matrix form $\theta_i=A_i\theta_{\max}$, where $A_i\in GL(6g-6,\mathbb{Z})$. Thus, one gets a matrix representation of the generator $\gamma_i$, given by the formula

$$\rho_{\mathbb{A}_\theta}(\gamma_i):=A_i.$$

The map $\rho_{\mathbb{A}_\theta} : G \to GL(6g-6,\mathbb{Z})$ extends to the rest of the group $G$ via its values on the generators; namely, for every $g \in G$ one sets $\rho_{\mathbb{A}_\theta}(g) = A_1^{k_1}\dots A_m^{k_m}$, whenever $g = \gamma_1^{k_1}\dots\gamma_m^{k_m}$. Let us verify, that the map $\rho_{\mathbb{A}_\theta}$ is a well defined homomorphism of groups $G$ and $GL(6g-6,\mathbb{Z})$. Indeed, let us write $g_1 = \gamma_1^{k_1}\dots\gamma_m^{k_m}$ and $g_2 = \gamma_1^{s_1}\dots\gamma_m^{s_m}$ for a pair of elements $g_1, g_2 \in G$; then their product $g_1g_2 = \gamma_1^{k_1}\dots\gamma_m^{k_m}\gamma_1^{s_1}\dots\gamma_m^{s_m} = \gamma_1^{l_1}\dots\gamma_m^{l_m}$, where the last equality is obtained by a reduction of words using the relations $r_1,\dots,r_n$. One can write relations $r_i$ in their matrix form $\rho_{\mathbb{A}_\theta}(r_i)$; thus, one gets the matrix equality $A_1^{l_1}\dots A_m^{l_m} = A_1^{k_1}\dots A_m^{k_m}A_1^{s_1}\dots A_m^{s_m}$. It is immediate from the last equation, that $\rho_{\mathbb{A}_\theta}(g_1g_2) = A_1^{l_1}\dots A_m^{l_m} = A_1^{k_1}\dots A_m^{k_m}A_1^{s_1}\dots A_m^{s_m} = \rho_{\mathbb{A}_\theta}(g_1)\rho_{\mathbb{A}_\theta}(g_2)$ for $\forall g_1, g_2 \in G$, i.e. $\rho_{\mathbb{A}_\theta}$ is a homomorphism. Lemma 5.4.1 follows. □

Let **AF-Toric-Aper** $\subset$ **AF-Toric** be a set consisting of the toric AF-algebras, whose Bratteli diagrams are *not* periodic; these are known as non-stationary toric AF-algebras (Section 3.5.2) and they are generic in the set **AF-Toric** endowed with the natural topology.

**Definition 3.4.2** *The action of group $G$ on the toric AF-algebra $\mathbb{A}_\theta \in$* **AF-Toric** *will be called free, if $\gamma(\mathbb{A}_\theta) = \mathbb{A}_\theta$ implies $\gamma = Id$.*

**Lemma 3.4.2** *If $\mathbb{A}_\theta \in$* **AF-Toric-Aper** *and the action of group $G$ on the $\mathbb{A}_\theta$ is free, then $\rho_{\mathbb{A}_\theta}$ is a faithful representation.*

*Proof.* Since the action of $G$ is free, to prove that $\rho_{\mathbb{A}_\theta}$ is faithful, it remains to show, that in the formula $\theta_i = A_i\theta_{\max}$, it holds $A_i = I$, if and only if, $\theta_i = \theta_{\max}$, where $I$ is the unit matrix. Indeed, it is immediate that $A_i = I$ implies $\theta_i = \theta_{\max}$. Suppose now that $\theta_i = \theta_{\max}$ and, let to the contrary, $A_i \neq I$. One gets $\theta_i = A_i\theta_{\max} = \theta_{\max}$. Such an equation has a non-trivial solution, if and only if, the vector $\theta_{\max}$ has a periodic Jacobi-Perron fraction; the period of such a fraction is given by the matrix $A_i$. This is impossible, since it has been assumed, that $\mathbb{A}_{\theta_{\max}} \in$ **AF-Toric-Aper**. The contradiction proves Lemma 5.4.2. □

Let $G = Mod\ (X)$, where $X$ is a surface of genus $g \geq 2$. The group $G$ is finitely presented, see [Dehn 1938] [19]; it acts on the Teichmueller space $T(g)$ by isomorphisms of the Riemann surfaces. Moreover, the action of $G$ is free on a generic set, $U \subset T(g)$, consisting of the Riemann surfaces with the trivial group of automorphisms. On the other hand, there exists a functor

$$F : \textbf{Alg-Gen} \longrightarrow \textbf{AF-Toric}$$

between the Riemann surfaces (complex algebraic curves) and toric AF-algebras, see Theorem 5.2.1.

**Lemma 3.4.3** *The pre-image* $F^{-1}$(**AF-Toric-Aper**) *is a generic set in the space* $T(g)$.

*Proof.* Note, that the set of stationary toric AF-algebras is a countable set. The functor $F$ is a surjective map, which is continuous with respect to the natural topology on the sets **Alg-Gen** and **AF-Toric**. Therefore, the pre-image of the complement of a countable set is a generic set. Lemma 5.4.3 follows. □

Consider the set $U \cap F^{-1}$(**AF-Toric-Aper**); this set is non-empty, since it is the intersection of two generic subsets of $T(g)$, see Lemma 5.4.3. Let

$$S \in U \cap F^{-1}(\textbf{AF-Toric-Aper})$$

be a point (a Riemann surface) in the above set. In view of Lemma 5.4.1, group $G$ acts on the toric AF-algebra $\mathbb{A}_\theta = F(S)$ by the stable isomorphisms. By the construction, the action is free and $\mathbb{A}_\theta \in$ **AF-Toric-Aper**. In view of Lemma 5.4.2, one gets a faithful representation $\rho = \rho_{\mathbb{A}_\theta}$ of the group $G \cong Mod\ (X)$ into the matrix group $GL(6g-6, \mathbb{Z})$. Theorem 5.4.1 is proved. □

**Guide to the literature.** The mapping class groups were introduced by M. Dehn [Dehn 1938] [19]. For a primer on the mapping class groups we refer the reader to the textbook [Farb & Margalit 2011] [25]. The Harvey conjecture was formulated in [Harvey 1979] [36]. Some infinite-dimensional (asymptotic) faithfulness of the mapping class groups was proved by [Anderson 2006] [1]. A faithful representation of $Mod\ (X)$ in the matrix group $GL(6g-6, \mathbb{Z})$ was constructed in [76].

# Exercises

1. Prove that the skew-symmetric relations

$$\begin{cases} x_3x_1 &= q_{13}x_1x_3, \\ x_4x_2 &= q_{24}x_2x_4, \\ x_4x_1 &= q_{14}x_1x_4, \\ x_3x_2 &= q_{23}x_2x_3, \\ x_2x_1 &= q_{12}x_1x_2, \\ x_4x_3 &= q_{34}x_3x_4, \end{cases}$$

   are invariant of the involution $x_1^* = x_2, x_3^* = x_4$, if and only if, the following restrictions on the constants $q_{ij}$ hold

$$\begin{cases} q_{13} &= (\bar{q}_{24})^{-1}, \\ q_{24} &= (\bar{q}_{13})^{-1}, \\ q_{14} &= (\bar{q}_{23})^{-1}, \\ q_{23} &= (\bar{q}_{14})^{-1}, \\ q_{12} &= \bar{q}_{12}, \\ q_{34} &= \bar{q}_{34}, \end{cases}$$

   where $\bar{q}_{ij}$ means the complex conjugate of $q_{ij} \in \mathbb{C} \setminus \{0\}$.

2. Prove that a family of free algebras $\mathbb{C}\langle x_1, x_2, x_3, x_4\rangle$ modulo an ideal generated by six skew-symmetric quadratic relations

$$\begin{cases} x_3x_1 &= \mu e^{2\pi i\theta}x_1x_3, \\ x_4x_2 &= \frac{1}{\mu}e^{2\pi i\theta}x_2x_4, \\ x_4x_1 &= \mu e^{-2\pi i\theta}x_1x_4, \\ x_3x_2 &= \frac{1}{\mu}e^{-2\pi i\theta}x_2x_3, \\ x_2x_1 &= x_1x_2, \\ x_4x_3 &= x_3x_4, \end{cases}$$

   consists of the pairwise non-isomorphic algebras for different values of $\theta \in S^1$ and $\mu \in (0, \infty)$.

3. Prove that the system of relations for noncommutative torus $\mathcal{A}_\theta$

$$\begin{cases} x_3x_1 & = e^{2\pi i\theta}x_1x_3, \\ x_4x_2 & = e^{2\pi i\theta}x_2x_4, \\ x_4x_1 & = e^{-2\pi i\theta}x_1x_4, \\ x_3x_2 & = e^{-2\pi i\theta}x_2x_3, \\ x_2x_1 & = x_1x_2 = e, \\ x_4x_3 & = x_3x_4 = e. \end{cases}$$

is equivalent to the system of relations

$$\begin{cases} x_3x_1x_4 & = & e^{2\pi i\theta}x_1, \\ x_4 & = & e^{2\pi i\theta}x_2x_4x_1, \\ x_4x_1x_3 & = & e^{-2\pi i\theta}x_1, \\ x_2 & = & e^{-2\pi i\theta}x_4x_2x_3, \\ x_1x_2 & = & x_2x_1 = e, \\ x_3x_4 & = & x_4x_3 = e. \end{cases}$$

(Hint: use the last two relations.)

4. Prove that the system of relations for the Sklyanin $*$-algebra plus the scaled unit relation, i.e.

$$\begin{cases} x_3x_1 & = \mu e^{2\pi i\theta}x_1x_3, \\ x_4x_2 & = \frac{1}{\mu}e^{2\pi i\theta}x_2x_4, \\ x_4x_1 & = \mu e^{-2\pi i\theta}x_1x_4, \\ x_3x_2 & = \frac{1}{\mu}e^{-2\pi i\theta}x_2x_3, \\ x_2x_1 & = x_1x_2 = \frac{1}{\mu}e, \\ x_4x_3 & = x_3x_4 = \frac{1}{\mu}e \end{cases}$$

is equivalent to the system

$$\begin{cases} x_3x_1x_4 & = e^{2\pi i\theta}x_1, \\ x_4 & = e^{2\pi i\theta}x_2x_4x_1, \\ x_4x_1x_3 & = e^{-2\pi i\theta}x_1, \\ x_2 & = e^{-2\pi i\theta}x_4x_2x_3, \\ x_2x_1 & = x_1x_2 = \frac{1}{\mu}e, \\ x_4x_3 & = x_3x_4 = \frac{1}{\mu}e. \end{cases}$$

(Hint: use multiplication and cancellation involving the last two equations.)

5. If **Proj-Alg** is the category of all complex projective varieties $X$ (of dimension $n$) and **C*-Serre** the category of all Serre $C^*$-algebras $\mathcal{A}_X$, then the formula $X \mapsto \mathcal{A}_X$ gives rise to a map

$$F : \textbf{Proj-Alg} \longrightarrow \textbf{C*-Serre}.$$

Prove that the map $F$ is actually a functor which takes isomorphisms between projective varieties to the stable isomorphisms (Morita equivalences) between the corresponding Serre $C^*$-algebras. (Hint: repeat the argument for elliptic curves given in Section 5.1.1.)

6. Prove Remark 5.3.5, i.e. that equations $b_k^{\alpha} t = t b_k$ and $t^* = t^{-1}$ imply equation $(b_k^*)^{\alpha} = (b_k^{\alpha})^*$.

# Chapter 4

# Number Theory

The most elegant functors (with values in NCG) are acting on the arithmetic schemes $X$. We start with the simplest case of $X$ being elliptic curve with complex multiplication by the number field $k = \mathbb{Q}(f\sqrt{-D})$; in this case $X \cong \mathcal{E}(K)$, where $K$ is the Hilbert class field of $k$ [Serre 1967] [92]. We prove in Section 6.1 that functor $F$ sends $\mathcal{E}(K)$ to noncommutative torus with real multiplication by the number field $\mathbb{Q}(f\sqrt{D})$. It is proved in Section 6.2 that the so-called *arithmetic complexity* of such a torus is linked by a simple formula to the rank of elliptic curve $\mathcal{E}(K)$ whenever $D \equiv 3 \textbf{ mod } 4$ is a prime number and $f = 1$. In Section 6.3 we introduce an $L$-function $L(\mathcal{A}_{RM}, s)$ associated to the noncommutative torus with real multiplication and prove that any such coincides with the classical Hasse-Weil function $L(\mathcal{E}_{CM}, s)$ of an elliptic curve with complex multiplication; a surprising *localization formula* tells us that the crossed products replace prime (or maximal) ideals familiar from the commutative algebra. In Section 6.4 a functor $F$ : **Alg-Num** $\to$ **NC-Tor** from a category of the finite Galois extensions $E$ of the field $\mathbb{Q}$ to the category of even-dimensional noncommutative tori with real multiplication $\mathcal{A}_{RM}^{2n}$ is defined. An $L$-function $L(\mathcal{A}_{RM}^{2n}, s)$ is constructed and it is conjectured that if $\mathcal{A}_{RM}^{2n} = F(E)$, then $L(\mathcal{A}_{RM}^{2n}, s) \equiv L(\sigma, s)$, where $L(\sigma, s)$ is the *Artin L-function* of $E$ corresponding to an irreducible representation $\sigma : Gal\ (E|\mathbb{Q}) \to GL_n(\mathbb{C})$. We prove the conjecture for $n = 1$ (resp., $n = 0$) and $E$ being the Hilbert class field of an imaginary quadratic field $k$ (resp., field $\mathbb{Q}$). Thus we deal with an analog of the *Langlands program*, where the "automorphic cuspidal representations of group $GL_n$" are replaced by the noncommutative tori $\mathcal{A}_{RM}^{2n}$, see [Gelbart 1984] [28] for an introduction to the Langlands program. In Section 6.5 we compute the number of points of projective variety $V(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ in terms of invariants of the *Serre $C^*$-algebra* associated to the complex projective variety $V(\mathbb{C})$, see Section 5.3.1; the calculation involves an explicit formula for the traces of *Frobenius map* of $V(\mathbb{F}_q)$ being linked to the *Weil Conjectures*, see e.g. [Hartshorne 1977] [35], Appendix C for an introduction. Finally, in Section 6.6 we apply our functor $F$ : **Ell** $\to$ **NC-Tor** to a problem of the *transcendental number theory*, see e.g. [Baker 1975] [5] for an introduction. Namely, we use the formula $F(\mathcal{E}_{CM}^{(-D,f)}) = \mathcal{A}_{RM}^{(D,f)}$ of Section 6.1 to prove that the transcendental function $\mathcal{J}(\theta,\varepsilon) = e^{2\pi i\theta+\log\log\varepsilon}$ takes algebraic values for the algebraic arguments $\theta$ and $\varepsilon$. Moreover, these values of $\mathcal{J}(\theta,\varepsilon)$ belong to the Hilbert class field of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ for all but a finite set of values of $D$.

## 4.1 Complex multiplication

We recall that an *elliptic curve* is the subset of the complex projective plane of the form $\mathcal{E}(\mathbb{C}) = \{(x,y,z) \in \mathbb{C}P^2 \mid y^2z = 4x^3 - g_2xz^2 - g_3z^3\}$, where $g_2$ and $g_3$ are some constant complex numbers. The *$j$-invariant* of $\mathcal{E}(\mathbb{C})$ is the complex number

$$j(\mathcal{E}(\mathbb{C})) = \frac{1728g_2^3}{g_2^3 - 27g_3^2},$$

which is constant only on isomorphic elliptic curves. The Weierstrass function $\wp(z)$ defines an isomorphism $\mathcal{E}(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ between elliptic curves and complex tori of modulus $\tau \in \mathbb{H} := \{z = x+iy \in \mathbb{C} \mid y > 0\}$, see Theorem 5.1.1; by $\mathcal{E}_\tau$ we understand an elliptic curve of complex modulus $\tau$.

**Definition 4.1.1** *By an isogeny between elliptic curves $\mathcal{E}_\tau$ and $\mathcal{E}_{\tau'}$ one understands an analytic map $\varphi : \mathcal{E}_\tau \to \mathcal{E}_{\tau'}$, such that $\varphi(0) = 0$. Clearly, the invertible isogeny corresponds to an isomorphism between elliptic curves.*

**Remark 4.1.1** The elliptic curves $\mathcal{E}_\tau$ and $\mathcal{E}_{\tau'}$ are isogenous if and only if

$$\tau' = \frac{a\tau+b}{c\tau+d} \quad \text{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \quad \text{with} \quad ad-bc > 0.$$

The case of an invertible matrix (i.e. $ad-bc=1$) corresponds to an isomorphism between elliptic curves. (We leave the proof to the reader. Hint: notice that $z \mapsto \alpha z$ is an invertible holomorphic map for each $\alpha \in \mathbb{C} - \{0\}$.)

An *endomorphism* of $\mathcal{E}_\tau$ is a multiplication of the lattice $L_\tau := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ by complex number $z$ such that

$$zL_\tau \subseteq L_\tau.$$

In other words, the endomorphism is an isogeny of the elliptic curve into itself. The sum and product of two endomorphisms is an endomorphism of $\mathcal{E}_\tau$; thus one gets a commutative ring of all endomorphisms of $\mathcal{E}_\tau$ denoted by $End\ (\mathcal{E}_\tau)$. Typically $End\ (\mathcal{E}_\tau) \cong \mathbb{Z}$, i.e. the only endomorphisms of $\mathcal{E}_\tau$ are the multiplication-by-$m$ endomorphisms; however, for a countable set of $\tau$

$$End\ (\mathcal{E}_\tau) \cong \mathbb{Z} + fO_k,$$

where $k = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field, $O_k$ its ring of integers and $f \geq 1$ is the conductor of a finite index subring of $O_k$. (The proof of this simple but fundamental fact is left to the reader.) It is easy to see that in such a case $\tau \in End\ (\mathcal{E}_\tau)$, i.e. complex modulus itself is an imaginary quadratic number.

**Definition 4.1.2** *Elliptic curve $\mathcal{E}_\tau$ is said to have complex multiplication if $End\ (\mathcal{E}_\tau) \cong \mathbb{Z} + fO_k$, i.e. $\tau$ is an imaginary quadratic number; such a curve will be denoted by $\mathcal{E}_{CM}^{(-D,f)}$.*

**Remark 4.1.2** There is a finite number of pairwise non-isomorphic elliptic curves with the same ring of non-trivial endomorphisms $R := End\ (\mathcal{E}_\tau)$; such a number is equal to $|Cl\ (R)|$, where $Cl\ (R)$ is the class group of ring $R$. This fact is extremely important, because the $j$-invariant $j(\mathcal{E}_{CM}^{(-D,f)})$ is known to be an algebraic number and, therefore, $Gal\ (K|k) \cong Cl\ (R)$, where $K = k(j(\mathcal{E}_{CM}^{(-D,f)}))$ and $Gal\ (K|k)$ is the Galois group of the field extension $K|k$. In other words, the number field $K$ is the *Hilbert class field* of imaginary quadratic field $k$. Moreover,

$$\mathcal{E}_{CM}^{(-D,f)} \cong \mathcal{E}(K),$$

i.e. the complex constants $g_2$ and $g_3$ in the cubic equation for $\mathcal{E}_{CM}^{(-D,f)}$ must belong to the number field $K$.

### 4.1.1 Functor on elliptic curves with complex multiplication

**Definition 4.1.3** *By* **Ell-Isgn** *we shall mean the category of all elliptic curves $\mathcal{E}_\tau$; the arrows of* **Ell-Isgn** *are identified with the isogenies between elliptic curves $\mathcal{E}_\tau$. We shall write* **NC-Tor-Homo** *to denote the category of all noncommutative tori $\mathcal{A}_\theta$; the arrows of* **NC-Tor-Homo** *are identified with the stable homomorphisms between noncommutative tori $\mathcal{A}_\theta$.*

**Remark 4.1.3** The noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are stably homomorphic if and only if

$$\theta' = \frac{a\theta + b}{c\theta + d} \quad \text{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \quad \text{with} \quad ad - bc > 0.$$

The case of an invertible matrix (i.e. $ad - bc = 1$) corresponds to a stable isomorphism (Morita equivalence) between noncommutative tori. (We leave the proof to the reader. Hint: follow and modify the argument of [Rieffel 1981] [88].)

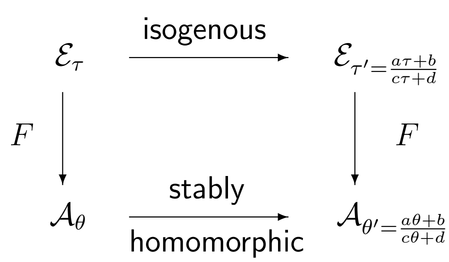



Figure 4.1: Functor on isogenous elliptic curves.

**Theorem 4.1.1 (Functor on isogenous elliptic curves)** *There exists a covariant functor*

$$F : \textbf{Ell-Isgn} \longrightarrow \textbf{NC-Tor-Homo},$$

*which maps isogenous elliptic curves $\mathcal{E}_\tau$ to the stably homomorphic noncommutative tori $\mathcal{A}_\theta$, see Fig. 6.1; the functor $F$ is non-injective and $Ker\ F \cong (0, \infty)$. In particular, $F$ maps isomorphic elliptic curves to the stably isomorphic (Morita equivalent) noncommutative tori.*

**Theorem 4.1.2 (Functor on elliptic curves with complex multiplication)** *If* $Isom\ (\mathcal{E}_{CM}) := \{\mathcal{E}_\tau \in \textbf{Ell-Isgn} \mid \mathcal{E}_\tau \cong \mathcal{E}_{CM}\}$ *is the isomorphism class of an elliptic curve with complex multiplication and* $\mathfrak{m}_{CM} := \mu_{CM}(\mathbb{Z} + \mathbb{Z}\theta_{CM}) \subset \mathbb{R}$ *is a* $\mathbb{Z}$*-module such that* $\mathcal{A}_{\theta_{CM}} = F(\mathcal{E}_{CM})$ *and* $\mu_{CM} \in Ker\ F$*, then:*

*(i)* $\mathfrak{m}_{CM}$ *is an invariant of* $Isom\ (\mathcal{E}_{CM})$*;*

*(ii)* $\mathfrak{m}_{CM}$ *is a full module in the real quadratic number field.*

*In particular,* $\mathcal{A}_{\theta_{CM}}$ *is a noncommutative torus with real multiplication.*

**Definition 4.1.4** *If* $\mathcal{A}_{RM}^{(D,f)}$ *is a noncommutative torus with real multiplication, then the Riemann surface* $X(\mathcal{A}_{RM}^{(D,f)})$ *is called associated to* $\mathcal{A}_{RM}^{(D,f)}$ *whenever the covering of geodesic spectrum of* $X(\mathcal{A}_{RM}^{(D,f)})$ *on the half-plane* $\mathbb{H}$ *contains the set* $\{\tilde{\gamma}(x,\bar{x})\ :\ \forall x \in \mathfrak{m}_{CM}\}$*, where*

$$\tilde{\gamma}(x,\bar{x}) = \frac{xe^{\frac{t}{2}} + i\bar{x}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} + ie^{-\frac{t}{2}}}, \qquad -\infty \le t \le \infty$$

*is the geodesic half-circle through the pair of conjugate quadratic irrationalities* $x, \bar{x} \in \mathfrak{m}_{CM} \subset \partial\mathbb{H}$*, see Definition 6.1.5.*

**Theorem 4.1.3 (Functor on noncommutative tori with real multiplication)** *For each square-free integer* $D > 1$ *and integer* $f \ge 1$ *there exists a holomorphic map* $F^{-1} : X(\mathcal{A}_{RM}^{(D,f)}) \to \mathcal{E}_{CM}^{(-D,f)}$*, where* $F(\mathcal{E}_{CM}^{(-D,f)}) = \mathcal{A}_{RM}^{(D,f)}$*.*

**Remark 4.1.4** Roughly speaking, Theorem 6.1.3 is an explicit form of functor $F$ constructed in Theorem 6.1.2; moreover, Theorem 6.1.3 says the $F$ is a bijection by constructing an explicit inverse functor $F^{-1}$.

## 4.1.2 Proof of Theorem 6.1.1

The proof is a modification of the one for Theorem 5.1.2; we freely use the notation and facts of the Teichmüller theory introduced in Section 5.1.2. Let $\phi = Re\ \omega$ be a 1-form defined by a holomorphic form $\omega$ on the complex torus $S$. Since $\omega$ is holomorphic, $\phi$ is a closed 1-form on topological torus $T^2$. The $\mathbb{R}$-isomorphism $h_q : H^0(S,\Omega) \to Hom\ (H_1(T^2);\mathbb{R})$, as explained, is given by the formulas:

$$\begin{cases} \lambda_1 &= \int_{\gamma_1}\phi \\ \lambda_2 &= \int_{\gamma_2}\phi, \end{cases}$$

where $\{\gamma_1, \gamma_2\}$ is a basis in the first homology group of $T^2$. We further assume that, after a proper choice of the basis, $\lambda_1, \lambda_2$ are positive real numbers. Denote by $\Phi_{T^2}$ the space of measured foliations on $T^2$. Each $\mathcal{F} \in \Phi_{T^2}$ is measure equivalent to a foliation by a family of the parallel lines of a slope $\theta$ and the invariant transverse measure $\mu$, see Fig. 6.2.

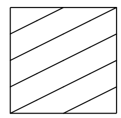


Figure 4.2: Measured foliation $\mathcal{F}$ on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

We use the notation $\mathcal{F}_\theta^\mu$ for such a foliation. There exists a simple relationship between the reals $(\lambda_1, \lambda_2)$ and $(\theta, \mu)$. Indeed, the closed 1-form $\phi = Const$ defines a measured foliation, $\mathcal{F}_\theta^\mu$, so that

$$\left\{\begin{array}{lll} \lambda_1 & = \int_{\gamma_1} \phi & = \int_0^1 \mu dx \\ \lambda_2 & = \int_{\gamma_2} \phi & = \int_0^1 \mu dy \end{array}\right. , \text{ where } \quad \frac{dy}{dx} = \theta.$$

By the integration:

$$\left\{\begin{array}{lll} \lambda_1 & = \int_0^1 \mu dx & = \mu \\ \lambda_2 & = \int_0^1 \mu\theta dx & = \mu\theta. \end{array}\right.$$

Thus, one gets $\mu = \lambda_1$ and $\theta = \frac{\lambda_2}{\lambda_1}$. Recall that the Hubbard-Masur theory establishes a homeomorphism $h : T_S(1) \to \Phi_{T^2}$, where $T_S(1) \cong \mathbb{H} = \{\tau : Im\ \tau > 0\}$ is the Teichmüller space of the torus, see Corollary 5.1.1. Denote by $\omega_N$ an invariant (Néron) differential of the complex torus $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$. It is well known that $\omega_1 = \int_{\gamma_1} \omega_N$ and $\omega_2 = \int_{\gamma_2} \omega_N$, where $\gamma_1$ and $\gamma_2$ are the meridians of the torus. Let $\pi$ be a projection acting by the formula $(\theta, \mu) \mapsto \theta$. An explicit formula for the functor $F$ : **Ell-Isgn** $\to$ **NC-Tor-Homo** is given by the composition $F = \pi \circ h$, where $h$ is the Hubbard-Masur homeomorphism. In other words, one gets the following explicit correspondence between the complex and noncommutative tori:

$$\mathcal{E}_\tau = \mathcal{E}_{(\int_{\gamma_2} \omega_N)/(\int_{\gamma_1} \omega_N)} \stackrel{\text{h}}{\longmapsto} \mathcal{F}^{\int_{\gamma_1} \phi}_{(\int_{\gamma_2} \phi)/(\int_{\gamma_1} \phi)} \stackrel{\pi}{\longmapsto} \mathcal{A}_{(\int_{\gamma_2} \phi)/(\int_{\gamma_1} \phi)} = \mathcal{A}_\theta,$$

where $\mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Let

$$\varphi : \mathcal{E}_\tau \longrightarrow \mathcal{E}_{\tau'}$$

be an isogeny of the elliptic curves. The action of $\varphi$ on the homology basis $\{\gamma_1, \gamma_2\}$ of $T^2$ is given by the formulas

$$\left\{\begin{array}{ll} \gamma_1' & = a\gamma_1 + b\gamma_2 \\ \gamma_2' & = c\gamma_1 + d\gamma_2 \end{array}\right. , \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}).$$

Recall that the functor $F$ : **Ell-Isgn** $\to$ **NC-Tor-Homo** is given by the formula

$$\tau = \frac{\int_{\gamma_2} \omega_N}{\int_{\gamma_1} \omega_N} \longmapsto \theta = \frac{\int_{\gamma_2} \phi}{\int_{\gamma_1} \phi},$$

where $\omega_N$ is an invariant differential on $\mathcal{E}_\tau$ and $\phi = Re\ \omega$ is a closed 1-form on $T^2$.

(i) From the left-hand side of the above equation, one obtains

$$\left\{\begin{array}{lllll} \omega_1' & = \int_{\gamma_1'} \omega_N & = \int_{a\gamma_1 + b\gamma_2} \omega_N & = a\int_{\gamma_1} \omega_N + b\int_{\gamma_2} \omega_N & = a\omega_1 + b\omega_2 \\ \omega_2' & = \int_{\gamma_2'} \omega_N & = \int_{c\gamma_1 + d\gamma_2} \omega_N & = c\int_{\gamma_1} \omega_N + d\int_{\gamma_2} \omega_N & = c\omega_1 + d\omega_2, \end{array}\right.$$

and therefore $\tau' = \frac{\int_{\gamma_2'} \omega_N}{\int_{\gamma_1'} \omega_N} = \frac{c + d\tau}{a + b\tau}$.

(ii) From the right-hand side, one obtains

$$\left\{\begin{array}{lllll} \lambda_1' & = \int_{\gamma_1'} \phi & = \int_{a\gamma_1 + b\gamma_2} \phi & = a\int_{\gamma_1} \phi + b\int_{\gamma_2} \phi & = a\lambda_1 + b\lambda_2 \\ \lambda_2' & = \int_{\gamma_2'} \phi & = \int_{c\gamma_1 + d\gamma_2} \phi & = c\int_{\gamma_1} \phi + d\int_{\gamma_2} \phi & = c\lambda_1 + d\lambda_2, \end{array}\right.$$

and therefore $\theta' = \frac{\int_{\gamma_2'} \phi}{\int_{\gamma_1'} \phi} = \frac{c + d\theta}{a + b\theta}$. Comparing (i) and (ii), one gets the conclusion of the first part of Theorem 6.1.1. To prove the second part, recall that the invertible isogeny is an isomorphism of the elliptic curves. In this case $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\theta' = \theta \ mod\ SL_2(\mathbb{Z})$. Therefore $F$ sends the isomorphic elliptic curves to the stably isomorphic noncommutative tori. The second part of Theorem 6.1.1 is proved. It follows from the proof that $F$ : **Ell-Isgn** $\to$ **NC-Tor-Homo** is a covariant functor. Indeed, $F$ preserves the morphisms and does not reverse the arrows: $F(\varphi_1\varphi_2) = \varphi_1\varphi_2 = F(\varphi_1)F(\varphi_2)$ for any pair of the isogenies $\varphi_1, \varphi_2 \in Mor$ (**Ell-Isgn**). Theorem 6.1.1 follows. $\square$

### 4.1.3 Proof of Theorem 6.1.2

**Lemma 4.1.1** *Let $\mathfrak{m} \subset \mathbb{R}$ be a module of the rank 2, i.e $\mathfrak{m} = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, where $\theta = \frac{\lambda_2}{\lambda_1} \notin \mathbb{Q}$. If $\mathfrak{m}' \subseteq \mathfrak{m}$ is a submodule of the rank 2, then $\mathfrak{m}' = k\mathfrak{m}$, where either:*

*(i) $k \in \mathbb{Z} - \{0\}$ and $\theta \in \mathbb{R} - \mathbb{Q}$, or*

*(ii) $k$ and $\theta$ are the irrational numbers of a quadratic number field.*

*Proof.* Any rank 2 submodule of $m$ can be written as $\mathfrak{m}' = \lambda_1'\mathbb{Z} + \lambda_2'\mathbb{Z}$, where

$$\begin{cases} \lambda_1' &= a\lambda_1 + b\lambda_2 \\ \lambda_2' &= c\lambda_1 + d\lambda_2 \end{cases} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}).$$

(i) Let us assume that $b \neq 0$. Let $\Delta = (a+d)^2 - 4(ad - bc)$ and $\Delta' = (a+d)^2 - 4bc$. We shall consider the following cases.

**Case 1: $\Delta > 0$ and $\Delta \neq m^2$, $m \in \mathbb{Z} - \{0\}$.** The real number $k$ can be determined from the equations:

$$\begin{cases} \lambda_1' &= k\lambda_1 &= a\lambda_1 + b\lambda_2 \\ \lambda_2' &= k\lambda_2 &= c\lambda_1 + d\lambda_2. \end{cases}$$

Since $\theta = \frac{\lambda_2}{\lambda_1}$, one gets the equation $\theta = \frac{c+d\theta}{a+b\theta}$ by taking the ratio of two equations above. A quadratic equation for $\theta$ writes as $b\theta^2 + (a-d)\theta - c = 0$. The discriminant of the equation coincides with $\Delta$ and therefore there exist real roots $\theta_{1,2} = \frac{d-a\pm\sqrt{\Delta}}{2b}$. Moreover, $k = a + b\theta = \frac{1}{2}(a + d \pm \sqrt{\Delta})$. Since $\Delta$ is not the square of an integer, $k$ and $\theta$ are irrationalities of the quadratic number field $\mathbb{Q}(\sqrt{\Delta})$.

**Case 2: $\Delta > 0$ and $\Delta = m^2$, $m \in \mathbb{Z} - \{0\}$.** Note that $\theta = \frac{a-d\pm|m|}{2c}$ is a rational number. Since $\theta$ does not satisfy the rank assumption of the lemma, the case should be omitted.

**Case 3: $\Delta = 0$.** The quadratic equation has a double root $\theta = \frac{a-d}{2c} \in \mathbb{Q}$. This case leads to a module of the rank 1, which is contrary to an assumption of the lemma.

**Case 4: $\Delta < 0$ and $\Delta' \neq m^2$, $m \in \mathbb{Z} - \{0\}$.** Let us define a new basis $\{\lambda_1'', \lambda_2''\}$ in $\mathfrak{m}'$ so that

$$\begin{cases} \lambda_1'' &= \lambda_1' \\ \lambda_2'' &= -\lambda_2'. \end{cases}$$

Then:

$$\begin{cases} \lambda_1'' &= a\lambda_1 + b\lambda_2 \\ \lambda_2'' &= -c\lambda_1 - d\lambda_2, \end{cases}$$

and $\theta = \frac{\lambda_2''}{\lambda_1''} = \frac{-c-d\theta}{a+b\theta}$. The quadratic equation for $\theta$ has the form $b\theta^2 + (a+d)\theta + c = 0$, whose discrimimant is $\Delta' = (a+d)^2 - 4bc$. Let us show that $\Delta' > 0$. Indeed, $\Delta = (a+d)^2 - 4(ad-bc) < 0$ and the evident inequality $-(a-d)^2 \leq 0$ have the same sign, and we shall add them up. After an obvious elimination, one gets $bc < 0$. Therefore $\Delta'$ is a sum of the two positive integers, which is always a positive integer. Thus, there exist the real roots $\theta_{1,2} = \frac{-a-d\pm\sqrt{\Delta'}}{2b}$. Moreover, $k = a + b\theta = \frac{1}{2}(a - d \pm \sqrt{\Delta'})$. Since $\Delta'$ is not the square of an integer, $k$ and $\theta$ are the irrational numbers in the quadratic field $\mathbb{Q}(\sqrt{\Delta'})$.

**Case 5: $\Delta < 0$ and $\Delta' = m^2$, $m \in \mathbb{Z} - \{0\}$.** Note that $\theta = \frac{-a-d\pm|m|}{2b}$ is a rational number. Since $\theta$ does not satisfy the rank assumption of the lemma, the case should be omitted.

(ii) Assume that $b = 0$.

**Case 1: $a - d \neq 0$.** The quadratic equation for $\theta$ degenerates to a linear eauation $(a-d)\theta + c = 0$. The root $\theta = \frac{c}{d-a} \in \mathbb{Q}$ does not satisfy the rank assumption again, and we omit the case.

**Case 2: $a = d$ and $c \neq 0$.** It is easy to see, that the set of the solutions for $\theta$ is an empty set.

**Case 3: $a = d$ and $c = 0$.** Finally, in this case all coefficients of the quadratic equation vanish, so that any $\theta \in \mathbb{R} - \mathbb{Q}$ is a solution. Note that $k = a = d \in \mathbb{Z}$. Thus, one gets case (i) of the lemma. Since there are no other possiblities left, Lemma 6.1.1 is proved. □

**Lemma 4.1.2** *Let $\mathcal{E}_{CM}$ be an elliptic curve with complex multiplication and consider a $\mathbb{Z}$-module $F(Isom\ (\mathcal{E}_{CM})) = \mu_{CM}(\mathbb{Z} + \mathbb{Z}\theta_{CM}) := \mathfrak{m}_{CM}$. Then:*

*(i) $\theta_{CM}$ is a quadratic irrationality,*

*(ii) $\mu_{CM} \in \mathbb{Q}$ (up to a choice of map $F$).*

*Proof.* (i) Since $\mathcal{E}_{CM}$ has complex multiplication, one gets $End\ (\mathcal{E}_{CM}) > \mathbb{Z}$. In particular, there exists a non-trivial isogeny

$$\varphi : \ \mathcal{E}_{CM} \longrightarrow \mathcal{E}_{CM},$$

i.e an endomorphism which is *not* the multiplication by $k \in \mathbb{Z}$. By Theorem 6.1.1 and Remark 6.1.3, each isogeny $\varphi$ defines a rank 2 submodule $\mathfrak{m}'$ of module $\mathfrak{m}_{CM}$. By Lemma 6.1.1, $\mathfrak{m}' = k\mathfrak{m}_{CM}$ for a $k \in \mathbb{R}$. Because $\varphi$ is a non-trivial endomorphism, we get $k \notin \mathbb{Z}$; thus, option (i) of Lemma 6.1.1 is excluded. Therefore, by the item (ii) of Lemma 6.1.1, real number $\theta_{CM}$ must be a quadratic irrationality.

(ii) Recall that $E_{\mathcal{F}} \subset Q - \{0\}$ is the space of holomorphic differentials on the complex torus, whose horizontal trajectory structure is equivalent to given measured foliation $\mathcal{F} = \mathcal{F}_\theta^\mu$. We shall vary $\mathcal{F}_\theta^\mu$, thus varying the Hubbard-Masur homeomorphism $h = h(\mathcal{F}_\theta^\mu) : E_{\mathcal{F}} \to T(1)$, see Section 6.1.2. Namely, consider a 1-parameter continuous family of such maps $h = h_\mu$, where $\theta = Const$ and $\mu \in \mathbb{R}$. Recall that $\mu_{CM} = \lambda_1 = \int_{\gamma_1} \phi$, where $\phi = Re\ \omega$ and $\omega \in E_{\mathcal{F}}$. The family $h_\mu$ generates a family $\omega_\mu = h_\mu^{-1}(C)$, where $C$ is a fixed point in $T(1)$. Denote by $\phi_\mu$ and $\lambda_1^\mu$ the corresponding families of the closed 1-forms and their periods, respectively. By the continuity, $\lambda_1^\mu$ takes on a rational value for a $\mu = \mu'$. (Actually, every neighborhood of $\mu_0$ contains such a $\mu'$.) Thus, $\mu_{CM} \in \mathbb{Q}$ for the Hubbard-Masur homeomorphism $h = h_{\mu'}$. Lemma 6.1.2 follows. $\square$

The claim (ii) of Theorem 6.1.2 follows from (i) of Lemma 6.1.2 and claim (i) of Theorem 6.1.2. To prove claim (i) of Theorem 6.1.2, notice that whenever $\mathcal{E}_1, \mathcal{E}_2 \in Isom(\mathcal{E}_{CM})$ the respective $\mathbb{Z}$-modules coincide, i.e. $\mathfrak{m}_1 = \mathfrak{m}_2$; this happens because an isomorphism between elliptic curves corresponds to a change of basis in the module $\mathfrak{m}$, see Theorem 6.1.1 and Remark 6.1.3. Theorem 6.1.2 is proved. $\square$

### 4.1.4 Proof of Theorem 6.1.3

Let us recall some classical facts and notation, and give an exact definition of the Riemann surface $X(\mathcal{A}_{RM}^{(D,f)})$. Let $N \geq 1$ be an integer; recall that $\Gamma_1(N)$ is a subgroup of the modular group $SL_2(\mathbb{Z})$ consisting of matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, d \equiv\ 1 \ mod\ N,\ c \equiv\ 0\ mod\ N \right\};$$

the corresponding Riemann surface $\mathbb{H}/\Gamma_1(N)$ will be denoted by $X_1(N)$. Consider the geodesic spectrum of $X_1(N)$, i.e the set $Spec\ X_1(N)$ consisting of all closed geodesics of the surface $X_1(N)$; each geodesic $\gamma \in Spec\ X_1(N)$ is the image under the covering map $\mathbb{H} \to \mathbb{H}/\Gamma_1(N)$ of a geodesic half-circle

$\tilde{\gamma} \in \mathbb{H}$ passing through the points $x$ and $\bar{x}$ fixed by the linear fractional transformation $x \mapsto \frac{ax+b}{cx+d}$, where matrix $(a,b,c,d) \in \Gamma_1(N)$. It is not hard to see, that $x$ and $\bar{x}$ are quadratic irrational numbers; the numbers are real when $|a+d| > 2$.

**Definition 4.1.5** *We shall say that the Riemann surface $X$ is associated to the noncommutative torus $\mathcal{A}_{RM}^{(D,f)}$, if $\{\tilde{\gamma}(x,\bar{x}) \ : \ \forall x \in \mathfrak{m}_{RM}^{(D,f)}\} \subset \widetilde{Spec}\ X$, where $\widetilde{Spec}\ X \subset \mathbb{H}$ is the set of geodesic half-circles covering the geodesic spectrum of $X$ and $\mathfrak{m}_{RM}^{(D,f)}$ is a $\mathbb{Z}$-module (a pseudo-lattice) in $\mathbb{R}$ generated by torus $\mathcal{A}_{RM}^{(D,f)}$; the associated Riemann surface will be denoted by $X(\mathcal{A}_{RM}^{(D,f)})$.*

**Lemma 4.1.3** *$X(\mathcal{A}_{RM}^{(D,f)}) \cong X_1(fD)$.*

*Proof.* Recall that $\mathfrak{m}_{RM}^{(D,f)}$ is a $\mathbb{Z}$-module (a pseudo-lattice) with real multiplication by an order $R$ in the real quadratic number field $\mathbb{Q}(\sqrt{D})$; it is known, that $\mathfrak{m}_{RM}^{(D,f)} \subseteq R$ and $R = \mathbb{Z} + (f\omega)\mathbb{Z}$, where $f \geq 1$ is the conductor of $R$ and

$$\omega = \begin{cases} \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \ mod\ 4, \\ \sqrt{D} & \text{if } D \equiv 2,3 \ mod\ 4, \end{cases}$$

see e.g. [Borevich & Shafarevich 1988] [11], pp. 130-131. Recall that matrix $(a,b,c,d) \in SL_2(\mathbb{Z})$ has a pair of real fixed points $x$ and $\bar{x}$ if and only if $|a+d| > 2$ (the hyperbolic matrix); the fixed points can be found from the equation $x = (ax+b)(cx+d)^{-1}$ by the formulas

$$x = \frac{a-d}{2c} + \sqrt{\frac{(a+d)^2-4}{4c^2}}, \qquad \bar{x} = \frac{a-d}{2c} - \sqrt{\frac{(a+d)^2-4}{4c^2}}.$$

Case I. If $D \equiv 1\ mod\ 4$, then the above formulas imply that $R = (1+\frac{f}{2})\mathbb{Z} + \frac{\sqrt{f^2D}}{2}\mathbb{Z}$. If $x \in \mathfrak{m}_{RM}^{(D,f)}$ is fixed point of a transformation $(a,b,c,d) \in SL_2(\mathbb{Z})$, then

$$\begin{cases} \frac{a-d}{2c} &= (1+\frac{f}{2})z_1 \\ \frac{(a+d)^2-4}{4c^2} &= \frac{f^2D}{4}z_2^2 \end{cases}$$

for some integer numbers $z_1$ and $z_2$. The second equation can be written in the form $(a+d)^2 - 4 = c^2f^2Dz_2^2$; we have therefore $(a+d)^2 \equiv 4\ mod\ (fD)$ and $a+d \equiv \pm 2\ mod\ (fD)$. Without loss of generality we assume $a+d \equiv 2\ mod\ (fD)$ since matrix $(a,b,c,d) \in SL_2(\mathbb{Z})$ can be multiplied by $-1$. Notice that the last equation admits a solution $a = d \equiv 1 \ mod \ (fD)$. The first equation yields us $\frac{a-d}{c} = (2+f)z_1$, where $c \neq 0$ since the matrix $(a,b,c,d)$ is hyperbolic. Notice that $a - d \equiv 0 \ mod \ (fD)$; since the ratio $\frac{a-d}{c}$ must be integer, we conclude that $c \equiv 0 \ mod \ (fD)$. Summing up, we get:

$$a \equiv 1 \ mod \ (fD), \quad d \equiv 1 \ mod \ (fD), \quad c \equiv 0 \ mod \ (fD).$$

Case II. If $D \equiv 2$ or $3 \ mod \ 4$, then $R = \mathbb{Z} + (\sqrt{f^2 D}) \ \mathbb{Z}$. If $x \in \mathfrak{m}_{RM}^{(D,f)}$ is fixed point of a transformation $(a,b,c,d) \in SL_2(\mathbb{Z})$, then

$$\begin{cases} \frac{a-d}{2c} & = & z_1 \\ \frac{(a+d)^2-4}{4c^2} & = & f^2 D z_2^2 \end{cases}$$

for some integer numbers $z_1$ and $z_2$. The second equation gives $(a+d)^2 - 4 = 4c^2 f^2 D z_2^2$; therefore $(a+d)^2 \equiv 4 \ mod \ (fD)$ and $a + d \equiv \pm 2 \ mod \ (fD)$. Again without loss of generality we assume $a + d \equiv 2 \ mod \ (fD)$ since matrix $(a,b,c,d) \in SL_2(\mathbb{Z})$ can be multiplied by $-1$. The last equation admits a solution $a = d \equiv 1 \ mod \ (fD)$. The first equation is $\frac{a-d}{c} = 2z_1$, where $c \neq 0$. Since $a - d \equiv 0 \ mod \ (fD)$ and the ratio $\frac{a-d}{c}$ must be integer, one concludes that $c \equiv 0 \ mod \ (fD)$. All together, one gets

$$a \equiv 1 \ mod \ (fD), \quad d \equiv 1 \ mod \ (fD), \quad c \equiv 0 \ mod \ (fD).$$

Since all possible cases are exhausted, Lemma 6.1.3 follows. □

**Remark 4.1.5** There exist other finite index subgroups of $SL_2(\mathbb{Z})$ whose geodesic spectrum contains the set $\{\tilde{\gamma}(x,\bar{x}) \ : \ \forall x \in \mathfrak{m}_{RM}^{(D,f)}\}$; however $\Gamma_1(fD)$ is a unique group with such a property among subgroups of the principal congruence group.

**Remark 4.1.6** Not all geodesics of $X_1(fD)$ have the above form; thus the set
$\{\tilde{\gamma}(x,\bar{x}) \ : \ \forall x \in \mathfrak{m}_{RM}^{(D,f)}\}$ is strictly included in the geodesic spectrum of modular curve $X_1(fD)$.

**Definition 4.1.6** *The group*

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, d \equiv \ 1 \ mod \ N, \ b, c \equiv \ 0 \ mod \ N \right\}$$

*is called a principal congruence group of level $N$; the corresponding compact modular curve will be denoted by $X(N) = \mathbb{H}/\Gamma(N)$.*

**Lemma 4.1.4 (Hecke)** *There exists a holomorphic map* $X(fD) \to \mathcal{E}_{CM}^{(-D,f)}$.

*Proof.* A detailed proof of this beautiful fact is given in [Hecke 1928] [38]. For the sake of clarity, we shall give an idea of the proof. Let $\mathfrak{R}$ be an order of conductor $f \geq 1$ in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$; consider an $L$-function attached to $\mathfrak{R}$

$$L(s,\psi) = \prod_{\mathfrak{P}\subset\mathfrak{R}} \frac{1}{1 - \frac{\psi(\mathfrak{P})}{N(\mathfrak{P})^s}}, \quad s \in \mathbb{C},$$

where $\mathfrak{P}$ is a prime ideal in $\mathfrak{R}$, $N(\mathfrak{P})$ its norm and $\psi$ a Grössencharacter. A crucial observation of Hecke says that the series $L(s,\psi)$ converges to a cusp form $w(s)$ of the principal congruence group $\Gamma(fD)$. By the Deuring Theorem, $L(\mathcal{E}_{CM}^{(-D,f)}, s) = L(s,\psi)L(s,\bar{\psi})$, where $L(\mathcal{E}_{CM}^{(-D,f)}, s)$ is the Hasse-Weil $L$-function of the elliptic curve and $\bar{\psi}$ a conjugate of the Grössencharacter, see e.g. [Silverman 1994] [94], p. 175; moreover $L(\mathcal{E}_{CM}^{(-D,f)}, s) = L(w,s)$, where $L(w,s) := \sum_{n=1}^{\infty} \frac{c_n}{n^s}$ and $c_n$ the Fourier coefficients of the cusp form $w(s)$. In other words, $\mathcal{E}_{CM}^{(-D,f)}$ is a modular elliptic curve. One can now apply the modularity principle: if $A_w$ is an abelian variety given by the periods of holomorphic differential $w(s)ds$ (and its conjugates) on $X(fD)$, then the diagram in Fig. 6.3 is commutative. The holomorphic map $X(fD) \to \mathcal{E}_{CM}^{(-D,f)}$ is obtained as a composition of the canonical embedding $X(fD) \to A_w$ with the subsequent holomorphic projection $A_w \to \mathcal{E}_{CM}^{(-D,f)}$. Lemma 6.1.4 is proved. □

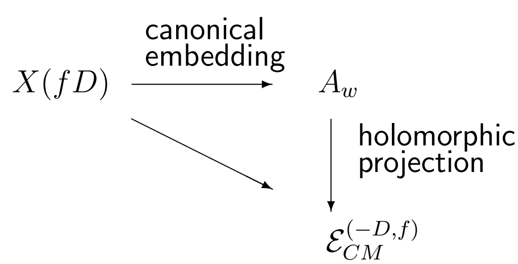



Figure 4.3: Hecke lemma.

**Lemma 4.1.5** *The functor* $F$ *acts by the formula* $\mathcal{E}_{CM}^{(-D,f)} \mapsto \mathcal{A}_{RM}^{(D,f)}$.

*Proof.* Let $L_{CM}$ be a lattice with complex multiplication by an order $\mathfrak{R} = \mathbb{Z} + (f\omega)\mathbb{Z}$ in the imaginary quadatic field $\mathbb{Q}(\sqrt{-D})$; the multiplication by $\alpha \in \mathfrak{R}$ generates an endomorphism $(a,b,c,d) \in M_2(\mathbb{Z})$ of the lattice $L_{CM}$. It is known from Section 6.1.3, **Case 4**, that the endomorphisms of lattice $L_{CM}$ and endomorphisms of the pseudo-lattice $\mathfrak{m}_{RM} = F(L_{CM})$ are related by the following explicit map

$$\left(\begin{matrix} a & b \\ c & d \end{matrix}\right) \in End\ (L_{CM}) \longmapsto \left(\begin{matrix} a & b \\ -c & -d \end{matrix}\right) \in End\ (\mathfrak{m}_{RM}),$$

Moreover, one can always assume $d = 0$ in a proper basis of $L_{CM}$. We shall consider the following two cases.

Case I. If $D \equiv 1 \ mod\ 4$ then we have $\mathfrak{R} = \mathbb{Z} + (\frac{f+\sqrt{-f^2D}}{2})\mathbb{Z}$; thus $\alpha = \frac{2m+fn}{2} + \sqrt{\frac{-f^2Dn^2}{4}}$ for some $m, n \in \mathbb{Z}$. Therefore multiplication by $\alpha$ corresponds to an endomorphism $(a,b,c,0) \in M_2(\mathbb{Z})$, where

$$\left\{\begin{array}{lll} a & = & Tr(\alpha) = \alpha + \bar{\alpha} = 2m + fn \\ b & = & -1 \\ c & = & N(\alpha) = \alpha\bar{\alpha} = \left(\frac{2m+fn}{2}\right)^2 + \frac{f^2Dn^2}{4}. \end{array}\right.$$

To calculate a primitive generator of endomorphisms of the lattice $L_{CM}$ one should find a multiplier $\alpha_0 \neq 0$ such that

$$|\alpha_0| = \min_{m.n\in\mathbb{Z}} |\alpha| = \min_{m.n\in\mathbb{Z}} \sqrt{N(\alpha)}.$$

From the equation for $c$ the minimum is attained at $m = -\frac{f}{2}$ and $n = 1$ if $f$ is even or $m = -f$ and $n = 2$ if $f$ is odd. Thus

$$\alpha_0 = \left\{\begin{array}{ll} \pm\frac{f}{2}\sqrt{-D}, & \text{if } f \text{ is even} \\ \pm f\sqrt{-D}, & \text{if } f \text{ is odd.} \end{array}\right.$$

To find the matrix form of the endomorphism $\alpha_0$, we shall substitute in the corresponding formula $a = d = 0, b = -1$ and $c = \frac{f^2D}{4}$ if $f$ is even or $c = f^2D$ if $f$ is odd. Thus functor $F$ maps the multiplier $\alpha_0$ into

$$F(\alpha_0) = \left\{\begin{array}{ll} \pm\frac{f}{2}\sqrt{D}, & \text{if } f \text{ is even} \\ \pm f\sqrt{D}, & \text{if } f \text{ is odd.} \end{array}\right.$$

Comparing the above equations, one verifies that formula $F(\mathcal{E}_{CM}^{(-D,f)}) = \mathcal{A}_{RM}^{(D,f)}$ is true in this case.

Case II. If $D \equiv 2$ or $3 \; mod \; 4$ then $\mathfrak{R} = \mathbb{Z} + (\sqrt{-f^2 D})\,\mathbb{Z}$; thus the multiplier $\alpha = m + \sqrt{-f^2 D n^2}$ for some $m, n \in \mathbb{Z}$. A multiplication by $\alpha$ corresponds to an endomorphism $(a, b, c, 0) \in M_2(\mathbb{Z})$, where

$$\begin{cases} a &= \quad Tr(\alpha) = \alpha + \bar{\alpha} = 2m \\ b &= \quad -1 \\ c &= \quad N(\alpha) = \alpha\bar{\alpha} = m^2 + f^2 D n^2. \end{cases}$$

We shall repeat the argument of Case I; then from the equation for $c$ the minimum of $|\alpha|$ is attained at $m = 0$ and $n = \pm 1$. Thus $\alpha_0 = \pm f\sqrt{-D}$. To find the matrix form of the endomorphism $\alpha_0$ we substitute in the corresponding equation $a = d = 0, b = -1$ and $c = f^2 D$. Thus functor $F$ maps the multiplier $\alpha_0 = \pm f\sqrt{-D}$ into $F(\alpha_0) = \pm f\sqrt{D}$. In other words, formula $F(\mathcal{E}_{CM}^{(-D,f)}) = \mathcal{A}_{RM}^{(D,f)}$ is true in this case as well. Since all possible cases are exhausted, Lemma 6.1.5 is proved. □

**Lemma 4.1.6** *For every $N \geq 1$ there exists a holomorphic map $X_1(N) \to X(N)$.*

*Proof.* Indeed, $\Gamma(N)$ is a normal subgroup of index $N$ of the group $\Gamma_1(N)$; therefore there exists a degree $N$ holomorphic map $X_1(N) \to X(N)$. Lemma 6.1.6 follows. □

Theorem 6.1.3 follows from Lemmas 6.1.3-6.1.5 and Lemma 6.1.6 for $N = fD$. □

**Guide to the literature.** D. Hilbert counted complex multiplication as not only the most beautiful part of mathematics but also of entire science; it surely does as it links complex analysis and number theory. One cannot beat [Serre 1967] [92] for an introduction, but more comprehensive [Silverman 1994] [94], Chapter 2 is the must. Real multiplication has been introduced in [Manin 2004] [52]. The link between the two was the subject of [66] and the inverse functor $F^{-1}$ was constructed in [71].

## 4.2 Ranks of the $K$-rational elliptic curves

We are working in the category of elliptic curves with complex multiplication; such curves were denoted by $\mathcal{E}_{CM}^{(-D,f)}$, where $f \geq 1$ is the conductor of an order in the imaginary quadratic field $k = \mathbb{Q}(\sqrt{-D})$. Recall that $\mathcal{E}_{CM}^{(-D,f)} \cong \mathcal{E}(K)$, where $K = k(j(\mathcal{E}_{CM}^{(-D,f)}))$ is the Hilbert class field of $k$, see e.g. [Serre 1967] [92]. In other words, we deal with a $K$-rational projective curve

$$\mathcal{E}(K) = \{(x,y,z) \in \mathbb{C}P^2 \mid y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3\},$$

where constants $g_2$ and $g_3$ belong to the number field $K$. It is well known, that any pair of points $p, p' \in \mathcal{E}(K)$ defines a sum $p + p' \in \mathcal{E}(K)$ and an inverse $-p \in \mathcal{E}(K)$ so that $\mathcal{E}(K)$ has the structure of an abelian group; the next result is now a standard fact, see e.g. [Tate 1974] [103], p. 192.

**Theorem 4.2.1 (Mordell-Néron)** *The $\mathcal{E}(K)$ is finitely generated abelian group.*

**Definition 4.2.1** *By $rk$ $(\mathcal{E}_{CM}^{(-D,f)})$ we understand the integer number equal to the rank of abelian group $\mathcal{E}(K)$.*

**Remark 4.2.1** The $rk$ $(\mathcal{E}_{CM}^{(-D,f)})$ is an invariant of the $K$-isomorphism class of $\mathcal{E}_{CM}^{(-D,f)}$ but not of the general isomorphism class; those variations of the rank are known as *twists* of $\mathcal{E}_{CM}^{(-D,f)}$. Of course, if two curves are $K$-isomorphic, they are also isomorphic over $\mathbb{C}$.

In what remains, we calculate $rk$ $(\mathcal{E}_{CM}^{(-D,f)})$ in terms of invariants of the noncommutative torus $\mathcal{A}_{RM}^{(D,f)} = F(\mathcal{E}_{CM}^{(-D,f)})$; one such invariant called an *arithmetic complexity* will be introduced below.

### 4.2.1 Arithmetic complexity of noncommutative tori

Let $\theta$ be a quadratic irrationality, i.e. irrational root of a quadratic polynomial $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{Z}$; denote by $Per$ $(\theta) := (\overline{a_1, a_2, \dots, a_P})$ the minimal period of continued fraction of $\theta$ taken up to a cyclic permutation. Fix $P$ and suppose for a moment that $\theta$ is a function of its period

$$\theta(x_0, x_1, \dots, x_P) = [x_0, \overline{x_1, \dots, x_P}],$$

where $x_i \geq 1$ are integer variables; then $\theta(x_0, \dots, x_P) \in \mathbb{Q} + \sqrt{\mathbb{Q}}$, where $\sqrt{\mathbb{Q}}$ are square roots of positive rationals. Consider a constraint (a restriction) $x_1 = x_{P-1}, x_2 = x_{P-2}, \dots, x_P = 2x_0$; then $\theta(x_0, x_1, x_2, \dots, x_2, x_1, 2x_0) \in \sqrt{\mathbb{Q}}$, see e.g. [Perron 1954] [82], p. 79. Notice, that in this case there are $\frac{1}{2}P + 1$

independent variables, if $P$ is even and $\frac{1}{2}(P+1)$, if $P$ is odd. The number of independent variables will further decrease, if $\theta$ is square root of an integer; let us introduce some notation. For a regular fraction $[a_0, a_1, \ldots]$ one associates the linear equations

$$\begin{cases} y_0 &= a_0 y_1 + y_2 \\ y_1 &= a_1 y_2 + y_3 \\ y_2 &= a_2 y_3 + y_4 \\ &\vdots \end{cases}$$

One can put above equations in the form

$$\begin{cases} y_j &= A_{i-1,j} y_{i+j} + a_{i+j} A_{i-2,j} y_{i+j+1} \\ y_{j+1} &= B_{i-1,j} y_{i+j} + a_{i+j} B_{i-2,j} y_{i+j+1}, \end{cases}$$

where the polynomials $A_{i,j}, B_{i,j} \in \mathbb{Z}[a_0, a_1, \ldots]$ are called *Muir's symbols*, see [Perron 1954] [82], p.10. The following lemma will play an important rôle.

**Lemma 4.2.1 ([Perron 1954] [82], pp. 88 and 107)** *There exists a square-free integer $D > 0$, such that*

$$[x_0, \overline{x_1, \ldots, x_1, x_P}] = \begin{cases} \sqrt{D}, & \text{if } x_P = 2x_0 \text{ and } D = 2,3 \textbf{ mod } 4, \\ \frac{\sqrt{D}+1}{2}, & \text{if } x_P = 2x_0 - 1 \text{ and } D = 1 \textbf{ mod } 4, \end{cases}$$

*if and only if $x_P$ satisfies the diophantine equation*

$$x_P = mA_{P-2,1} - (-1)^P A_{P-3,1} B_{P-3,1},$$

*for an integer $m > 0$; moreover, in this case $D = \frac{1}{4}x_P^2 + mA_{P-3,1} - (-1)^P B_{P-3,1}^2$.*

Let $(x_0^*, \ldots, x_P^*)$ be a solution of the diophantine equation of Lemma 6.2.1. By *dimension*, $d$, of this solution one understands the maximal number of variables $x_i$, such that for every $s \in \mathbb{Z}$ there exists a solution of the above diophantine equation of the form $(x_0, \ldots, x_i^* + s, \ldots, x_P)$. In geometric terms, $d$ is equal to dimension of a connected component through the point $(x_0^*, \ldots, x_P^*)$ of an affine variety $V_m$ (i.e. depending on $m$) defined by the diophantine equation. For the sake of clarity, let us consider a simple example.

**Example 4.2.1 ([Perron 1954] [82], p. 90)** If $P = 4$, then Muir's symbols are: $A_{P-3,1} = A_{1,1} = x_1x_2 + 1$, $B_{P-3,1} = B_{1,1} = x_2$ and $A_{P-2,1} = A_{2,1} = x_1x_2x_3 + x_1 + x_3 = x_1^2x_2 + 2x_1$, since $x_3 = x_1$. Thus, our diophantine equation takes the form

$$2x_0 = m(x_1^2x_2 + 2x_1) - x_2(x_1x_2 + 1),$$

and, therefore, $\sqrt{x_0^2 + m(x_1x_2+1) - x_2^2} = [x_0, \overline{x_1, x_2, x_1, 2x_0}]$. First, let us show that the affine variety defined by the last equation is not connected. Indeed, by Lemma 6.2.1, parameter $m$ must be integer for all (integer) values of $x_0, x_1$ and $x_2$. This is not possible in general, since from our last equation one obtains $m = (2x_0 + x_2(x_1x_2+1))(x_1^2x_2 + 2x_1)^{-1}$ is a rational number. However, a restriction to $x_1 = 1,\ x_2 = x_0 - 1$ defines a (maximal) connected component of the variety corresponding to our equation, since in this case $m = x_0$ is always an integer. Thus, one gets a family of solutions of of the form $\sqrt{(x_0+1)^2 - 2} = [x_0, \overline{1, x_0 - 1, 1, 2x_0}]$, where each solution has dimension $d = 1$.

**Definition 4.2.2** *By an arithmetic complexity $c(\mathcal{A}_{RM}^{(D,f)})$ of the noncommutative torus $\mathcal{A}_{RM}^{(D,f)}$ one understands an integer equal to dimension $d$ of solution $(x_0^*, \dots, x_P^*)$ of diophantine equation $x_P = mA_{P-2,1} - (-1)^P A_{P-3,1}B_{P-3,1}$; if $\mathcal{A}_\theta$ has no real multiplication, then the arithmetic complexity is assumed to be infinite.*

### 4.2.2 $\mathbb{Q}$-curves

For the sake of simplicity, we shall restrict our considerations to a family of elliptic curves $\mathcal{E}_{CM}^{(-D,f)}$ known as the $\mathbb{Q}$-*curves*; a general result exists only in a conjectural form so far, see *Exercises, problems and conjectures.*

**Definition 4.2.3 ([Gross 1980] [29])** *Let $(\mathcal{E}_{CM}^{(-D,f)})^\sigma$, $\sigma \in Gal\ (k|\mathbb{Q})$ be the Galois conjugate of the curve $\mathcal{E}_{CM}^{(-D,f)}$; by a $\mathbb{Q}$-curve one understands $\mathcal{E}_{CM}^{(-D,f)}$, such that there exists an isogeny between $(\mathcal{E}_{CM}^{(-D,f)})^\sigma$ and $\mathcal{E}_{CM}^{(-D,f)}$ for each $\sigma \in Gal\ (k|\mathbb{Q})$.*

**Remark 4.2.2** The curve $\mathcal{E}_{CM}^{(-p,1)}$ is a $\mathbb{Q}$-curve, whenever $p \equiv 3 \mathbf{\ mod\ } 4$ is a prime number, see [Gross 1980] [29], p. 33; we shall write $\mathfrak{P}_{3 \mathbf{\ mod\ } 4}$ to denote the set of all such primes.

**Remark 4.2.3** The rank of $\mathcal{E}_{CM}^{(-p,1)}$ is always divisible by $2h_k$, where $h_k$ is the class number of number field $k := \mathbb{Q}(\sqrt{-p})$, see [Gross 1980] [29], p. 49.

**Definition 4.2.4** *By a $\mathbb{Q}$-rank of $\mathcal{E}_{CM}^{(-p,1)}$ one understands the integer*

$$rk_{\mathbb{Q}}(\mathcal{E}_{CM}^{(-p,1)}) := \frac{1}{2h_k}\ rk\ (\mathcal{E}_{CM}^{(-p,1)}).$$

The following result links invariants of noncommutative tori and geometry of the $K$-rational elliptic curves; namely, the arithmetic complexity plus one is equal to the $\mathbb{Q}$-rank of the corresponding elliptic curve.

**Theorem 4.2.2** $rk_{\mathbb{Q}}\ (\mathcal{E}_{CM}^{(-p,1)}) + 1 = c(\mathcal{A}_{RM}^{(p,1)})$ *whenever* $p \equiv 3$ **mod** $4$.

**Remark 4.2.4** The general formula $rk\ (\mathcal{E}_{CM}^{(-D,f)}) + 1 = c(\mathcal{A}_{RM}^{(D,f)})$ for all $D \geq 2$ and $f \geq 1$ is known as the *rank conjecture*, see *Exercises, problems and conjectures.*

### 4.2.3 Proof of Theorem 6.2.2

**Lemma 4.2.2** *If* $[x_0, \overline{x_1, \dots, x_k, \dots, x_1, 2x_0}] \in \sqrt{\mathfrak{P}_{3\ \mathbf{mod}\ 4}}$ *, then:*

*(i)* $P = 2k$ *is an even number, such that:*

*(a)* $P \equiv 2$ **mod** $4$*, if* $p \equiv 3$ **mod** $8$*;*

*(b)* $P \equiv 0$ **mod** $4$*, if* $p \equiv 7$ **mod** $8$*;*

*(ii) either of two is true:*

*(a)* $x_k = x_0$ *(a culminating period);*

*(b)* $x_k = x_0 - 1$ *and* $x_{k-1} = 1$ *(an almost-culminating period).*

*Proof.* (i) Recall that if $p \neq 2$ is a prime, then one and only one of the following diophantine equations is solvable:

$$\begin{cases} x^2 - py^2 &= -1, \\ x^2 - py^2 &= 2, \\ x^2 - py^2 &= -2, \end{cases}$$

see e.g. [Perron 1954] [82], Satz 3.21. Since $p \equiv 3$ **mod** $4$, one concludes that $x^2 - py^2 = -1$ is not solvable [Perron 1954] [82], Satz 3.23-24; this happens if and only if $P = 2k$ is even (for otherwise the continued fraction of $\sqrt{p}$ would provide a solution).

It is known, that for even periods $P = 2k$ the convergents $A_i/B_i$ satisfy the diophantine equation $A_{k-1}^2 - pB_{k-1}^2 = (-1)^k\ 2$, see [Perron 1954] [82], p.103; thus if $P \equiv 0$ **mod** $4$, the equation $x^2 - py^2 = 2$ is solvable and if $P \equiv 2$ **mod** $4$, then the equation $x^2 - py^2 = -2$ is sovable. But equation $x^2 - py^2 = 2$ (equation $x^2 - py^2 = -2$, resp.) is solvable if and only if $p \equiv 7$ **mod** $8$ ($p \equiv 3$ **mod** $8$, resp.), see [Perron 1954] [82], Satz 3.23 (Satz 3.24, resp.). Item (i) follows.

(ii) The equation $A_{k-1}^2 - pB_{k-1}^2 = (-1)^k\ 2$ is a special case of equation $A_{k-1}^2 - pB_{k-1}^2 = (-1)^k\ Q_k$, where $Q_k$ is the full quotient of continued fraction [Perron 1954] [82], p.92; therefore, $Q_k = 2$. One can now apply [Perron 1954] [82], Satz 3.15, which says that for $P = 2k$ and $Q_k = 2$ the continued fraction of $\sqrt{\mathfrak{P}_{3\ \mathbf{mod}\ 4}}$ is either culminating (i.e. $x_k = x_0$) or almost-culminating (i.e. $x_k = x_0 - 1$ and $x_{k-1} = 1$). Lemma 6.2.2 follows. □

**Lemma 4.2.3** *If $p \equiv 3$ **mod** $8$, then $c(\mathcal{A}_{RM}^{(p,1)}) = 2$.*

*Proof.* The proof proceeds by induction in period $P$, which is in this case $P \equiv 2$ **mod** $4$ by Lemma 6.2.2. We shall start with $P = 6$, since $P = 2$ reduces to it, see item (i) below.

(i) Let $P = 6$ be a culminating period; then the diophantine equation in Definition 6.2.2 admits a general solution $[x_0, \overline{x_1, 2x_1, x_0, , 2x_1, x_1, 2x_0}] = \sqrt{x_0^2 + 4nx_1 + 2}$, where $x_0 = n(2x_1^2 + 1) + x_1$, see [Perron 1954] [82], p. 101. The solution depends on two integer variables $x_1$ and $n$, which is the maximal possible number of variables in this case; therefore, the dimension of the solution is $d = 2$, so as complexity of the corresponding torus. Notice that the case $P = 2$ is obtained from $P = 6$ by restriction to $n = 0$; thus the complexity for $P = 2$ is equal to 2.

(ii) Let $P = 6$ be an almost-culminating period; then the diophantine equation in Definition 6.2.2 has a solution $[3s + 1, \overline{2, 1, 3s, 1, 2, 6s + 2}] = \sqrt{(3s + 1)^2 + 2s + 1}$, where $s$ is an integer variable [Perron 1954] [82], p. 103. We encourage the reader to verify, that this solution is a restriction of solution (i) to $x_1 = -1$ and $n = s + 1$; thus, the dimension of our solution is $d = 2$, so as the complexity of the corresponding torus.

(iii) Suppose a solution $[x_0, \overline{x_1, \ldots, x_{k-1}, x_k, x_{k-1}, \ldots, x_1, 2x_0}]$ with the (culminating or almost-culminating) period $P_0 \equiv 3$ **mod** $8$ has dimension $d = 2$; let us show that a solution

$$[x_0, \overline{y_1, x_1, \ldots, x_{k-1}, y_{k-1}, x_k, y_{k-1}, x_{k-1}, \ldots, x_1, y_1, 2x_0}]$$

with period $P_0 + 4$ has also dimension $d = 2$. According to [Weber 1926] [108], if above fraction is a solution to the diophantine equation in Definition 6.2.2, then either (i) $y_{k-1} = 2y_1$ or (ii) $y_{k-1} = 2y_1 + 1$ and $x_1 = 1$. We proceed by showing that case (i) is not possible for the square roots of prime numbers.

Indeed, let to the contrary $y_{k-1} = 2y_1$; then the following system of equations must be compatible:

$$\begin{cases} A_{k-1}^2 & - & pB_{k-1}^2 = -2, \\ A_{k-1} & = & 2y_1 A_{k-2} + A_{k-3}, \\ B_{k-1} & = & 2y_1 B_{k-2} + B_{k-3}, \end{cases}$$

where $A_i, B_i$ are convergents and the first equation is solvable since $p \equiv 3 \ \mathbf{mod}\ 8$. From the first equation, both convergents $A_{k-1}$ and $B_{k-1}$ are odd numbers. (They are both odd or even, but even excluded, since $A_{k-1}$ and $B_{k-1}$ are relatively prime.) From the last two equations, the convergents $A_{k-3}$ and $B_{k-3}$ are also odd. Then the convergents $A_{k-2}$ and $B_{k-2}$ must be even, since among six consequent convergents $A_{k-1}, B_{k-1}, A_{k-2}, B_{k-2}, A_{k-3}, B_{k-3}$ there are always two even; but this is not possible, because $A_{k-2}$ and $B_{k-2}$ are relatively prime. Thus, $y_{k-1} \neq 2y_1$.

Therefore the above equations give a solution of the diophantine equation in Definition 6.2.2 if and only if $y_{k-1} = 2y_1 + 1$ and $x_1 = 1$; the dimension of such a solution coincides with the dimension of solution

$$[x_0, \overline{x_1, \ldots, x_{k-1}, x_k, x_{k-1}, \ldots, x_1, 2x_0}],$$

since for two new integer variables $y_1$ and $y_{k-1}$ one gets two new constraints. Thus, the dimension of the above solution is $d = 2$, so as the complexity of the corresponding torus. Lemma 6.2.3 follows. □

**Lemma 4.2.4** *If* $p \equiv 7 \ \mathbf{mod}\ 8$, *then* $c(\mathcal{A}_{RM}^{(p,1)}) = 1$.

*Proof.* The proof proceeds by induction in period $P \equiv 0 \ \mathbf{mod}\ 4$, see Lemma 6.2.2; we start with $P = 4$.

(i) Let $P = 4$ be a culminating period; then equation in Definition 6.2.2 admits a solution $[x_0, \overline{x_1, x_2, x_1, 2x_0}] = \sqrt{x_0^2 + m(x_1x_2 + 1) - x_2^2}$, where $x_2 = x_0$, see Example 6.2.1 for the details. Since the polynomial $m(x_0x_1 + 1)$ under the square root represents a prime number, we have $m = 1$; the latter equation is not solvable in integers $x_0$ and $x_1$, since $m = x_0(x_0x_1 + 3)x_1^{-1}(x_0x_1 + 2)^{-1}$. Thus, there are no solutions of the diophantine equation in Definition 6.2.2 with the culminating period $P = 4$.

(ii) Let $P = 4$ be an almost-culminating period; then equation in Definition 6.2.2 admits a solution $[x_0, \overline{1, x_0 - 1, 1, 2x_0}] = \sqrt{(x_0 + 1)^2 - 2}$. The dimension of this solution was proved to be $d = 1$, see Example 6.2.1; thus, the complexity of the corresponding torus is equal to 1.

(iii) Suppose a solution $[x_0, \overline{x_1, \dots, x_{k-1}, x_k, x_{k-1}, \dots, x_1, 2x_0}]$ with the (culminating or almost-culminating) period $P_0 \equiv 7 \textbf{ mod } 8$ has dimension $d = 1$. It can be shown by the same argument as in Lemma 6.2.3, that for a solution of the form

$$[x_0, \overline{y_1, x_1, \dots, x_{k-1}, y_{k-1}, x_k, y_{k-1}, x_{k-1}, \dots, x_1, y_1, 2x_0}]$$

having the period $P_0 + 4$ the dimension remains the same, i.e. $d = 1$; we leave details to the reader. Thus, complexity of the corresponding torus is equal to 1. Lemma 6.2.4 follows. □

**Lemma 4.2.5 ([Gross 1980] [29], p. 78)**

$$rk_{\mathbb{Q}}\ (\mathcal{E}_{CM}^{(-p,1)}) = \begin{cases} 1, & if\ \ p \equiv 3 \textbf{ mod } 8 \\ 0, & if\ \ p \equiv 7 \textbf{ mod } 8. \end{cases} \tag{4.1}$$

Theorem 6.2.2 follows from Lemmas 6.2.3-6.2.5. □

### 4.2.4 Numerical examples

To illustrate Theorem 6.2.2 by numerical examples, we refer the reader to Fig. 6.4 with all $\mathbb{Q}$-curves $\mathcal{E}_{CM}^{(-p,1)}$ for $p < 100$; notice, that there are infinitely many pairwise non-isomorphic $\mathbb{Q}$-curves [Gross 1980] [29].

**Guide to the literature.** The problem of ranks of rational elliptic curves was raised by [Poincaré 1901] [86], p. 493. It was proved by [Mordell 1922] [56] that ranks of rational and by [Néron 1952] [60] that ranks of the $K$-rational elliptic curves are always finite. The ranks of individual elliptic curves are calculated by the method of *descent*, see e.g. [Cassels 1966] [14], p.205. More conceptual approach uses an analytic object called the *Hasse-Weil L-function* $L(\mathcal{E}_\tau, s)$; it was conjectured by B. J. Birch and H. P. F. Swinnerton-Dyer that the order of zero of such a function at $s = 1$ is equal to the rank of $\mathcal{E}_\tau$, see e.g. [Tate 1974] [103], p. 198. The *rank conjecture* involving invariants of noncommutative tori was formulated in [66] and proved in [72] for the $\mathbb{Q}$-curves $\mathcal{E}_{CM}^{(-p,1)}$ with prime $p \equiv 3 \textbf{ mod } 4$.

| $p \equiv 3$ **mod** 4 | $rk_{\mathbb{Q}}(\mathcal{E}_{CM}^{(-p,1)})$ | $\sqrt{p}$ | $c(\mathcal{A}_{RM}^{(p,1)})$ |
|---|---|---|---|
| 3 | 1 | $[1, \overline{1, 2}]$ | 2 |
| 7 | 0 | $[2, \overline{1, 1, 1, 4}]$ | 1 |
| 11 | 1 | $[3, \overline{3, 6}]$ | 2 |
| 19 | 1 | $[4, \overline{2, 1, 3, 1, 2, 8}]$ | 2 |
| 23 | 0 | $[4, \overline{1, 3, 1, 8}]$ | 1 |
| 31 | 0 | $[5, \overline{1, 1, 3, 5, 3, 1, 1, 10}]$ | 1 |
| 43 | 1 | $[6, \overline{1, 1, 3, 1, 5, 1, 3, 1, 1, 12}]$ | 2 |
| 47 | 0 | $[6, \overline{1, 5, 1, 12}]$ | 1 |
| 59 | 1 | $[7, \overline{1, 2, 7, 2, 1, 14}]$ | 2 |
| 67 | 1 | $[8, \overline{5, 2, 1, 1, 7, 1, 1, 2, 5, 16}]$ | 2 |
| 71 | 0 | $[8, \overline{2, 2, 1, 7, 1, 2, 2, 16}]$ | 1 |
| 79 | 0 | $[8, \overline{1, 7, 1, 16}]$ | 1 |
| 83 | 1 | $[9, \overline{9, 18}]$ | 2 |

Figure 4.4: The $\mathbb{Q}$-curves $\mathcal{E}_{CM}^{(-p,1)}$ with $p < 100$.

## 4.3 Non-commutative reciprocity

In the world of $L$-functions each equivalence between two $L$-functions is called a *reciprocity*, see e.g. [Gelbart 1984] [28]. In this section we shall introduce an $L$-function $L(\mathcal{A}_{RM}, s)$ associated to the noncommutative torus with real multiplication and prove that any such coincides with the classical Hasse-Weil function $L(\mathcal{E}_{CM}, s)$ of an elliptic curve with complex multiplication. The necessary and sufficient condition for such a reciprocity is the relation

$$\mathcal{A}_{RM} = F(\mathcal{E}_{CM}),$$

where $F$ : **Ell** $\to$ **NC-Tor** is the functor introduced in Section 6.1.1; we shall call such a relation a *non-commutative reciprocity*, because it involves invariants of the non-commutative algebra $\mathcal{A}_{RM}$. The reciprocity provides us with explicit *localization formulas* for the torus $\mathcal{A}_{RM}$ at each prime number $p$; we shall use these formulas in the sequel.

**Remark 4.3.1** The reader can think of the non-commutative reciprocity as an analog of the Eichler-Shimura theory; recall that such a theory identifies the $L$-function coming from certain cusp form (of weight two) and the Hasse-Weil function of a rational elliptic curve, see e.g. [Knapp 1992] [44], Chapter XI.

### 4.3.1 $L$-function of noncommutative tori

Let $p$ be a prime number and $\mathcal{E}_{CM}$ be an elliptic curve with complex multiplication; denote by $\mathcal{E}_{CM}(\mathbb{F}_p)$ localization of the $\mathcal{E}_{CM}$ at the prime ideal $\mathfrak{P}$ over $p$, see e.g. [Silverman 1994] [94], p.171. We are looking for a proper concept of localization of the algebra $\mathcal{A}_{RM} = F(\mathcal{E}_{CM})$ corresponding to the localization $\mathcal{E}_{CM}(\mathbb{F}_p)$ of elliptic curve $\mathcal{E}_{CM}$ at prime $p$. To attain the goal, recall that the cardinals $|\mathcal{E}_{CM}(\mathbb{F}_p)|$ generate the Hasse-Weil function $L(\mathcal{E}_{CM}, s)$ of the curve $\mathcal{E}_{CM}$, see e.g. [Silverman 1994] [94], p.172; thus, we have to define an $L$-function of the noncommutative torus $\mathcal{A}_{RM} = F(\mathcal{E}_{CM})$ equal to the Hasse-Weil function of the curve $\mathcal{E}_{CM}$.

**Definition 4.3.1** *If $\mathcal{A}_{RM}$ is a noncommutative torus with real multiplication, consider an integer matrix*

$$A = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix},$$

*where $\overline{(a_1, \dots, a_k)}$ is the minimal period of continued fraction of a quadratic irrationality $\theta$ corresponding to $\mathcal{A}_{RM}$. For each prime $p$ consider an integer matrix*

$$L_p := \begin{pmatrix} tr\ (A^{\pi(p)}) & p \\ -1 & 0 \end{pmatrix},$$

*where $tr\ (\bullet)$ is the trace of a matrix and $\pi(n)$ is an integer-valued function defined in the Supplement 6.3.3. By a local zeta function of torus $\mathcal{A}_{RM}$ one understands the analytic function*

$$\zeta_p(\mathcal{A}_{RM}, z) := \exp\left(\sum_{n=1}^{\infty} \frac{|K_0(\mathcal{O}_{\varepsilon_n})|}{n}\ z^n\right),\ \varepsilon_n = \begin{cases} L_p^n, & \text{if } p \nmid tr^2(A) - 4 \\ 1 - \alpha^n, & \text{if } p \mid tr^2(A) - 4, \end{cases}$$

*where $\alpha \in \{-1, 0, 1\}$, $\mathcal{O}_{\varepsilon_n} = \mathcal{A}_{RM} \rtimes_{\varepsilon_n} \mathbb{Z}$ is the Cuntz-Krieger algebra and $K_0(\bullet)$ its $K_0$-group, see Section 3.7. By an $L$-function of the noncommutative torus $\mathcal{A}_{RM}$ we understand the analytic function*

$$L(\mathcal{A}_{RM}, s) := \prod_p \zeta_p(\mathcal{A}_{RM}, p^{-s}), \quad s \in \mathbb{C},$$

*where $p$ runs through the set of all prime numbers.*

**Theorem 4.3.1** *The following conditions are equivalent:*

*(i)* $\mathcal{A}_{RM} = F(\mathcal{E}_{CM})$;

*(ii)*

$$\begin{cases} L(\mathcal{A}_{RM}, s) & \equiv & L(\mathcal{E}_{CM}, s), \\ K_0(\mathcal{O}_{\varepsilon_n}) & \cong & \mathcal{E}_{CM}(\mathbb{F}_{p^n}), \end{cases}$$

*where* $F$ : **Ell** $\to$ **NC-Tor** *is the functor defined in Section 6.1.1 and* $L(\mathcal{E}_{CM}, s)$ *is the Hasse-Weil L-function of elliptic curve* $\mathcal{E}_{CM}$.

**Remark 4.3.2 (Non-commutative localization)** Theorem 6.3.1 implies a localization formula for the torus $\mathcal{A}_{RM}$ at a prime $p$, since the Cuntz-Krieger algebra $\mathcal{O}_{\varepsilon_n} \cong \mathcal{A}_{RM} \rtimes_{\varepsilon_n} \mathbb{Z}$ can be viewed as a non-commutative coordinate ring of elliptic curve $\mathcal{E}_{CM}(\mathbb{F}_{p^n})$. Thus, to localize a non-commutative ring one takes its crossed product rather than taking its prime (or maximal) ideal as prescribed by the familiar formula for commutative rings.

### 4.3.2 Proof of Theorem 6.3.1

Let $p$ be such, that $\mathcal{E}_{CM}$ has a good reduction at $\mathfrak{P}$; the corresponding local zeta function $\zeta_p(\mathcal{E}_{CM}, z) = (1 - tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P}))z + pz^2)^{-1}$, where $\psi_{\mathcal{E}(K)}$ is the Grössencharacter on $K$ and $tr$ is the trace of algebraic number. We have to prove, that $\zeta_p(\mathcal{E}_{CM}, z) = \zeta_p(\mathcal{A}_{RM}, z) := (1 - tr\ (A^{\pi(p)})z + pz^2)^{-1}$; the last equality is a consequence of definition of $\zeta_p(\mathcal{A}_{RM}, z)$. Let $\mathcal{E}_{CM} \cong \mathbb{C}/L_{CM}$, where $L_{CM} = \mathbb{Z} + \mathbb{Z}\tau$ is a lattice in the complex plane [Silverman 1994] [101], pp. 95-96; let $K_0(\mathcal{A}_{RM}) \cong \mathfrak{m}_{RM}$, where $\mathfrak{m}_{RM} = \mathbb{Z} + \mathbb{Z}\theta$ is a *pseudo-lattice* in $\mathbb{R}$, see [Manin 2004] [52]. Roughly speaking, we construct an invertible element (a unit) $u$ of the ring $End\ (\mathfrak{m}_{RM})$ attached to pseudo-lattice $\mathfrak{m}_{RM} = F(L_{CM})$, such that

$$tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})) = tr\ (u) = tr\ (A^{\pi(p)}).$$

The latter will be achieved with the help of an explicit formula connecting endomorphisms of lattice $L_{CM}$ with such of the pseudo-lattice $\mathfrak{m}_{RM}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in End\ (L_{CM}) \longmapsto \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \in End\ (\mathfrak{m}_{RM}),$$

see proof of Lemma 6.1.5 for the details. We shall split the proof into a series of lemmas, starting with the following elementary lemma.

**Lemma 4.3.1** *Let $A = (a, b, c, d)$ be an integer matrix with $ad - bc \neq 0$ and $b = 1$. Then $A$ is similar to the matrix $(a + d, 1, c - ad, 0)$.*

*Proof.* Indeed, take a matrix $(1, 0, d, 1) \in SL_2(\mathbb{Z})$. The matrix realizes the similarity, i.e.

$$\begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} a+d & 1 \\ c-ad & 0 \end{pmatrix}.$$

Lemma 6.3.1 follows. □

**Lemma 4.3.2** *The matrix $A = (a+d, 1, c-ad, 0)$ is similar to its transpose $A^t = (a + d, c - ad, 1, 0)$.*

*Proof.* We shall use the following criterion: the (integer) matrices $A$ and $B$ are similar, if and only if the characteristic matrices $xI - A$ and $xI - B$ have the same Smith normal form. The calculation for the matrix $xI - A$ gives

$$\begin{pmatrix} x-a-d & -1 \\ ad-c & x \end{pmatrix} \sim \begin{pmatrix} x-a-d & -1 \\ x^2-(a+d)x+ad-c & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & x^2-(a+d)x+ad-c \end{pmatrix},$$

where $\sim$ are the elementary operations between the rows (columns) of the matrix. Similarly, a calculation for the matrix $xI - A^t$ gives

$$\begin{pmatrix} x-a-d & ad-c \\ -1 & x \end{pmatrix} \sim \begin{pmatrix} x-a-d & x^2-(a+d)x+ad-c \\ -1 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & x^2-(a+d)x+ad-c \end{pmatrix}.$$

Thus, $(xI - A) \sim (xI - A^t)$ and Lemma 6.3.2 follows. □

**Corollary 4.3.1** *The matrices $(a, 1, c, d)$ and $(a+d, c-ad, 1, 0)$ are similar.*

*Proof.* The Corollary 6.3.1 follows from Lemmas 6.3.1–6.3.2. □

Recall that if $\mathcal{E}_{CM}^{(-D,f)}$ is an elliptic curve with complex multiplication by order $R = \mathbb{Z} + fO_k$ in imaginary quadratic field $k = \mathbb{Q}(\sqrt{-D})$, then $\mathcal{A}_{RM}^{(D,f)} = F(\mathcal{E}_{CM}^{(-D,f)})$ is the noncommutative torus with real multiplication by the order $\mathfrak{R} = \mathbb{Z} + fO_{\mathfrak{k}}$ in real quadratic field $\mathfrak{k} = \mathbb{Q}(\sqrt{D})$.

**Lemma 4.3.3** *Each $\alpha \in R$ goes under $F$ into an $\omega \in \mathfrak{R}$, such that $tr\ (\alpha) = tr\ (\omega)$, where $tr\ (x) = x + \bar{x}$ is the trace of an algebraic number $x$.*

*Proof.* Recall that each $\alpha \in R$ can be written in a matrix form for a given base $\{\omega_1, \omega_2\}$ of the lattice $L_{CM}$. Namely,

$$\begin{cases} \alpha\omega_1 & = a\omega_1 + b\omega_2 \\ \alpha\omega_2 & = c\omega_1 + d\omega_2, \end{cases}$$

where $(a, b, c, d)$ is an integer matrix with $ad - bc \neq 0$. and $tr\ (\alpha) = a + d$. The first equation implies $\alpha = a + b\tau$; since both $\alpha$ and $\tau$ are algebraic integers, one concludes that $b = 1$. In view of Corollary 6.3.1, in a base $\{\omega_1', \omega_2'\}$, the $\alpha$ has a matrix form $(a + d, c - ad, 1, 0)$. To calculate a real quadratic $\omega \in \mathfrak{R}$ corresponding to $\alpha$, recall an explicit formula obtained in the proof of Lemma 6.1.5; namely, each endomorphism $(a, b, c, d)$ of the lattice $L_{CM}$ gives rise to the endomorphism $(a, b, -c, -d)$ of pseudo-lattice $\mathfrak{m}_{RM} = F(L_{CM})$. Thus, one gets a map:

$$F: \ \begin{pmatrix} a+d & c-ad \\ 1 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} a+d & c-ad \\ -1 & 0 \end{pmatrix}.$$

In other words, for a given base $\{\lambda_1, \lambda_2\}$ of the pseudo-lattice $\mathbb{Z} + \mathbb{Z}\theta$ one can write

$$\begin{cases} \omega\lambda_1 & = (a + d)\lambda_1 + (c - ad)\lambda_2 \\ \omega\lambda_2 & = -\lambda_1. \end{cases}$$

It is easy to verify, that $\omega$ is a real quadratic integer with $tr\ (\omega) = a + d$. The latter coincides with the $tr\ (\alpha)$. Lemma 6.3.3 follows. $\square$

Let $\omega \in \mathfrak{R}$ be an endomorphism of the pseudo-lattice $\mathfrak{m}_{RM} = \mathbb{Z} + \mathbb{Z}\theta$ of degree $deg\ (\omega) := \omega\bar{\omega} = n$. The endomorphism maps $\mathfrak{m}_{RM}$ to a sub-lattice $\mathfrak{m}_0 \subset \mathfrak{m}_{RM}$ of index $n$; any such has the form $\mathfrak{m}_0 = \mathbb{Z} + (n\theta)\mathbb{Z}$, see e.g. [Borevich & Shafarevich 1966] [11], p.131. Moreover, $\omega$ generates an automorphism, $u$, of the pseudo-lattice $\mathfrak{m}_0$; the traces of $\omega$ and $u$ are related.

**Lemma 4.3.4** *$tr\ (u) = tr\ (\omega)$.*

*Proof.* Let us calculate the action of endomorphism $\omega = (a + d, c - ad, -1, 0)$ on the pseudo-lattice $\mathfrak{m}_0 = \mathbb{Z} + (n\theta)\mathbb{Z}$. Since $deg\ (\omega) = c - ad = n$, one gets

$$\begin{pmatrix} a+d & n \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \theta \end{pmatrix} = \begin{pmatrix} a+d & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ n\theta \end{pmatrix},$$

where $\{1,\theta\}$ and $\{1,n\theta\}$ are bases of the pseudo-lattices $\mathfrak{m}_{RM}$ and $\mathfrak{m}_0$, respectively, and $u=(a+d,1,-1,0)$ is an automorphism of $\mathfrak{m}_0$. It is easy to see, that $tr\ (u)=a+d=tr\ (\omega)$. Lemma 6.3.4 follows. □

**Remark 4.3.3 (Second proof of Lemma 6.3.4)** There exists a canonical proof of Lemma 6.3.4 based on the notion of a subshift of finite type [Wagoner 1999] [106]; we shall give such a proof below, since it generalizes to pseudo-lattices of any rank. Consider a dimension group ([Blackadar 1986] [10], p.55) corresponding to the endomorphism $\omega$ of lattice $\mathbb{Z}^2$, i.e. the limit $G(\omega)$:

$$\mathbb{Z}^2 \overset{\omega}{\to} \mathbb{Z}^2 \overset{\omega}{\to} \mathbb{Z}^2 \overset{\omega}{\to} \dots$$

It is known that $G(\omega)\cong\mathbb{Z}[\frac{1}{\lambda}]$, where $\lambda>1$ is the Perron-Frobenius eigenvalue of $\omega$. We shall write $\hat{\omega}$ to denote the shift automorphism of dimension group $G(\omega)$, ([Effros 1981] [21], p. 37) and $\zeta_\omega(t)=\exp\left(\sum_{k=1}^{\infty}\frac{tr\ (\omega^k)}{k}t^k\right)$ and $\zeta_{\hat{\omega}}(t)=\exp\left(\sum_{k=1}^{\infty}\frac{tr\ (\hat{\omega}^k)}{k}t^k\right)$ the corresponding Artin-Mazur zeta functions [106], p. 273. Since the Artin-Mazur zeta function of the subshift of finite type is an invariant of shift equivalence, we conclude that $\zeta_\omega(t)\equiv\zeta_{\hat{\omega}}(t)$; in particular, $tr\ (\omega)=tr\ (\hat{\omega})$. Hence the matrix form of $\hat{\omega}=(a+d,1,-1,0)=u$ and, therefore, $tr\ (u)=tr\ (\omega)$. Lemma 6.3.4 is proved by a different method. □

**Lemma 4.3.5** *The automorphism $u$ is a unit of the ring $\mathfrak{R}_0:=End\ (\mathfrak{m}_0)$; it is the fundamental unit of $\mathfrak{R}_0$, whenever $n=p$ is a prime number and $tr\ (u)=tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P}))$, where $(\psi_{\mathcal{E}(K)}(\mathfrak{P}))$ is the Grössencharacter associated to prime $p$, see Supplement 6.3.3.*

*Proof.* (i) Since $deg\ (u)=1$, the element $u$ is invertible and, therefore, a unit of the ring $\mathfrak{R}_0$; in general, unit $u$ is not the fundamental unit of $\mathfrak{R}_0$, since it is possible that $u=\varepsilon^a$, where $\varepsilon$ is another unit of $\mathfrak{R}_0$ and $a\ge 1$.

(ii) When $n=p$ is a prime number, then we let $\psi_{\mathcal{E}(K)}(\mathfrak{P})$ be the corresponding Grössencharacter on $K$ attached to an elliptic curve $\mathcal{E}_{CM}\cong\mathcal{E}(K)$, see Supplement 6.3.3 for the notation. The Grössencharacter can be identified with a complex number $\alpha\in k$ of the imaginary quadratic field $k$ associated to the complex multiplication. Let $tr\ (u)=tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P}))$ and suppose to the contrary, that $u$ is not the fundamental unit of $\mathfrak{R}_0$, i.e. $u=\varepsilon^a$ for a unit $\varepsilon\in\mathfrak{R}_0$ and an integer $a\ge 1$. Then there exists a Grössencharacter $\psi'_{\mathcal{E}(K)}(\mathfrak{P})$, such that

$$tr\ (\psi'_{\mathcal{E}(K)}(\mathfrak{P})) < tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})).$$

Since $tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})) = q_{\mathfrak{P}} + 1 - \#\tilde{E}(\mathbb{F}_{\mathfrak{P}})$, one concludes that $\#\tilde{E}(\mathbb{F}'_{\mathfrak{P}}) > \#\tilde{E}(\mathbb{F}_{\mathfrak{P}})$; in other words, there exists a non-trivial extension $\mathbb{F}'_{\mathfrak{P}} \supset \mathbb{F}_{\mathfrak{P}}$ of the finite field $\mathbb{F}_{\mathfrak{P}}$. The latter is impossible, since any extension of $\mathbb{F}_{\mathfrak{P}}$ has the form $\mathbb{F}_{\mathfrak{P}^n}$ for some $n \geq 1$; thus $a = 1$, i.e. unit $u$ is the fundamental unit of the ring $\mathfrak{R}_0$. Lemma 6.3.5 is proved. □

**Lemma 4.3.6** $tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})) = tr\ (A^{\pi(p)})$.

*Proof.* Recall that the fundamental unit of the order $\mathfrak{R}_0$ is given by the formula $\varepsilon_p = \varepsilon^{\pi(p)}$, where $\varepsilon$ is the fundamental unit of the ring $O_{\mathfrak{k}}$ and $\pi(p)$ an integer number, see Hasse's Lemma 6.3.10 of Supplement 6.3.3. On the other hand, matrix $A = \prod_{i=1}^{n}(a_i, 1, 1, 0)$, where $\theta = \overline{(a_1, \ldots, a_n)}$ is a purely periodic continued fraction. Therefore

$$A\begin{pmatrix}1\\ \theta\end{pmatrix} = \varepsilon\begin{pmatrix}1\\ \theta\end{pmatrix},$$

where $\varepsilon > 1$ is the fundamental unit of the real quadratic field $\mathfrak{k} = \mathbb{Q}(\theta)$. In other words, $A$ is the matrix form of the fundamental unit $\varepsilon$. Therefore the matrix form of the fundamental unit $\varepsilon_p = \varepsilon^{\pi(p)}$ of $\mathfrak{R}_0$ is given by matrix $A^{\pi(p)}$. One can apply Lemma 6.3.5 and get

$$tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})) = tr\ (\varepsilon_p) = tr\ (A^{\pi(p)}).$$

Lemma 6.3.6 follows. □

One can finish the proof of Theorem 6.3.1 by comparing the local $L$-series of the Hasse-Weil $L$-function for the $\mathcal{E}_{CM}$ with that of the local zeta for the $\mathcal{A}_{RM}$. The local $L$-series for $\mathcal{E}_{CM}$ are $L_{\mathfrak{P}}(\mathcal{E}(K), T) = 1 - a_{\mathfrak{P}}T + q_{\mathfrak{P}}T^2$ if the $\mathcal{E}_{CM}$ has a good reduction at $\mathfrak{P}$ and $L_{\mathfrak{P}}(\mathcal{E}(K), T) = 1 - \alpha T$ otherwise; here

$$\left\{\begin{array}{lll} q_{\mathfrak{P}} & = & N_{\mathbb{Q}}^{K}\mathfrak{P} = \#\mathbb{F}_{\mathfrak{P}} = p, \\ a_{\mathfrak{P}} & = & q_{\mathfrak{P}} + 1 - \#\tilde{E}(\mathbb{F}_{\mathfrak{P}}) = tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})), \\ \alpha & \in & \{-1, 0, 1\}. \end{array}\right.$$

Therefore,

$$L_{\mathfrak{P}}(\mathcal{E}_{CM}, T) = \begin{cases} 1 - tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P}))T + pT^2, & \text{for good reduction} \\ 1 - \alpha T, & \text{for bad reduction.} \end{cases}$$

**Lemma 4.3.7** *For $\mathcal{A}_{RM} = F(\mathcal{E}_{CM})$, it holds*

$$\zeta_p^{-1}(\mathcal{A}_{RM}, T) = 1 - tr\ (A^{\pi(p)})T + pT^2,$$

*whenever $p \nmid tr^2(A) - 4$.*

*Proof.* By the formula $K_0(\mathcal{O}_B) = \mathbb{Z}^2/(I - B^t)\mathbb{Z}^2$, one gets

$$|K_0(\mathcal{O}_{L_p^n})| = \left|\frac{\mathbb{Z}^2}{(I - (L_p^n)^t)\mathbb{Z}^2}\right| = |det(I - (L_p^n)^t)| = |Fix\ (L_p^n)|,$$

where $Fix\ (L_p^n)$ is the set of (geometric) fixed points of the endomorphism $L_p^n : \mathbb{Z}^2 \to \mathbb{Z}^2$. Thus,

$$\zeta_p(\mathcal{A}_{RM}, z) = \exp\left(\sum_{n=1}^{\infty} \frac{|Fix\ (L_p^n)|}{n}\ z^n\right), \quad z \in \mathbb{C}.$$

But the latter series is an Artin-Mazur zeta function of the endomorphism $L_p$; it converges to a rational function $det^{-1}(I - zL_p)$, see e.g. [Hartshorn 1977] [35], p.455. Thus, $\zeta_p(\mathcal{A}_{RM}, z) = det^{-1}(I - zL_p)$. The substitution $L_p = (tr\ (A^{\pi(p)}), p, -1, 0)$ gives us

$$det\ (I - zL_p) = det\ \begin{pmatrix} 1 - tr\ (A^{\pi(p)})z & -pz \\ z & 1 \end{pmatrix} = 1 - tr\ (A^{\pi(p)})z + pz^2.$$

Put $z = T$ and get $\zeta_p(\mathcal{A}_{RM}, T) = (1 - tr\ (A^{\pi(p)})T + pT^2)^{-1}$, which is a conclusion of Lemma 6.3.7. □

**Lemma 4.3.8** *For $\mathcal{A}_{RM} = F(\mathcal{E}_{CM})$, it holds*

$$\zeta_p^{-1}(\mathcal{A}_{RM}, T) = 1 - \alpha T,$$

*whenever $p \mid tr^2(A) - 4$.*

*Proof.* Indeed, $K_0(\mathcal{O}_{1-\alpha^n}) = \mathbb{Z}/(1 - 1 + \alpha^n)\mathbb{Z} = \mathbb{Z}/\alpha^n\mathbb{Z}$. Thus, $|K_0(\mathcal{O}_{1-\alpha^n})| = det\ (\alpha^n) = \alpha^n$. By the definition,

$$\zeta_p(\mathcal{A}_{RM}, z) = \exp\left(\sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(\alpha z)^n}{n}\right) = \frac{1}{1 - \alpha z}.$$

The substitution $z = T$ gives the conclusion of Lemma 6.3.8. □

**Lemma 4.3.9** *Let $\mathfrak{P} \subset K$ be a prime ideal over $p$; then $\mathcal{E}_{CM} = \mathcal{E}(K)$ has a bad reduction at $\mathfrak{P}$ if and only if $p \mid tr^2(A) - 4$.*

*Proof.* Let $k$ be a field of complex multiplication of the $\mathcal{E}_{CM}$; its discriminant we shall write as $\Delta_k < 0$. It is known, that whenever $p \mid \Delta_k$, the $\mathcal{E}_{CM}$ has a bad reduction at the prime ideal $\mathfrak{P}$ over $p$. On the other hand, the explicit formula for functor $F$ applied to the matrix $L_p$ gives us $F$ : $(tr\ (A^{\pi(p)}), p, -1, 0) \mapsto (tr\ (A^{\pi(p)}), p, 1, 0)$, see proof of Lemma 6.1.5. The characteristic polynomials of the above matrices are $x^2 - tr\ (A^{\pi(p)})x + p$ and $x^2 - tr\ (A^{\pi(p)})x - p$, respectively. They generate an imaginary (resp., a real) quadratic field $k$ (resp., $\mathfrak{k}$) with the discriminant $\Delta_k = tr^2(A^{\pi(p)}) - 4p < 0$ (resp., $\Delta_{\mathfrak{k}} = tr^2(A^{\pi(p)}) + 4p > 0$). Thus, $\Delta_{\mathfrak{k}} - \Delta_k = 8p$. It is easy to see, that $p \mid \Delta_{\mathfrak{k}}$ if and only if $p \mid \Delta_k$. It remains to express the discriminant $\Delta_{\mathfrak{k}}$ in terms of the matrix $A$. Since the characteristic polynomial for $A$ is $x^2 - tr\ (A)x + 1$, it follows that $\Delta_{\mathfrak{k}} = tr^2(A) - 4$. Lemma 6.3.9 follows. □

Let us prove that the first part of Theorem 6.3.1 implies the first claim of its second part; notice, that the critical piece of information is provided by Lemma 6.3.6, which says that $tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})) = tr\ (A^{\pi(p)})$. Thus, Lemmas 6.3.7–6.3.9 imply that $L_{\mathfrak{P}}(\mathcal{E}_{CM}, T) \equiv \zeta_p^{-1}(\mathcal{A}_{RM}, T)$. The first claim of part (ii) of Theorem 6.3.1 follows.

A. Let $p$ be a good prime. Let us prove the second claim of part (ii) of Theorem 6.3.1 in the case $n = 1$. From the left side: $K_0(\mathcal{A}_{RM} \rtimes_{L_p} \mathbb{Z}) \cong K_0(\mathcal{O}_{L_p}) \cong \mathbb{Z}^2/(I - L_p^t)\mathbb{Z}^2$, where $L_p = (tr\ (A^{\pi(p)}), p, -1, 0)$. To calculate the abelian group $\mathbb{Z}^2/(I - L_p^t)\mathbb{Z}^2$, we shall use a reduction of the matrix $I - L_p^t$ to the Smith normal form:

$$I - L_p^t = \begin{pmatrix} 1 - tr\ (A^{\pi(p)}) & 1 \\ -p & 1 \end{pmatrix} \sim \begin{pmatrix} 1 + p - tr\ (A^{\pi(p)}) & 0 \\ -p & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 + p - tr\ (A^{\pi(p)}) \end{pmatrix}.$$

Therefore, $K_0(\mathcal{O}_{L_p}) \cong \mathbb{Z}_{1+p-tr\ (A^{\pi(p)})}$. From the right side, the $\mathcal{E}_{CM}(\mathbb{F}_{\mathfrak{P}})$ is an elliptic curve over the field of characteristic $p$. Recall, that the chord and tangent law turns the $\mathcal{E}_{CM}(\mathbb{F}_{\mathfrak{P}})$ into a finite abelian group. The group is cyclic and has the order $1+q_{\mathfrak{P}}-a_{\mathfrak{P}}$. But $q_{\mathfrak{P}} = p$ and $a_{\mathfrak{P}} = tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})) = tr\ (A^{\pi(p)})$, see Lemma 6.3.6. Thus, $\mathcal{E}_{CM}(\mathbb{F}_{\mathfrak{P}}) \cong \mathbb{Z}_{1+p-tr\ (A^{\pi(p)})}$; therefore $K_0(\mathcal{O}_{L_p}) \cong$

$\mathcal{E}_{CM}(\mathbb{F}_p)$. The general case $n \geq 1$ is treated likewise. Repeating the argument of Lemmas 6.3.1–6.3.2, it follows that

$$L_p^n = \begin{pmatrix} tr\ (A^{n\pi(p)}) & p^n \\ -1 & 0 \end{pmatrix}.$$

Then one gets $K_0(\mathcal{O}_{L_p^n}) \cong \mathbb{Z}_{1+p^n - tr\ (A^{n\pi(p)})}$ on the left side. From the right side, $|\mathcal{E}_{CM}(\mathbb{F}_{p^n})| = 1 + p^n - tr\ (\psi^n_{\mathcal{E}(K)}(\mathfrak{P}))$; but a repetition of the argument of Lemma 6.3.6 yields us $tr\ (\psi^n_{\mathcal{E}(K)}(\mathfrak{P})) = tr\ (A^{n\pi(p)})$. Comparing the left and right sides, one gets that $K_0(\mathcal{O}_{L_p^n}) \cong \mathcal{E}_{CM}(\mathbb{F}_{p^n})$. This argument finishes the proof of the second claim of part (ii) of Theorem 6.3.1 for the good primes.

B. Let $p$ be a bad prime. From the proof of Lemma 6.3.8, one gets for the left side $K_0(\mathcal{O}_{\varepsilon_n}) \cong \mathbb{Z}_{\alpha^n}$. From the right side, it holds $|\mathcal{E}_{CM}(\mathbb{F}_{p^n})| = 1 + q_{\mathfrak{P}} - a_{\mathfrak{P}}$, where $q_{\mathfrak{P}} = 0$ and $a_{\mathfrak{P}} = tr\ (\varepsilon_n) = \varepsilon_n$. Thus, $|\mathcal{E}_{CM}(\mathbb{F}_{p^n})| = 1 - \varepsilon_n = 1 - (1 - \alpha^n) = \alpha^n$. Comparing the left and right sides, we conclude that $K_0(\mathcal{O}_{\varepsilon_n}) \cong \mathcal{E}_{CM}(\mathbb{F}_{p^n})$ at the bad primes.

All cases are exhausted; thus part (i) of Theorem 6.3.1 implies its part (ii). The proof of converse consists in a step by step claims similar to just proved and is left to the reader. Theorem 6.3.1 is proved. $\square$

### 4.3.3 Supplement: Grössencharacters, units and $\pi(n)$

We shall briefly review the well known facts about complex multiplication and units in subrings of the ring of integers in algebraic number fields; for the detailed account, we refer the reader to [Silverman 1994] [94] and [Hasse 1950] [37], respectively.

#### Grössencharacters

Let $\mathcal{E}_{CM} \cong \mathcal{E}(K)$ be elliptic curve with complex multiplication and $K \cong k(j(\mathcal{E}_{CM}))$ the Hilbert class field attached to $\mathcal{E}_{CM}$. For each prime ideal $\mathfrak{P}$ of $K$, let $\mathbb{F}_{\mathfrak{P}}$ be a residue field of $K$ at $\mathfrak{P}$ and $q_{\mathfrak{P}} = N^K_{\mathbb{Q}}\mathfrak{P} = \#\mathbb{F}_{\mathfrak{P}}$, where $N^K_{\mathbb{Q}}$ is the norm of the ideal $\mathfrak{P}$. If $\mathcal{E}(K)$ has a good reduction at $\mathfrak{P}$, one defines $a_{\mathfrak{P}} = q_{\mathfrak{P}} + 1 - \#\tilde{\mathcal{E}}(\mathbb{F}_{\mathfrak{P}})$, where $\tilde{\mathcal{E}}(\mathbb{F}_{\mathfrak{P}})$ is a reduction of $\mathcal{E}(K)$ modulo the prime ideal $\mathfrak{P}$. If $\mathcal{E}(K)$ has good reduction at $\mathfrak{P}$, the polynomial

$$L_{\mathfrak{P}}(\mathcal{E}(K), T) = 1 - a_{\mathfrak{P}}T + q_{\mathfrak{P}}T^2,$$

is called the *local L-series* of $\mathcal{E}(K)$ at $\mathfrak{P}$. If $\mathcal{E}(K)$ has bad reduction at $\mathfrak{P}$, the local $L$-series are $L_{\mathfrak{P}}(\mathcal{E}(K),T) = 1 - T$ (resp. $L_{\mathfrak{P}}(\mathcal{E}(K),T) = 1 + T$; $L_{\mathfrak{P}}(\mathcal{E}(K),T) = 1$) if $\mathcal{E}(K)$ has split multiplicative reduction at $\mathfrak{P}$ (if $\mathcal{E}(K)$ has non-split multiplicative reduction at $\mathfrak{P}$; if $\mathcal{E}(K)$ has additive reduction at $\mathfrak{P}$).

**Definition 4.3.2** *By the Hasse-Weil L-function of elliptic curve $\mathcal{E}(K)$ one understands the global L-series defined by the Euler product*

$$L(\mathcal{E}(K),s) = \prod_{\mathfrak{P}} \, [L_{\mathfrak{P}}(\mathcal{E}(K), q_{\mathfrak{P}}^{-s})]^{-1}.$$

**Definition 4.3.3** *If $A_K^*$ be the idele group of the number field $K$, then by a Grössencharacter on $K$ one understands a continuous homomorphism*

$$\psi : A_K^* \longrightarrow \mathbb{C}^*$$

*with the property $\psi(K^*) = 1$; the asterisk denotes the group of invertible elements of the corresponding ring. The Hecke L-series attached to the Grössencharacter $\psi : A_K^* \to \mathbb{C}^*$ is defined by the Euler product*

$$L(s,\psi) = \prod_{\mathfrak{P}} (1 - \psi(\mathfrak{P}) q_{\mathfrak{P}}^{-s})^{-1},$$

*where the product is taken over all prime ideals of $K$.*

**Remark 4.3.4** For a prime ideal $\mathfrak{P}$ of field $K$ at which $\mathcal{E}(K)$ has good reduction and $\tilde{\mathcal{E}}(\mathbb{F}_{\mathfrak{P}})$ being the reduction of $\mathcal{E}(K)$ at $\mathfrak{P}$, we let

$$\phi_{\mathfrak{P}} : \tilde{\mathcal{E}}(\mathbb{F}_{\mathfrak{P}}) \longrightarrow \tilde{\mathcal{E}}(\mathbb{F}_{\mathfrak{P}})$$

denote the associated *Frobenius map*; if $\psi_{\mathcal{E}(K)} : A_K^* \to k^*$ is the Grössencharacter attached to the $\mathcal{E}_{CM}$, then the diagram in Fig. 6.5 is known to be commutative, see [Silverman 1994] [94], p.174. In particular, $\psi_{\mathcal{E}(K)}(\mathfrak{P})$ is an endomorphism of the $\mathcal{E}(K)$ given by the complex number $\alpha_{\mathcal{E}(K)}(\mathfrak{P}) \in R$, where $R = \mathbb{Z} + fO_k$ is an order in imaginary quadratic field $k$ . If $\overline{\psi}_{\mathcal{E}(K)}(\mathfrak{P})$ is the conjugate Grössencharacter viewed as a complex number, then the *Deuring Theorem* says that the Hasse-Weil $L$-function of the $\mathcal{E}(K)$ is related to the Hecke $L$-series of the $\psi_{\mathcal{E}(K)}$ by the formula

$$L(\mathcal{E}(K),s) \equiv L(s,\psi_{\mathcal{E}(K)}) L(s,\overline{\psi}_{\mathcal{E}(K)}).$$

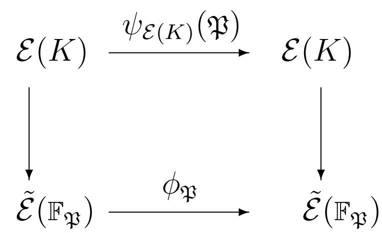



Figure 4.5: The Grössencharacter $\psi_{\mathcal{E}(K)}(\mathfrak{P})$.

### Units and function $\pi(n)$

Let $\mathfrak{k} = \mathbb{Q}(\sqrt{D})$ be a real quadratic number field and $O_{\mathfrak{k}}$ its ring of integers. For rational integer $n \geq 1$ we shall write $\mathfrak{R}_n \subseteq O_{\mathfrak{k}}$ to denote an order (i.e. a subring containing 1) of $O_{\mathfrak{k}}$. The order $\mathfrak{R}_n$ has a basis $\{1, n\omega\}$, where

$$\omega = \begin{cases} \frac{\sqrt{D}+1}{2} & \text{if } D \equiv 1 \ mod\ 4, \\ \sqrt{D} & \text{if } D \equiv 2, 3 \ mod\ 4. \end{cases}$$

In other words, $\mathfrak{R}_n = \mathbb{Z} + (n\omega)\mathbb{Z}$. It is clear, that $\mathfrak{R}_1 = O_{\mathfrak{k}}$ and the fundamental unit of $O_{\mathfrak{k}}$ we shall denote by $\varepsilon$. Each $\mathfrak{R}_n$ has its own fundamental unit, which we shall write as $\varepsilon_n$; notice that $\varepsilon_n \neq \varepsilon$ unless $n = 1$. There exists the well-known formula, which relates $\varepsilon_n$ to the fundamental unit $\varepsilon$, see e.g. [Hasse 1950] [37], p.297. Denote by $\mathfrak{G}_n := U(O_{\mathfrak{k}}/nO_{\mathfrak{k}})$ the multiplicative group of invertible elements (units) of the residue ring $O_{\mathfrak{k}}/nO_{\mathfrak{k}}$; clearly, all units of $O_{\mathfrak{k}}$ map (under the natural *mod n* homomorphism) to $\mathfrak{G}_n$. Likewise let $\mathfrak{g}_n := U(\mathfrak{R}_n/n\mathfrak{R}_n)$ be the group of units of the residue ring $\mathfrak{R}_n/n\mathfrak{R}_n$; it is not hard to prove ([Hasse 1950] [37], p.296), that $\mathfrak{g}_n \cong U(\mathbb{Z}/n\mathbb{Z})$ the "rational" unit group of the residue ring $\mathbb{Z}/n\mathbb{Z}$. Similarly, all units of the order $\mathfrak{R}_n$ map to $\mathfrak{g}_n$. Since units of $\mathfrak{R}_n$ are also units of $O_{\mathfrak{k}}$ (but not vice versa), $\mathfrak{g}_n$ is a subgroup of $\mathfrak{G}_n$; in particular, $|\mathfrak{G}_n|/|\mathfrak{g}_n|$ is an integer number and $|\mathfrak{g}_n| = \varphi(n)$, where $\varphi(n)$ is the Euler totient function. In general, the following formula is true

$$\frac{|\mathfrak{G}_n|}{|\mathfrak{g}_n|} = n \prod_{p_i | n} \left(1 - \left(\frac{D}{p_i}\right) \frac{1}{p_i}\right),$$

where $\left(\frac{D}{p_i}\right)$ is the Legendre symbol, see [Hasse 1950] [37], p. 351.

**Definition 4.3.4** *By the function* $\pi(n)$ *one understands the least integer number dividing* $|\mathfrak{G}_n|/|\mathfrak{g}_n|$ *and such that* $\varepsilon^{\pi(n)}$ *is a unit of* $\mathfrak{R}_n$*, i.e. belongs to* $\mathfrak{g}_n$*.*

**Lemma 4.3.10 ([Hasse 1950] [37], p.298)** $\varepsilon_n = \varepsilon^{\pi(n)}$.

**Remark 4.3.5** Lemma 6.3.10 asserts existence of the number $\pi(n)$ as one of the divisors of $|\mathfrak{G}_n|/|\mathfrak{g}_n|$, yet no analytic formula for $\pi(n)$ is known; it would be rather interesting to have such a formula.

**Remark 4.3.6** In the special case $n = p$ is a prime number, the following formula is true

$$\frac{|\mathfrak{G}_p|}{|\mathfrak{g}_p|} = p - \left(\frac{D}{p}\right).$$

**Guide to the literature.** The Hasse-Weil $L$-functions $L(\mathcal{E}(K), s)$ of the $K$-rational elliptic curves are covered in the textbooks by [Husemöller 1986] [42], [Knapp 1992] [44] and [Silverman 1994] [94]; see also the survey [Tate 1974] [103]. The reciprocity of $L(\mathcal{E}(K), s)$ with an $L$-function obtained from certain cusp form of weight two is subject of the *Eichler-Shimura theory*, see e.g. [Knapp 1992] [44], Chapter XI; such a reciprocity coupled with the *Shimura-Taniyama Conjecture* was critical to solution of the Fermat Last Theorem by A. Wiles. The *non-commutative reciprocity* of $L(\mathcal{E}(K), s)$ with an $L$-function obtained from a noncommutative torus with real multiplication was proved in [77].

## 4.4 Langlands program for noncommutative tori

We dealt with functors on the arithmetic schemes $X$ so far. In this section we shall define a functor $F$ on the category of all finite Galois extensions $E$ of the field $\mathbb{Q}$; the functor ranges in a category of the even-dimensional noncommutative tori with real multiplication. For such a torus, $\mathcal{A}_{RM}^{2n}$, we construct an $L$-function $L(\mathcal{A}_{RM}^{2n}, s)$; it is conjectured that for each $n \geq 1$ and each irreducible representation

$$\sigma : Gal\ (E|\mathbb{Q}) \longrightarrow GL_n(\mathbb{C}),$$

the corresponding *Artin L-function* $L(\sigma, s)$ coincides with $L(\mathcal{A}_{RM}^{2n}, s)$, whenever $\mathcal{A}_{RM}^{2n} = F(E)$. Our main result Theorem 6.4.1 says that the conjecture is true for $n = 1$ (resp., $n = 0$) and $E$ being the Hilbert class field of an imaginary quadratic field $k$ (resp., the rational field $\mathbb{Q}$) . Thus we are dealing with an analog of the *Langlands program*, where the "automorphic cuspidal representations of group $GL_n$" are replaced by the noncommutative tori $\mathcal{A}_{RM}^{2n}$, see [Gelbart 1984] [28] for an introduction.

### 4.4.1 $L(\mathcal{A}_{RM}^{2n}, s)$

The higher-dimensional noncommutative tori were introduced in Section 3.4.1; let us recall some notation. Let $\Theta = (\theta_{ij})$ be a real skew symmetric matrix of even dimension $2n$; by $\mathcal{A}_{\Theta}^{2n}$ we shall mean the even-dimensional noncommutaive torus defined by matrix $\Theta$, i.e. a universal $C^*$-algebra on the unitary generators $u_1, \dots, u_{2n}$ and relations

$$u_j u_i = e^{2\pi i \theta_{ij}} u_i u_j, \quad 1 \leq i, j \leq 2n.$$

It is known, that by the orthogonal linear transformations every (generic) real even-dimensional skew symmetric matrix can be brought to the normal form

$$\Theta_0 = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix}$$

where $\theta_i > 0$ are linearly independent over $\mathbb{Q}$. We shall consider the noncommutative tori $\mathcal{A}_{\Theta_0}^{2n}$, given by matrix in the above normal form; we refer to the family as a *normal family*. Recall that $K_0(\mathcal{A}_{\Theta_0}^{2n}) \cong \mathbb{Z}^{2^{2n-1}}$ and the positive cone $K_0^+(\mathcal{A}_{\Theta_0}^{2n})$ is given by the pseudo-lattice

$$\mathbb{Z} + \theta_1 \mathbb{Z} + \dots + \theta_n \mathbb{Z} + \sum_{i=n+1}^{2^{2n-1}} p_i(\theta) \mathbb{Z} \ \subset \ \mathbb{R},$$

where $p_i(\theta) \in \mathbb{Z}[1, \theta_1, \dots, \theta_n]$, see e.g. [Elliott 1982] [23].

**Definition 4.4.1** *The noncommutative torus $\mathcal{A}^{2n}_{\Theta_0}$ is said to have real multiplication if the endomorphism ring End $(K_0^+(\mathcal{A}^{2n}_{\Theta_0}))$ is non-trivial, i.e. exceeds the ring $\mathbb{Z}$; we shall denote such a torus by $\mathcal{A}^{2n}_{RM}$.*

**Remark 4.4.1** It is easy to see that if $\mathcal{A}^{2n}_{\Theta_0}$ has real multiplication, then $\theta_i$ are algebraic integers; we leave the proof to the reader. (Hint: each endomorphism of $K_0^+(\mathcal{A}^{2n}_{\Theta_0}) \cong \mathbb{Z}+\theta_1\mathbb{Z}+\ldots+\theta_n\mathbb{Z}+\sum_{i=n+1}^{2^{2n-1}} p_i(\theta)\mathbb{Z}$ is multiplication by a real number; thus the endomorphism is described by an integer matrix, which defines a polynomial equation involving $\theta_i$.)

**Remark 4.4.2** Remark 6.4.1 says that $\theta_i$ are algebraic integers whenever $\mathcal{A}^{2n}_{\Theta_0}$ has real multiplication; so will be the values of polynomials $p_i(\theta)$ in this case. Since such values belong to the number field $\mathbb{Q}(\theta_1,\ldots,\theta_n)$, one concludes that

$$K_0^+(\mathcal{A}^{2n}_{RM}) \cong \mathbb{Z}+\theta_1\mathbb{Z}+\ldots+\theta_n\mathbb{Z} \subset \mathbb{R}.$$

Let $A \in GL_{n+1}(\mathbb{Z})$ be a positive matrix such that

$$A \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} = \lambda_A \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix},$$

where $\lambda_A$ is the Perron-Frobenius eigenvalue of $A$; in other words, $A$ is a matrix corresponding to the *shift automorphism* $\sigma_A$ of $K_0^+(\mathcal{A}^{2n}_{RM})$ regarded as a stationary dimension group, see Definition 3.5.4. For each prime number $p$, consider the characteristic polynomial of matrix $A^{\pi(p)}$, where $\pi(n)$ is the integer-valued function introduced in Section 6.3.3; in other words,

$$\mathbf{Char}(A^{\pi(p)}) := \det(xI - A^{\pi(p)}) = x^{n+1} - a_1x^n - \ldots - a_nx - 1 \in \mathbb{Z}[x].$$

**Definition 4.4.2** *By a local zeta function of the noncommutative torus $\mathcal{A}^{2n}_{RM}$ we understand the function*

$$\zeta_p(\mathcal{A}^{2n}_{RM}, z) := \frac{1}{1 - a_1z + a_2z^2 - \ldots - a_nz^n + pz^{n+1}}, \quad z \in \mathbb{C}.$$

**Remark 4.4.3** To explain the structure of $\zeta_p(\mathcal{A}_{RM}^{2n}, z)$, consider the *companion matrix*

$$J = \begin{pmatrix} a_1 & 1 & \dots & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

of polynomial **Char** $(A^{\pi(p)}) = x^{n+1} - a_1x^n - \ldots - a_nx - 1$, i.e. the matrix $J$ such that $\det(xI - J) = x^{n+1} - a_1x^n - \ldots - a_nx - 1$. It is not hard to see, that the non-negative integer matrix $J$ corresponds to the *shift automorphism* of a stationary dimension group

$$\mathbb{Z}^{n+1} \overset{J_p}{\longrightarrow} \mathbb{Z}^{n+1} \overset{J_p}{\longrightarrow} \mathbb{Z}^{n+1} \overset{J_p}{\longrightarrow} \dots$$

where

$$J_p = \begin{pmatrix} a_1 & 1 & \dots & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \dots & 0 & 1 \\ p & 0 & \dots & 0 & 0 \end{pmatrix}.$$

On the other hand, the companion matrix of polynomial **Char** $(\sigma(Fr_p)) = \det(xI - \sigma(Fr_p)) = x^{n+1} - a_1x^n + \ldots - a_nx + p$ has the form

$$W_p = \begin{pmatrix} a_1 & 1 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \dots & 0 & 1 \\ -p & 0 & \dots & 0 & 0 \end{pmatrix},$$

see Section 6.4.3 for the meaning of $\sigma(Fr_p)$. Thus the action of functor $F$ : **Alg-Num** $\to$ **NC-Tor** on the corresponding companion matrices $W_p$ and $J_p$ is given by the formula

$$F : \begin{pmatrix} a_1 & 1 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \dots & 0 & 1 \\ -p & 0 & \dots & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 1 & \dots & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \dots & 0 & 1 \\ p & 0 & \dots & 0 & 0 \end{pmatrix}.$$

It remains to compare our formula for $\zeta_p(\mathcal{A}_{RM}^{2n}, z)$ with the well-known formula for the Artin zeta function

$$\zeta_p(\sigma_n, z) = \frac{1}{\det(I_n - \sigma_n(Fr_p)z)},$$

where $z = x^{-1}$, see [Gelbart 1984] [28], p. 181.

**Definition 4.4.3** *By an L-function of the noncommutative torus $\mathcal{A}_{RM}^{2n}$ one understand the product*

$$L(\mathcal{A}_{RM}^{2n}, s) := \prod_p \zeta_p(\mathcal{A}_{RM}^{2n}, p^{-s}), \quad s \in \mathbb{C},$$

*over all but a finite number of primes p.*

**Conjecture 4.4.1 (Langlands conjecture for noncommutative tori)** *For each finite extension E of the field of rational numbers $\mathbb{Q}$ with the Galois group Gal $(E|\mathbb{Q})$ and each irreducible representation*

$$\sigma_{n+1} : Gal\ (E|\mathbb{Q}) \to GL_{n+1}(\mathbb{C}),$$

*there exists a $2n$-dimensional noncommutative torus with real multiplication $\mathcal{A}_{RM}^{2n}$, such that*

$$L(\sigma_{n+1}, s) \equiv L(\mathcal{A}_{RM}^{2n}, s),$$

*where $L(\sigma_{n+1}, s)$ is the Artin L-function attached to representation $\sigma_{n+1}$ and $L(\mathcal{A}_{RM}^{2n}, s)$ is the L-function of the noncommutative torus $\mathcal{A}_{RM}^{2n}$.*

**Remark 4.4.4** Roughly speaking, Conjecture 6.4.1 says that the Galois extensions (abelian or not) of the field $\mathbb{Q}$ are in a one-to-one correspondence with the even-dimensional noncommutative tori with real multiplication. In the context of the Langlands program, the noncommutative torus $\mathcal{A}_{RM}^{2n}$ can be regarded as an analog of the "automorphic cuspidal representation $\pi_{\sigma_{n+1}}$ of the group $GL(n+1)$". This appearance of $\mathcal{A}_{RM}^{2n}$ is *not* random because the noncommutative tori classify the irreducible infinite-dimensional representations of the Lie group $GL(n+1)$, see the remarkable paper by [Poguntke 1983] [85]; such representations are known to be at the heart of the Langlands philosophy, see [Gelbart 1984] [28] .

**Theorem 4.4.1** *Conjecture 6.4.1 is true for $n = 1$ (resp., $n = 0$) and E abelian extension of an imaginary quadratic field k (resp., the rational field $\mathbb{Q}$).*

### 4.4.2 Proof of Theorem 6.4.1

**Case** $n = 1$

Roughly speaking, this case is equivalent to Theorem 6.3.1; it was a model example for Conjecture 6.4.1. Using the Grössencharacters, one can identify the Artin $L$-function for abelian extensions of the imaginary quadratic fields $k$ with the Hasse-Weil $L$-function $L(\mathcal{E}_{CM}, s)$, where $\mathcal{E}_{CM}$ is an elliptic curve with complex multiplication by $k$; but Theorem 6.3.1 says that $L(\mathcal{E}_{CM}, s) \equiv L(\mathcal{A}_{RM}, s)$, where $L(\mathcal{A}_{RM}, s)$ is the special case $n = 1$ of our function $L(\mathcal{A}_{RM}^{2n}, s)$.

To give the details, let $k$ be an imaginary quadratic field and let $\mathcal{E}_{CM}$ be an elliptic curve with complex multiplication by (an order) in $k$. By the theory of complex multiplication, the Hilbert class field $K$ of $k$ is given by the $j$-invariant of $\mathcal{E}_{CM}$, i.e.

$$K \cong k(j(\mathcal{E}_{CM})),$$

and $Gal\ (K|k) \cong Cl\ (k)$, where $Cl\ (k)$ is the ideal class group of $k$; moreover,

$$\mathcal{E}_{CM} \cong \mathcal{E}(K),$$

see e.g. [Silverman 1994] [94]. Recall that functor $F : \mathbf{Ell} \to \mathbf{NC\text{-}Tor}$ maps $\mathcal{E}_{CM}$ to a two-dimensional noncommutive torus with real multiplication $\mathcal{A}_{RM}^2$. To calculate $L(\mathcal{A}_{RM}^2, s)$, let $A \in GL_2(\mathbb{Z})$ be positive matrix corresponding the *shift automorphism* of $\mathcal{A}_{RM}^2$, i.e.

$$A \begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lambda_A \begin{pmatrix} 1 \\ \theta \end{pmatrix},$$

where $\theta$ is a quadratic irrationality and $\lambda_A$ the Perron-Frobenius eigenvalue of $A$. If $p$ is a prime, then the characteristic polynomial of matrix $A^{\pi(p)}$ can be written as **Char** $A^{\pi(p)} = x^2 - tr\ (A^{\pi(p)})x - 1$; therefore, the local zeta function of torus $\mathcal{A}_{RM}^2$ has the form

$$\zeta_p(\mathcal{A}_{RM}^2, z) = \frac{1}{1 - tr\ (A^{\pi(p)})z + pz^2}.$$

On the other hand, Lemma 6.3.6 says that

$$tr\ (A^{\pi(p)}) = tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P})),$$

where $\psi_{\mathcal{E}(K)}$ is the Grössencharacter on $K$ and $\mathfrak{P}$ the prime ideal of $K$ over $p$. But we know that the local zeta function of $\mathcal{E}_{CM}$ has the form

$$\zeta_p(\mathcal{E}_{CM}, z) = \frac{1}{1 - tr\ (\psi_{\mathcal{E}(K)}(\mathfrak{P}))z + pz^2};$$

thus for each prime $p$ it holds $\zeta_p(\mathcal{A}^2_{RM}, z) = \zeta_p(\mathcal{E}_{CM}, z)$. Leaving aside the bad primes, one derives the following important equality of the $L$-functions

$$L(\mathcal{A}^2_{RM}, s) \quad \equiv \quad L(\mathcal{E}_{CM}, s),$$

where $L(\mathcal{E}_{CM}, s)$ is the Hasse-Weil $L$-function of elliptic curve $\mathcal{E}_{CM}$. Case $n = 1$ of Theorem 6.4.1 becomes an implication of the following lemma.

**Lemma 4.4.1** *$L(\mathcal{E}_{CM}, s) \equiv L(\sigma_2, s)$, where $L(\sigma_2, s)$ the Artin $L$-function for an irreducible representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$.*

*Proof.* The Deuring theorem says that

$$L(\mathcal{E}_{CM}, s) = L(\psi_K, s)L(\overline{\psi}_K, s),$$

where $L(\psi_K, s)$ is the Hecke $L$-series attached to the Grössencharacter $\psi : \mathbb{A}^*_K \to \mathbb{C}^*$; here $\mathbb{A}^*_K$ denotes the adele ring of the field $K$ and the bar means a complex conjugation, see e.g. [Silverman 1994] [94], p.175. Because our elliptic curve has complex multiplication, the group $Gal\ (K|k)$ is abelian; one can apply the result of [Knapp 1997] [45], Theorem 5.1, which says that the Hecke $L$-series $L(\sigma_1 \circ \theta_{K|k}, s)$ equals the Artin $L$-function $L(\sigma_1, s)$, where $\psi_K = \sigma \circ \theta_{K|k}$ is the Grössencharacter and $\theta_{K|k} : \mathbb{A}^*_K \to Gal\ (K|k)$ the canonical homomorphism. Thus one gets

$$L(\mathcal{E}_{CM}, s) \equiv L(\sigma_1, s)L(\overline{\sigma}_1, s),$$

where $\overline{\sigma}_1 : Gal\ (K|k) \to \mathbb{C}$ means a (complex) conjugate representation of the Galois group. Consider the local factors of the Artin $L$-functions $L(\sigma_1, s)$ and $L(\overline{\sigma}_1, s)$; it is immediate, that they are $(1 - \sigma_1(Fr_p)p^{-s})^{-1}$ and $(1 - \overline{\sigma}_1(Fr_p)p^{-s})^{-1}$, respectively. Let us consider a representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$, such that

$$\sigma_2(Fr_p) = \begin{pmatrix} \sigma_1(Fr_p) & 0 \\ 0 & \overline{\sigma}_1(Fr_p) \end{pmatrix}.$$

It can be verified, that $det^{-1}(I_2 - \sigma_2(Fr_p)p^{-s}) = (1 - \sigma_1(Fr_p)p^{-s})^{-1}(1 - \overline{\sigma}_1(Fr_p)p^{-s})^{-1}$, i.e. $L(\sigma_2, s) = L(\sigma_1, s)L(\overline{\sigma}_1, s)$. Lemma 6.4.1 follows. □

From lemma 6.4.1 and $L(\mathcal{A}^2_{RM}, s) \equiv L(\mathcal{E}_{CM}, s)$, one gets

$$L(\mathcal{A}^2_{RM}, s) \equiv L(\sigma_2, s)$$

for an irreducible representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$. It remains to notice that $L(\sigma_2, s) = L(\sigma_2', s)$, where $\sigma_2' : Gal\ (K|\mathbb{Q}) \to GL_2(\mathbb{C})$, see e.g. [Artin 1924] [3], Section 3. Case $n = 1$ of Theorem 6.4.1 is proved □

**Case** $n = 0$

When $n = 0$, one gets a one-dimensional (degenerate) noncommutative torus; such an object, $\mathcal{A}_{\mathbb{Q}}$, can be obtained from the 2-dimensional torus $\mathcal{A}^2_\theta$ by forcing $\theta = p/q \in \mathbb{Q}$ be a rational number (hence our notation). One can always assume $\theta = 0$ and, thus

$$K_0^+(\mathcal{A}_{\mathbb{Q}}) \cong \mathbb{Z}.$$

The group of automorphisms of $\mathbb{Z}$-module $K_0^+(\mathcal{A}_{\mathbb{Q}}) \cong \mathbb{Z}$ is trivial, i.e. the multiplication by $\pm 1$; hence matrix $A$ corresponding to the shift automorphisms is either 1 or $-1$. Since $A$ must be positive, one gets $A = 1$. However, $A = 1$ is not a primitive; indeed, for any $N > 1$ matrix $A' = \zeta_N$ gives us $A = (A')^N$, where $\zeta_N = e^{\frac{2\pi i}{N}}$ is the $N$-th root of unity. Therefore, one gets

$$A = \zeta_N.$$

Since for the field $\mathbb{Q}$ it holds $\pi(n) = n$, one obtains $tr\ (A^{\pi(p)}) = tr\ (A^p) = \zeta_N^p$. A degenerate noncommutative torus, corresponding to the matrix $A = \zeta_N$, we shall write as $\mathcal{A}^N_{\mathbb{Q}}$.

Suppose that $Gal\ (K|\mathbb{Q})$ is abelian and let $\sigma : Gal\ (K|\mathbb{Q}) \to \mathbb{C}^\times$ be a homomorphism. By the *Artin reciprocity*, there exists an integer $N_\sigma$ and the Dirichlet character

$$\chi_\sigma : (\mathbb{Z}/N_\sigma\mathbb{Z})^\times \to \mathbb{C}^\times,$$

such that $\sigma(Fr_p) = \chi_\sigma(p)$, see e.g. [Gelbart 1984] [28]. On the other hand, it is verified directly, that $\zeta^p_{N_\sigma} = e^{\frac{2\pi i}{N_\sigma}p} = \chi_\sigma(p)$. Therefore **Char** $(A^p) = \chi_\sigma(p)x - 1$ and one gets

$$\zeta_p(\mathcal{A}^{N_\sigma}_{\mathbb{Q}}, z) = \frac{1}{1 - \chi_\sigma(p)z},$$

where $\chi_\sigma(p)$ is the Dirichlet character. Therefore, $L(\mathcal{A}_{\mathbb{Q}}^{N_\sigma}, s) \equiv L(s, \chi_\sigma)$ is the Dirichlet $L$-series; such a series, by construction, coincides with the Artin $L$-series of the representation $\sigma : Gal\ (K|\mathbb{Q}) \to \mathbb{C}^\times$. Case $n = 0$ of Theorem 6.4.1 is proved. □

### 4.4.3 Supplement: Artin $L$-function

The *Class Field Theory* (CFT) studies algebraic extensions of the number fields; the objective of CFT is a description of arithmetic of the extension $E$ in terms of arithmetic of the ground field $k$ and the Galois group $Gal\ (E|k)$ of the extension. Unless $Gal\ (E|k)$ is abelian, the CFT is out of reach so far; yet a series of conjectures called the *Langlands program* (LP) are designed to attain the goals of CFT. We refer the interested reader to [Gelbart 1984] [28] for an introduction to the CFT and LP; roughly speaking, the LP consists in an $n$-dimensional generalization of the Artin reciprocity based on the ideas and methods of representation theory of the locally compact Lie groups. The centerpiece of LP is the *Artin L-function* attached to representation $\sigma : Gal\ (E|k) \to GL_n(\mathbb{C})$ of the Galois group of $E$; we shall give a brief account of this $L$-function following the survey by [Gelbart 1984] [28].

The fundamental problem in algebraic number theory is to describe how an ordinary prime $p$ factors into prime ideals $\mathfrak{P}$ in the ring of integers of an arbitrary finite extensions $E$ of the rational field $\mathbb{Q}$. Let $O_E$ be the ring of integers of the extension $E$ and $pO_E$ a principal ideal; it is known that

$$pO_E = \prod \mathfrak{P}_i,$$

where $\mathfrak{P}_i$ are prime ideals of $O_E$. If $E$ is the Galois extension of $\mathbb{Q}$ and $Gal\ (E|\mathbb{Q})$ is the corresponding Galois group, then each automorphism $g \in Gal\ (E|\mathbb{Q})$ "moves around" the ideals $\mathfrak{P}_i$ in the prime decomposition of $p$ over $E$. An *isotropy subgroup* of $Gal\ (E|\mathbb{Q})$ (for given $p$) consists of the elements of $Gal\ (E|\mathbb{Q})$ which fix all the ideals $\mathfrak{P}_i$. For simplicity, we shall assume that $p$ is *unramified* in $E$, i.e. all $\mathfrak{P}_i$ are distinct; in this case the isotropy subgroup are *cyclic*. The (conjugacy class of) generator in the cyclic isotropy subgroup of $Gal\ (E|\mathbb{Q})$ corresponding to $p$ is called the *Frobenius element* and denoted by $Fr_p$. The element $Fr_p \in Gal\ (E|\mathbb{Q})$ describes completely the factorization of $p$ over $E$ and the major goal of the CFT is to express $Fr_p$ in terms of arithmetic of the ground field $\mathbb{Q}$. To handle this hard problem, it was suggested by E. Artin to consider the $n$-dimensional irreducible representations

$$\sigma_n : Gal\ (E|\mathbb{Q}) \longrightarrow GL_n(\mathbb{C}),$$

of the Galois group $Gal\ (E|\mathbb{Q})$, see [Artin 1924] [3]. The idea was to use the characteristic polynomial **Char** $(\sigma_n(Fr_p)) := \det(I_n - \sigma_n(Fr_p)z)$ of the matrix $\sigma_n(Fr_p)$; the polynomial is independent of the similarity class of $\sigma_n(Fr_p)$ in the group $GL_n(\mathbb{C})$ and provides an *intrinsic* description of the Frobenius element $Fr_p$.

**Definition 4.4.4** *By an Artin zeta function of representation* $\sigma_n$ *one understands the function*

$$\zeta_p(\sigma_n, z) := \frac{1}{\det(I_n - \sigma_n(Fr_p)z)}, \quad z \in \mathbb{C}.$$

*By an Artin L-function of representation* $\sigma_n$ *one understands the product*

$$L(\sigma_n, s) := \prod_p \zeta_p(\sigma_n, p^{-s}), \quad s \in \mathbb{C},$$

*over all but a finite set of primes* $p$.

**Remark 4.4.5 (Artin reciprocity)** If $n = 1$ and $Gal\ (E|\mathbb{Q}) \cong \mathbb{Z}/N\mathbb{Z}$ is abelian, then the calculation of the Artin $L$-function gives the equality

$$L(\chi, s) = \prod_p \frac{1}{1 - \chi(p)p^{-s}},$$

where $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}^{\times}$ is the Dirichlet character; the RHS of the equality is known as the *Dirichlet L-series* for $\chi$. Thus one gets a formula

$$\sigma(Fr_p) = \chi(p)$$

called the *Artin reciprocity law*; the formula generalizes many classical reciprocity results known for the particular values of $N$.

**Guide to the literature.** The Artin $L$-function first appeared in [Artin 1924] [3]. The origins of the Langlands program (and philosophy) can be found in his letter to André Weil, see [Langlands 1960's] [48]. An excellent introduction to the Langlands program has been written by [Gelbart 1984] [28]. The Langlands program for the even-dimensional noncommutative tori was the subject of [68].

## 4.5 Projective varieties over finite fields

In Section 5.3 we constructed a covariant functor

$$F : \textbf{Proj-Alg} \longrightarrow \textbf{C*-Serre}$$

from the category of complex projective varieties $V(\mathbb{C})$ to a category of the *Serre $C^*$-algebras* $\mathcal{A}_V$. Provided $V(\mathbb{C}) \cong V(K)$ for a number field $K \subset \mathbb{C}$, one can reduce variety $V(K)$ modulo a prime ideal $\mathfrak{P} \subset K$ over the prime product $q = p^r$; the reduction corresponds to a projective variety $V(\mathbb{F}_q)$ defined over the *finite field* $\mathbb{F}_q$. In this section we express the geometric invariant $|V(\mathbb{F}_q)|$ of $V(\mathbb{F}_q)$ (the number of points of variety $V(\mathbb{F}_q)$) in terms of the noncommutative invariants of the Serre $C^*$-algebra $\mathcal{A}_V$; the obtained formula shows interesting links to the *Weil Conjectures*, see e.g. [Hartshorne 1977] [35], Appendix C for an introduction. We test our formula on the concrete families of complex multiplication and rational elliptic curves.

### 4.5.1 Traces of Frobenius endomorphisms

The number of solutions of a system of polynomial equations over a finite field is an important invariant of the system and an old problem dating back to Gauss. Recall that if $\mathbb{F}_q$ is a field with $q = p^r$ elements and $V(\mathbb{F}_q)$ a smooth $n$-dimensional projective variety over $\mathbb{F}_q$, then one can define a zeta function $Z(V;t) := \exp\left(\sum_{r=1}^{\infty} |V(\mathbb{F}_{q^r})| \frac{t^r}{r}\right)$; the function is rational, i.e.

$$Z(V;t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)},$$

where $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$ and for each $1 \le i \le 2n-1$ the polynomial $P_i(t) \in \mathbb{Z}[t]$ can be written as $P_i(t) = \prod_{j=1}^{deg\ P_i(t)} (1 - \alpha_{ij} t)$ so that $\alpha_{ij}$ are algebraic integers with $|\alpha_{ij}| = q^{\frac{i}{2}}$, see e.g. [Hartshorne 1977] [35], pp. 454-457. The $P_i(t)$ can be viewed as characteristic polynomial of the Frobenius endomorphism $Fr_q^i$ of the $i$-th $\ell$-adic cohomology group $H^i(V)$; such an endomorphism is induced by the map acting on points of variety $V(\mathbb{F}_q)$ according to the formula $(a_1, \dots, a_n) \mapsto (a_1^q, \dots, a_n^q)$; we assume throughout the *Standard Conjectures*, see [Grothendieck 1968] [30]. If $V(\mathbb{F}_q)$ is defined by a system of polynomial equations, then the number of solutions of the system is given by the formula

$$|V(\mathbb{F}_q)| = \sum_{i=0}^{2n} (-1)^i \ tr\ (Fr_q^i),$$

where $tr$ is the trace of Frobenius endomorphism, see [Hartshorne 1977] [35], *loc. cit.*

Let $V(K)$ be a complex projective variety defined over an algebraic number field $K \subset \mathbb{C}$; suppose that projective variety $V(\mathbb{F}_q)$ is the reduction of $V(K)$ modulo the prime ideal $\mathfrak{P} \subset K$ corresponding to $q = p^r$. Denote by $\mathcal{A}_V$ the *Serre $C^*$-algebra* of projective variety $V(K)$, see Section 5.3.1. Consider the stable $C^*$-algebra of $\mathcal{A}_V$, i.e. the $C^*$-algebra $\mathcal{A}_V \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators on $\mathcal{H}$. Let $\tau : \mathcal{A}_V \otimes \mathcal{K} \to \mathbb{R}$ be the unique normalized trace (tracial state) on $\mathcal{A}_V \otimes \mathcal{K}$, i.e. a positive linear functional of norm 1 such that $\tau(yx) = \tau(xy)$ for all $x, y \in \mathcal{A}_V \otimes \mathcal{K}$, see [Blackadar 1986] [10], p. 31. Recall that $\mathcal{A}_V$ is the crossed product $C^*$-algebra of the form $\mathcal{A}_V \cong C(V) \rtimes \mathbb{Z}$, where $C(V)$ is the commutative $C^*$-algebra of complex valued functions on $V$ and the product is taken by an automorphism of algebra $C(V)$ induced by the map $\sigma : V \to V$, see Lemma 5.3.2. From the Pimsner-Voiculescu six term exact sequence for crossed products, one gets the short exact sequence of algebraic $K$-groups

$$0 \to K_0(C(V)) \overset{i_*}{\to} K_0(\mathcal{A}_V) \to K_1(C(V)) \to 0,$$

where map $i_*$ is induced by an embedding of $C(V)$ into $\mathcal{A}_V$, see [Blackadar 1986] [10], p. 83 for the details. We have $K_0(C(V)) \cong K^0(V)$ and $K_1(C(V)) \cong K^{-1}(V)$, where $K^0$ and $K^{-1}$ are the topological $K$-groups of variety $V$, see [Blackadar 1986] [10], p. 80. By the Chern character formula, one gets

$$\left\{ \begin{array}{lcl} K^0(V) \otimes \mathbb{Q} & \cong & H^{even}(V; \mathbb{Q}) \\ K^{-1}(V) \otimes \mathbb{Q} & \cong & H^{odd}(V; \mathbb{Q}), \end{array} \right.$$

where $H^{even}$ ($H^{odd}$) is the direct sum of even (odd, resp.) cohomology groups of $V$.

**Remark 4.5.1** It is known, that $K_0(\mathcal{A}_V \otimes \mathcal{K}) \cong K_0(\mathcal{A}_V)$ because of stability of the $K_0$-group with respect to tensor products by the algebra $\mathcal{K}$, see e.g. [Blackadar 1986] [10], p. 32.

Thus one gets the commutative diagram shown in Fig. 6.6, where $\tau_*$ denotes a homomorphism induced on $K_0$ by the canonical trace $\tau$ on the $C^*$-algebra $\mathcal{A}_V \otimes \mathcal{K}$.

$$\begin{array}{ccccc} H^{even}(V)\otimes\mathbb{Q} & \xrightarrow{i_*} & K_0(\mathcal{A}_V\otimes\mathcal{K})\otimes\mathbb{Q} & \longrightarrow & H^{odd}(V)\otimes\mathbb{Q} \\ & \searrow & \downarrow \tau_* & \swarrow & \\ & & \mathbb{R} & & \end{array}$$

Figure 4.6: $K$-theory of the Serre $C^*$-algebra $\mathcal{A}_V$.

Because $H^{even}(V) := \oplus_{i=0}^{n} H^{2i}(V)$ and $H^{odd}(V) := \oplus_{i=1}^{n} H^{2i-1}(V)$, one gets for each $0 \le i \le 2n$ an injective homomorphism

$$H^i(V) \to \mathbb{R}$$

and we shall denote by $\Lambda_i$ an additive abelian subgroup of real numbers defined by the homomorphism.

**Remark 4.5.2** The $\Lambda_i$ is called a *pseudo-lattice* [Manin 2004] [52], Section 1.

Recall that endomorphisms of a pseudo-lattice are given as multiplication of points of $\Lambda_i$ by the real numbers $\alpha$ such that $\alpha\Lambda_i \subseteq \Lambda_i$. It is known that $End\ (\Lambda_i) \cong \mathbb{Z}$ or $End\ (\Lambda_i) \otimes \mathbb{Q}$ is a real algebraic number field such that $\Lambda_i \subset End\ (\Lambda_i) \otimes \mathbb{Q}$, see e.g. [Manin 2004] [52], Lemma 1.1.1 for the case of quadratic fields. We shall write $\varepsilon_i$ to denote the unit of the order in the field $K_i := End\ (\Lambda_i) \otimes \mathbb{Q}$, which induces the shift automorphism of $\Lambda_i$, see [Effros 1981] [21], p. 38 for the details and terminology. Let $p$ be a "good prime" in the reduction $V(\mathbb{F}_q)$ of complex projective variety $V(K)$ modulo a prime ideal over $q = p^r$. Consider a sub-lattice $\Lambda_i^q$ of $\Lambda_i$ of the index $q$; by an index of the sub-lattice we understand its index as an abelian subgroup of $\Lambda_i$. We shall write $\pi_i(q)$ to denote an integer, such that multiplication by $\varepsilon_i^{\pi_i(q)}$ induces the shift automorphism of $\Lambda_i^q$. The trace of an algebraic number will be written as $tr\ (\bullet)$. The following result relates invariants $\varepsilon_i$ and $\pi_i(q)$ of the $C^*$-algebra $\mathcal{A}_V$ to the cardinality of the set $V(\mathbb{F}_q)$.

**Theorem 4.5.1 (Noncommutative invariant of projective varieties over finite fields)**

$$|V(\mathbb{F}_q)| = \sum_{i=0}^{2n} (-1)^i\ tr\ \left(\varepsilon_i^{\pi_i(q)}\right)$$

.

### 4.5.2 Proof of Theorem 6.5.1

**Lemma 4.5.1** *There exists a symplectic unitary matrix $\Theta_q^i \in Sp\ (deg\ P_i;\ \mathbb{R})$, such that*

$$Fr_q^i = q^{\frac{i}{2}}\Theta_q^i.$$

*Proof.* Recall that the eigenvalues of $Fr_q^i$ have absolute value $q^{\frac{i}{2}}$; they come in the complex conjugate pairs. On the other hand, symplectic unitary matrices in group $Sp\ (deg\ P_i;\ \mathbb{R})$ are known to have eigenvalues of absolute value 1 coming in complex conjugate pairs. Since the spectrum of a matrix defines the similarity class of matrix, one can write the characteristic polynomial of $Fr_q^i$ in the form

$$P_i(t) = det\ (I - q^{\frac{i}{2}}\Theta_q^i t),$$

where matrix $\Theta_q^i \in Sp\ (deg\ P_i;\ \mathbb{Z})$ and its eigenvalues have absolute value 1. It remains to compare the above equation with the formula

$$P_i(t) = det\ (I - Fr_q^i t),$$

i.e. $Fr_q^i = q^{\frac{i}{2}}\Theta_q^i$. Lemma 6.5.1 follows. □

**Lemma 4.5.2** *Using a symplectic transformation one can bring matrix $\Theta_q^i$ to the block form*

$$\Theta_q^i = \begin{pmatrix} A & I \\ -I & 0 \end{pmatrix},$$

*where $A$ is a positive symmetric and $I$ the identity matrix.*

*Proof.* Let us write $\Theta_q^i$ in the block form

$$\Theta_q^i = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where matrices $A, B, C, D$ are invertible and their transpose $A^T, B^T, C^T, D^T$ satisfy the symplectic equations

$$\begin{cases} A^T D - C^T B &= I, \\ A^T C - C^T A &= 0, \\ B^T D - D^T B &= 0. \end{cases}$$

Recall that symplectic matrices correspond to the linear fractional transformations $\tau \mapsto \frac{A\tau+B}{C\tau+D}$ of the Siegel half-space $\mathbb{H}_n = \{\tau = (\tau_j) \in \mathbb{C}^{\frac{n(n+1)}{2}} \mid \Im(\tau_j) >$

$0\}$ consisting of symmetric $n \times n$ matrices, see e.g. [Mumford 1983] [58], p. 173. One can always multiply the nominator and denominator of such a transformation by $B^{-1}$ without affecting the transformation; thus with no loss of generality, we can assume that $B = I$. We shall consider the symplectic matrix $T$ and its inverse $T^{-1}$ given by the formulas

$$T = \begin{pmatrix} I & 0 \\ D & I \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix}.$$

It is verified directly, that

$$T^{-1}\Theta_q^i T = \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix} \begin{pmatrix} A & I \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D & I \end{pmatrix} = \begin{pmatrix} A+D & I \\ C-DA & 0 \end{pmatrix}.$$

The system of symplectic equations with $B = I$ implies the following two equations

$$A^T D - C^T = I \quad \text{and} \quad D = D^T.$$

Applying transposition to the both parts of the first equation of the above equations, one gets $(A^T D - C^T)^T = I^T$ and, therefore, $D^T A - C = I$. But the second equation says that $D^T = D$; thus one arrives at the equation $DA - C = I$. The latter gives us $C - DA = -I$, which we substitute in the above equations and get (in a new notation) the conclusion of Lemma 6.5.2. Finally, the middle of the symplectic equations with $C = -I$ implies $A = A^T$, i.e. $A$ is a symmetric matrix. Since the eigenvalues of symmetric matrix are always real and in view of $tr\ (A) > 0$ (because $tr\ (Fr_q^i) > 0$), one concludes that $A$ is similar to a positive matrix, see e.g. [Handelman 1981] [32], Theorem 1. Lemma 6.5.2 follows. $\square$

**Lemma 4.5.3** *The symplectic unitary transformation $\Theta_q^i$ of $H^i(V;\mathbb{Z})$ descends to an automorphism of $\Lambda_i$ given by the matrix*

$$M_q^i = \begin{pmatrix} A & I \\ I & 0 \end{pmatrix}.$$

**Remark 4.5.3** In other words, Lemma 6.5.3 says that functor $F$ : **Proj-Alg** $\to$ **C*-Serre** acts between matrices $\Theta_q^i$ and $M_q^i$ according to the formula

$$F : \begin{pmatrix} A & I \\ -I & 0 \end{pmatrix} \longmapsto \begin{pmatrix} A & I \\ I & 0 \end{pmatrix}.$$

*Proof.* Since $\Lambda_i \subset K_i$ there exists a basis of $\Lambda_i$ consisting of algebraic numbers; denote by $(\mu_1, \dots, \mu_k;\ \nu_1, \dots, \nu_k)$ a basis of $\Lambda_i$ consisting of positive algebraic numbers $\mu_i > 0$ and $\nu_i > 0$. Using the injective homomorphism $\tau_*$, one can descend $\Theta_q^i$ to an automorphism of $\Lambda_i$ so that

$$\begin{pmatrix} \mu' \\ \nu' \end{pmatrix} = \begin{pmatrix} A & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} A\mu + \nu \\ -\mu \end{pmatrix},$$

where $\mu = (\mu_1, \dots, \mu_k)$ and $\nu = (\nu_1, \dots, \nu_k)$. Because vectors $\mu$ and $\nu$ consist of positive entries and $A$ is a positive matrix, it is immediate that $\mu' = A\mu + \nu > 0$ while $\nu' = -\mu < 0$.

**Remark 4.5.4** All automorphisms in the (Markov) category of pseudo-lattices come from multiplication of the basis vector $(\mu_1, \dots, \mu_k;\ \nu_1, \dots, \nu_k)$ of $\Lambda_i$ by an algebraic unit $\lambda > 0$ of field $K_i$; in particular, any such an automorphism must be given by a non-negative matrix, whose Perron-Frobenius eigenvalue coincides with $\lambda$. Thus for any automorphism of $\Lambda_i$ it must hold $\mu' > 0$ and $\nu' > 0$.

In view of the above, we shall consider an automorphism of $\Lambda_i$ given by matrix $M_q^i = (A, I, I, 0)$; clearly, for $M_q^i$ it holds $\mu' = A\mu + \nu > 0$ and $\nu' = \mu > 0$. Therefore $M_q^i$ is a non-negative matrix satisfying the necessary condition to belong to the Markov category. It is also a sufficient one, because the similarity class of $M_q^i$ contains a representative whose Perron-Frobenius eigenvector can be taken for a basis $(\mu, \nu)$ of $\Lambda_i$. This argument finishes the proof of Lemma 6.5.3. $\square$

**Corollary 4.5.1** *tr* $(M_q^i) = tr\ (\Theta_q^i)$.

*Proof.* This fact is an implication of the above formulas and a direct computation $tr\ (M_q^i) = tr\ (A) = tr\ (\Theta_q^i)$. $\square$

**Definition 4.5.1** *We shall call* $q^{\frac{i}{2}} M_q^i$ *a Markov endomorphism of* $\Lambda_i$ *and denote it by* $Mk_q^i$.

**Lemma 4.5.4** *tr* $(Mk_q^i) = tr\ (Fr_q^i)$.

*Proof.* Corollary 6.5.1 says that $tr\ (M_q^i) = tr\ (\Theta_q^i)$, and therefore

$$\begin{aligned} tr\ (Mk_q^i) &= tr\ (q^{\frac{i}{2}} M_q^i) = q^{\frac{i}{2}}\ tr\ (M_q^i) = \\ &= q^{\frac{i}{2}}\ tr\ (\Theta_q^i) = tr\ (q^{\frac{i}{2}} \Theta_q^i) = tr\ (Fr_q^i). \end{aligned}$$

In words, Frobenius and Markov endomorphisms have the same trace, i.e. $tr\ (Mk_q^i) = tr\ (Fr_q^i)$. Lemma 6.5.4 follows. $\square$

**Remark 4.5.5** Notice that, unless $i$ or $r$ are even, neither $\Theta_q^i$ nor $M_q^i$ are integer matrices; yet $Fr_q^i$ and $Mk_q^i$ are always integer matrices.

**Lemma 4.5.5** *There exists an algebraic unit $\omega_i \in K_i$ such that:*

*(i) $\omega_i$ corresponds to the shift automorphism of an index $q$ sub-lattice of pseudo-lattice $\Lambda_i$;*

*(ii) $tr\ (\omega_i) = tr\ (Mk_q^i)$.*

*Proof.* (i) To prove Lemma 6.5.5, we shall use the notion of a stationary dimension group and the corresponding shift automorphism; we refer the reader to [Effros 1981] [21], p. 37 and [Handelman 1981] [32], p.57 for the notation and details on stationary dimension groups and a survey of [Wagoner 1999] [106] for the general theory of subshifts of finite type. Consider a stationary dimension group, $G(Mk_q^i)$, generated by the Markov endomorphism $Mk_q^i$

$$\mathbb{Z}^{b_i} \overset{Mk_q^i}{\to} \mathbb{Z}^{b_i} \overset{Mk_q^i}{\to} \mathbb{Z}^{b_i} \overset{Mk_q^i}{\to} \dots,$$

where $b_i = deg\ P_i(t)$. Let $\lambda_M$ be the Perron-Frobenius eigenvalue of matrix $M_q^i$. It is known, that $G(Mk_q^i)$ is order-isomorphic to a dense additive abelian subgroup $\mathbb{Z}[\frac{1}{\lambda_M}]$ of $\mathbb{R}$; here $\mathbb{Z}[x]$ is the set of all polynomials in one variable with the integer coefficients. Let $\widehat{Mk_q^i}$ be a shift automorphism of $G(Mk_q^i)$ [Effros 1981] [21], p. 37. To calculate the automorphism, notice that multiplication of $\mathbb{Z}[\frac{1}{\lambda_M}]$ by $\lambda_M$ induces an automorphism of dimension group $\mathbb{Z}[\frac{1}{\lambda_M}]$. Since the determinant of matrix $M_q^i$ (i.e. the degree of Markov endomorphism) is equal to $q^n$, one concludes that such an automorphism corresponds to a unit of the endomorphism ring of a sub-lattice of $\Lambda_i$ of index $q^n$. We shall denote such a unit by $\omega_i$. Clearly, $\omega_i$ generates the required shift automorphism $\widehat{Mk_q^i}$ through multiplication of dimension group $\mathbb{Z}[\frac{1}{\lambda_M}]$ by the algebraic number $\omega_i$. Item (i) of Lemma 6.5.5 follows.

(ii) Consider the Artin-Mazur zeta function of $Mk_q^i$

$$\zeta_{Mk_q^i}(t) = \exp\left(\sum_{k=1}^{\infty} \frac{tr\ \left[(Mk_q^i)^k\right]}{k} t^k\right)$$

and such of $\widehat{Mk_q^i}$

$$\zeta_{\widehat{Mk_q^i}}(t) = \exp\left(\sum_{k=1}^{\infty} \frac{tr\ \left[(\widehat{Mk_q^i})^k\right]}{k} t^k\right).$$

Since $Mk_q^i$ and $\widehat{Mk_q^i}$ are shift equivalent matrices, one concludes that $\zeta_{Mk_q^i}(t) \equiv \zeta_{\widehat{Mk_q^i}}(t)$, see [Wagoner 1999] [106], p. 273. In particular,

$$tr\ (Mk_q^i) = tr\ (\widehat{Mk_q^i}).$$

But $tr\ (\widehat{Mk_q^i}) = tr\ (\omega_i)$, where on the right hand side is the trace of an algebraic number. In view of the above, one gets the conclusion of item (ii) of Lemma 6.5.5. $\square$

**Lemma 4.5.6** *There exists a positive integer $\pi_i(q)$, such that*

$$\omega_i = \varepsilon_i^{\pi_i(q)},$$

*where $\varepsilon_i \in End\ (\Lambda_i)$ is the fundamental unit corresponding to the shift automorphism of pseudo-lattice $\Lambda_i$.*

*Proof.* Given an automorphism $\omega_i$ of a finite-index sub-lattice of $\Lambda_i$ one can extend $\omega_i$ to an automorphism of entire $\Lambda_i$, since $\omega_i\Lambda_i = \Lambda_i$. Therefore each unit of (endomorphism ring of) a sub-lattice is also a unit of the host pseudo-lattice. Notice that the converse statement is false in general. On the other hand, by virtue of the Dirichlet Unit Theorem each unit of $End\ (\Lambda_i)$ is a product of a finite number of (powers of) fundamental units of $End\ (\Lambda_i)$. We shall denote by $\pi_i(q)$ the least positive integer, such that $\varepsilon_i^{\pi_i(q)}$ is the shift automorphism of a sub-lattice of index $q$ of pseudo-lattice $\Lambda_i$. The number $\pi_i(q)$ exists and uniquely defined, albeit no general formula for its calculation is known, see Remark 6.5.6. It is clear from construction, that $\pi_i(q)$ satisfies the claim of Lemma 6.5.6. $\square$

**Remark 4.5.6** No general formula for the number $\pi_i(q)$ as a function of $q$ is known; however, if the rank of $\Lambda_i$ is two (i.e. $n = 1$), then there are classical results recorded in e.g. [Hasse 1950] [37], p.298; see also Section 5.5.3.

Theorem 6.5.1 follows from Lemmas 6.5.4-6.5.6 and the known formula $|V(\mathbb{F}_q)| = \sum_{i=0}^{2n}(-1)^i\ tr\ (Fr_q^i)$. $\square$

### 4.5.3 Examples

Let $V(\mathbb{C}) \cong \mathcal{E}_\tau$ be an elliptic curve; it is well known that its Serre $C^*$-algebra $\mathcal{A}_{\mathcal{E}_\tau}$ is isomorphic to the noncommutative torus $\mathcal{A}_\theta$ with the unit scaled by a constant $0 < \log\mu < \infty$. Furthermore, $K_0(\mathcal{A}_\theta) \cong K_1(\mathcal{A}_\theta) \cong \mathbb{Z}^2$ and the canonical trace $\tau$ on $\mathcal{A}_\theta$ gives us the following formula

$$\tau_*(K_0(\mathcal{A}_{\mathcal{E}_\tau} \otimes \mathcal{K})) = \mu(\mathbb{Z} + \mathbb{Z}\theta).$$

Because $H^0(\mathcal{E}_\tau;\mathbb{Z}) = H^2(\mathcal{E}_\tau;\mathbb{Z}) \cong \mathbb{Z}$ while $H^1(\mathcal{E}_\tau;\mathbb{Z}) \cong \mathbb{Z}^2$, one gets the following pseudo-lattices

$$\Lambda_0 = \Lambda_2 \cong \mathbb{Z} \quad \text{and} \quad \Lambda_1 \cong \mu(\mathbb{Z} + \mathbb{Z}\theta).$$

For the sake of simplicity, we shall focus on the following families of elliptic curves.

#### Complex multiplication

Suppose that $\mathcal{E}_\tau$ has complex multiplication; recall that such a curve was denoted by $\mathcal{E}_{CM}^{(-D,f)}$, i.e. the endomorphism ring of $\mathcal{E}_\tau$ is an order of conductor $f \geq 1$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. By the results of Section 5.1 on elliptic curve $\mathcal{E}_{CM}^{(-D,f)}$, the formulas for $\Lambda_i$ are as follows

$$\Lambda_0 = \Lambda_2 \cong \mathbb{Z} \quad \text{and} \quad \Lambda_1 = \varepsilon[\mathbb{Z} + (f\omega)\mathbb{Z}],$$

where $\omega = \frac{1}{2}(1+\sqrt{D})$ if $D \equiv 1 \, mod \, 4$ and $D \neq 1$ or $\omega = \sqrt{D}$ if $D \equiv 2, 3 \, mod \, 4$ and $\varepsilon > 1$ is the fundamental unit of order $\mathbb{Z} + (f\omega)\mathbb{Z}$.

**Remark 4.5.7** The reader can verify, that $\Lambda_1 \subset K_1$, where $K_1 \cong \mathbb{Q}(\sqrt{D})$.

Let $p$ be a good prime. Consider a localization $\mathcal{E}(\mathbb{F}_p)$ of curve $\mathcal{E}_{CM}^{(-D,f)} \cong \mathcal{E}(K)$ at the prime ideal $\mathfrak{P}$ over $p$. It is well known, that the Frobenius endomorphism of elliptic curve with complex multiplication is defined by the Grössencharacter; the latter is a complex number $\alpha_{\mathfrak{P}} \in \mathbb{Q}(\sqrt{-D})$ of absolute value $\sqrt{p}$. Moreover, multiplication of the lattice $L_{CM} = \mathbb{Z}+\mathbb{Z}\tau$ by $\alpha_{\mathfrak{P}}$ induces the Frobenius endomorphism $Fr_p^1$ on $H^1(\mathcal{E}(K);\mathbb{Z})$, see e.g. [Silverman 1994] [94], p. 174. Thus one arrives at the following matrix form for the Frobenius

& Markov endomorphisms and the shift automorphism, respectively:

$$\begin{cases} Fr_p^1 & = \begin{pmatrix} tr\ (\alpha_{\mathfrak{P}}) & p \\ -1 & 0 \end{pmatrix}, \\ Mk_p^1 & = \begin{pmatrix} tr\ (\alpha_{\mathfrak{P}}) & p \\ 1 & 0 \end{pmatrix}, \\ \widehat{Mk_p^1} & = \begin{pmatrix} tr\ (\alpha_{\mathfrak{P}}) & 1 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

To calculate positive integer $\pi_1(p)$ appearing in Theorem **??**, denote by $\left(\frac{D}{p}\right)$ the Legendre symbol of $D$ and $p$. A classical result of the theory of real quadratic fields asserts that $\pi_1(p)$ must be one of the divisors of the integer number

$$p - \left(\frac{D}{p}\right),$$

see e.g. [Hasse 1950] [37], p. 298. Thus the trace of Frobenius endomorphism on $H^1(\mathcal{E}(K);\mathbb{Z})$ is given by the formula

$$tr\ (\alpha_{\mathfrak{P}}) = tr\ (\varepsilon^{\pi_1(p)}).$$

The right hand side of the above equation can be further simplified, since

$$tr\ (\varepsilon^{\pi_1(p)}) = 2T_{\pi_1(p)}\left[\ \frac{1}{2}\ tr\ (\varepsilon)\right],$$

where $T_{\pi_1(p)}(x)$ is the Chebyshev polynomial (of the first kind) of degree $\pi_1(p)$. Thus one obtains a formula for the number of (projective) solutions of a cubic equation over field $\mathbb{F}_p$ in terms of invariants of pseudo-lattice $\Lambda_1$

$$|\mathcal{E}(\mathbb{F}_p)| = 1 + p - 2T_{\pi_1(p)}\left[\ \frac{1}{2}\ tr\ (\varepsilon)\right].$$

### Rational elliptic curve

Let $b \geq 3$ be an integer and consider a rational elliptic curve $\mathcal{E}(\mathbb{Q}) \subset \mathbb{C}P^2$ given by the homogeneous Legendre equation

$$y^2 z = x(x-z)\left(x - \frac{b-2}{b+2}z\right).$$

The Serre $C^*$-algebra of projective variety $V \cong \mathcal{E}(\mathbb{Q})$ is isomorphic (modulo an ideal) to the Cuntz-Krieger algebra $\mathcal{O}_B$, where

$$B = \begin{pmatrix} b-1 & 1 \\ b-2 & 1 \end{pmatrix},$$

see [78]. Recall that $\mathcal{O}_B \otimes \mathcal{K}$ is the crossed product $C^*$-algebra of a stationary AF $C^*$-algebra by its shift automorphism, see [Blackadar 1986] [10], p. 104; the AF $C^*$-algebra has the following dimension group

$$\mathbb{Z}^2 \xrightarrow{B^T} \mathbb{Z}^2 \xrightarrow{B^T} \mathbb{Z}^2 \xrightarrow{B^T} \dots,$$

where $B^T$ is the transpose of matrix $B$. Because $\mu$ must be a positive eigenvalue of matrix $B^T$, one gets

$$\mu = \frac{2-b+\sqrt{b^2-4}}{2}.$$

Likewise, since $\theta$ must be the corresponding positive eigenvector $(1,\theta)$ of the same matrix, one gets

$$\theta = \frac{1}{2}\left(\sqrt{\frac{b+2}{b-2}} - 1\right).$$

Therefore, pseudo-lattices $\Lambda_i$ are $\Lambda_0 = \Lambda_2 \cong \mathbb{Z}$ and

$$\Lambda_1 \cong \frac{2-b+\sqrt{b^2-4}}{2}\left[\mathbb{Z} + \frac{1}{2}\left(\sqrt{\frac{b+2}{b-2}} - 1\right)\mathbb{Z}\right].$$

**Remark 4.5.8** The pseudo-lattice $\Lambda_1 \subset K_1$, where $K_1 = \mathbb{Q}(\sqrt{b^2-4})$.

Let $p$ be a good prime and let $\mathcal{E}(\mathbb{F}_p)$ be the reduction of our rational elliptic curve modulo $p$. It follows from Section 6.3.3, that $\pi_1(p)$ as one of the divisors of integer number

$$p - \left(\frac{b^2-4}{p}\right).$$

Unlike the case of complex multiplication, the Grössencharacter is no longer available for $\mathcal{E}(\mathbb{Q})$; yet the trace of Frobenius endomorphism can be computed using Theorem 6.5.1, i.e.

$$tr\ (Fr_p^1) = tr\ \left[(B^T)^{\pi_1(p)}\right].$$

Using the Chebyshev polynomials, one can write the last equation in the form

$$tr\ (Fr_p^1) = 2T_{\pi_1(p)}\left[\frac{1}{2}\ tr\ (B^T)\right].$$

Since $tr\ (B^T) = b$, one gets

$$tr\ (Fr_p^1) = 2T_{\pi_1(p)}\left(\frac{b}{2}\right).$$

Thus one obtains a formula for the number of solutions of equation $y^2z = x(x-z)\left(x - \frac{b-2}{b+2}z\right)$ over field $\mathbb{F}_p$ in terms of the *noncommutative invariants* of pseudo-lattice $\Lambda_1$ of the form

$$|\mathcal{E}(\mathbb{F}_p)| = 1 + p - 2T_{\pi_1(p)}\left(\frac{b}{2}\right).$$

We shall conclude by a concrete example comparing the obtained formula with the known results for rational elliptic curves in the Legendre form, see e.g. [Hartshorne 1977] [35], p. 333 and [Kirwan 1992] [43], pp. 49-50.

**Example 4.5.1 (Comparison to classical invariants)** Suppose that $b \equiv 2 \ mod\ 4$. Recall that the $j$-invariant takes the same value on $\lambda$, $1-\lambda$ and $\frac{1}{\lambda}$, see e.g. [Hartshorne 1977] [35], p. 320. Therefore, one can bring equation $y^2z = x(x-z)\left(x - \frac{b-2}{b+2}z\right)$ to the form

$$y^2z = x(x-z)(x-\lambda z),$$

where $\lambda = \frac{1}{4}(b+2) \in \{2, 3, 4, \dots\}$. Notice that for the above curve

$$tr\ (B^T) = b = 2(2\lambda - 1).$$

To calculate $tr\ (Fr_p^1)$ for our elliptic curve, recall that in view of last equality, one gets

$$tr\ (Fr_p^1) = 2\ T_{\pi_1(p)}(2\lambda - 1).$$

It will be useful to express Chebyshev polynomial $T_{\pi_1(p)}(2\lambda-1)$ in terms of the hypergeometric function ${}_2F_1(a, b; c; z)$; the standard formula brings our last equation to the form

$$tr\ (Fr_p^1) = 2\ {}_2F_1(-\pi_1(p),\ \pi_1(p);\ \frac{1}{2};\ 1-\lambda).$$

We leave to the reader to prove the identity

$$2\ {}_2F_1(-\pi_1(p),\ \pi_1(p);\ \tfrac{1}{2};\ 1-\lambda) =$$

$$= (-1)^{\pi_1(p)}\ {}_2F_1(\pi_1(p)+1,\ \pi_1(p)+1;\ 1;\ \lambda).$$

In the last formula

$${}_2F_1(\pi_1(p)+1,\ \pi_1(p)+1;\ 1;\ \lambda) = \sum_{r=0}^{\pi_1(p)} \binom{\pi_1(p)}{r}^2 \lambda^r,$$

see [Carlitz 1966] [13], p. 328. Recall that $\pi_1(p)$ is a divisor of $p - \left(\frac{b^2-4}{p}\right)$, which in our case takes the value $\frac{p-1}{2}$. Bringing together the above formulas, one gets

$$|\mathcal{E}(\mathbb{F}_p)| = 1 + p + (-1)^{\frac{p-1}{2}} \sum_{r=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{r}^2 \lambda^r.$$

The reader is encouraged to compare the obtained formula with the classical result in [Hartshorne 1977] [35], p. 333 and [Kirwan 1992] [43], pp. 49-50; notice also an intriguing relation with the *Hasse invariant*.

**Guide to the literature.** The *Weil Conjectures* (WC) were formulated in [Weil 1949] [109]; along with the Langalands Program, the WC shaped the modern look of number theory. The theory of *motives* was elaborated by [Grothendieck 1968] [30] to solve the WC. An excellent introduction to the WC can be found in [Hartshorne 1977] [35], Appendix C. The related noncommutative invariants were calculated in [70].

## 4.6 Transcendental number theory

The functor

$$F : \mathbf{Ell} \longrightarrow \mathbf{NC\text{-}Tor}$$

constructed in Section 5.1 has an amazing application in the *transcendental number theory*, see e.g. [Baker 1975] [5] for an introduction. Namely, we shall use the formula $F(\mathcal{E}_{CM}^{(-D,f)}) = \mathcal{A}_{RM}^{(D,f)}$ obtained in Section 6.1 to prove that the transcendental function

$$\mathcal{J}(\theta,\varepsilon) := e^{2\pi i\theta+\log\log\varepsilon}$$

takes algebraic values for the algebraic arguments $\theta$ and $\varepsilon$. Moreover, these values of $\mathcal{J}(\theta,\varepsilon)$ belong to the Hilbert class field of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ for all but a finite set of values of $D$.

### 4.6.1 Algebraic values of transcendental functions

Recall that an old and difficult problem of number theory is to determine if given irrational value of a transcendental function is algebraic or transcendental for certain algebraic arguments; the algebraic values are particularly remarkable and worthy of thorough investigation, see [Hilbert 1902] [40], p. 456. Only few general results are known, see e.g. [Baker 1975] [5]. We shall mention the famous Gelfond-Schneider Theorem saying that $e^{\beta\log\alpha}$ is a transcendental number, whenever $\alpha\not\in\{0,1\}$ is an algebraic and $\beta$ an irrational algebraic number. In contrast, Klein's invariant $j(\tau)$ is known to take algebraic values whenever $\tau\in\mathbb{H}:=\{x+iy\in\mathbb{C}\mid y>0\}$ is an imaginary quadratic number. In follows we shall focus on algebraic values of the transcendental function

$$\mathcal{J}(\theta,\varepsilon) := \{e^{2\pi i\theta+\log\log\varepsilon} \mid -\infty<\theta<\infty,\ 1<\varepsilon<\infty\}$$

for real arguments $\theta$ and $\varepsilon$.

**Remark 4.6.1** The $\mathcal{J}(\theta,\varepsilon)$ can be viewed as an extension of Klein's invariant $j(\tau)$ to the boundary of half-plane $\mathbb{H}$; hence our notation.

Let $K=\mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of class number $h\geq 1$; let $\{\mathcal{E}_1,\dots\mathcal{E}_h\}$ be pairwise non-isomorphic elliptic curves with complex multiplication by the ring of integers of field $K$. For $1\leq i\leq h$ we shall write $\mathcal{A}_{\theta_i}=F(\mathcal{E}_i)$ to denote the noncommutative torus, where $F:$ **Ell** $\to$ **NC-Tor** is the functor defined in Section 5.1. It follows from Theorem 6.1.2 that each $\theta_i$ is a quadratic irrationality of the field $\mathbb{Q}(\sqrt{D})$. We shall write $\overline{(a_1^{(i)},\dots,a_n^{(i)})}$ to denote the period of continued fraction for $\theta_i$ and for each $\theta_i$ we shall consider the matrix

$$A_i=\begin{pmatrix} a_1^{(i)} & 1\\ 1 & 0\end{pmatrix}\dots\begin{pmatrix} a_n^{(i)} & 1\\ 1 & 0\end{pmatrix}.$$

**Remark 4.6.2** In other words, matrix $A_i$ corresponds to the *shift automorphism* $\sigma_{A_i}$ of the dimension group $K_0^+(\mathcal{A}_{\theta_i}) \cong \mathbb{Z} + \mathbb{Z}\theta_i$, see Section 3.5.2.

Let $\varepsilon_i > 1$ be the Perron-Frobenius eigenvalue of matrix $A_i$; it is easy to see, that $\varepsilon_i$ is a quadratic irrationality of the field $\mathbb{Q}(\sqrt{D})$.

**Theorem 4.6.1** *If* $D \notin \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ *is a square-free positive integer, then* $\{\mathcal{J}(\theta_i, \varepsilon_i) \mid 1 \le i \le h\}$ *are conjugate algebraic numbers. Moreover, such numbers are generators of the Hilbert class field of imaginary quadratic field* $\mathbb{Q}(\sqrt{-D})$.

## 4.6.2 Proof of Theorem 6.6.1

The idea of proof is remarkably simple; indeed, recall that the system of defining relations

$$\begin{cases} x_3x_1 &= e^{2\pi i\theta}x_1x_3,\\ x_4x_2 &= e^{2\pi i\theta}x_2x_4,\\ x_4x_1 &= e^{-2\pi i\theta}x_1x_4,\\ x_3x_2 &= e^{-2\pi i\theta}x_2x_3,\\ x_2x_1 &= x_1x_2 = e,\\ x_4x_3 &= x_3x_4 = e, \end{cases}$$

for the noncommutative torus $\mathcal{A}_\theta$ and defining relations

$$\begin{cases} x_3x_1 &= \mu e^{2\pi i\theta}x_1x_3,\\ x_4x_2 &= \frac{1}{\mu}e^{2\pi i\theta}x_2x_4,\\ x_4x_1 &= \mu e^{-2\pi i\theta}x_1x_4,\\ x_3x_2 &= \frac{1}{\mu}e^{-2\pi i\theta}x_2x_3,\\ x_2x_1 &= x_1x_2,\\ x_4x_3 &= x_3x_4, \end{cases}$$

for the Sklyanin $*$-algebra $S(q_{13})$ with $q_{13} = \mu e^{2\pi i\theta} \in \mathbb{C}$ are identical modulo the "scaled unit relation"

$$x_1x_2 = x_3x_4 = \frac{1}{\mu}e,$$

see Lemma 5.1.3. On the other hand, we know that if our elliptic curve is isomorphic to $\mathcal{E}_{CM}^{(-D,f)}$, then:

(i) $q_{13} = \mu e^{2\pi i\theta} \in K$, where $K = k(j(\mathcal{E}_{CM}^{(-D,f)}))$ is the Hilbert class field of the imaginary quadratic field $k = \mathbb{Q}(\sqrt{-D})$;

(ii) $\theta \in \mathbb{Q}(\sqrt{D})$, since Theorem 6.1.3 says that $F(\mathcal{E}_{CM}^{(-D,f)}) = \mathcal{A}_{RM}^{(D,f)}$.

Thus one gets the following inclusion

$$K \ni \mu e^{2\pi i\theta}, \quad \text{where } \theta \in \mathbb{Q}(\sqrt{D}).$$

The only missing piece of data is the constant $\mu$; however, the following lemma fills in the gap.

**Lemma 4.6.1** *For each noncommutative torus $\mathcal{A}_{RM}^{(D,f)}$ the constant $\mu = \log\varepsilon$, where $\varepsilon > 1$ is the Perron-Frobenius eigenvalue of positive integer matrix*

$$A = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

*and $\overline{(a_1, \dots, a_n)}$ is the period of continued fraction of the corresponding quadratic irrationality $\theta$, see also Remark 6.6.2.*

*Proof.* (i) Recall that the range of the canonical trace $\tau$ on projections of algebra $\mathcal{A}_\theta \otimes \mathcal{K}$ is given by pseudo-lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\theta$ [Rieffel 1990] [89], p. 195. Because $\tau(\frac{1}{\mu}e) = \frac{1}{\mu}\tau(e) = \frac{1}{\mu}$, the pseudo-lattice corresponding to the algebra $\mathcal{A}_\theta$ with a scaled unit can be written as $\Lambda_\mu = \mu(\mathbb{Z} + \mathbb{Z}\theta)$.

(ii) To express $\mu$ in terms of the inner invariants of pseudo-lattice $\Lambda$, denote by $R$ the ring of endomorphisms of $\Lambda$ and by $U_R \subset R$ the multiplicative group of automorphisms (units) of $\Lambda$. For each $\varepsilon, \varepsilon' \in U_R$ it must hold

$$\mu(\varepsilon\varepsilon'\Lambda) = \mu(\varepsilon\varepsilon')\Lambda = \mu(\varepsilon)\Lambda + \mu(\varepsilon')\Lambda,$$

since $\mu$ is an additive functional on the pseudo-lattice $\Lambda$. Canceling $\Lambda$ in the above equation, one gets

$$\mu(\varepsilon\varepsilon') = \mu(\varepsilon) + \mu(\varepsilon'), \qquad \forall \varepsilon, \varepsilon' \in U_R.$$

The only real-valued function on $U_R$ with such a property is the logarithmic function (a *regulator* of $U_R$); thus $\mu(\varepsilon) = \log\varepsilon$.

(iii) Notice that $U_R$ is generated by a single element $\varepsilon$. To calculate the generator, recall that pseudo-lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\theta$ is isomorphic to a pseudo-lattice $\Lambda' = \mathbb{Z} + \mathbb{Z}\theta'$, where $\theta' = \overline{(a_1, \dots, a_n)}$ is purely periodic continued fraction and $(a_1, \dots, a_n)$ is the period of continued fraction of $\theta$. From the standard facts of the theory of continued fractions, one gets that $\varepsilon$ coincides with the Perron-Frobenius eigenvalue of matrix

$$A = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, $\varepsilon > 1$ and it is an invariant of the stable isomorphism class of algebra $\mathcal{A}_\theta$. Lemma 6.6.1 is proved. $\square$

**Remark 4.6.3 (Second proof of Lemma 6.6.1)** Lemma 6.6.1 follows from a purely measure-theoretic argument. Indeed, if $h_x : \mathbb{R} \to \mathbb{R}$ is a "stretch-out" automorphism of real line $\mathbb{R}$ given by the formula $t \mapsto tx$, $\forall t \in \mathbb{R}$, then the only $h_x$-invariant measure $\mu$ on $\mathbb{R}$ is the "scale-back" measure $d\mu = \frac{1}{t} dt$. Taking the antiderivative and integrating between $t_0 = 1$ and $t_1 = x$, one gets

$$\mu = \log x.$$

It remains to notice that for pseudo-lattice $K_0^+(\mathcal{A}_{RM}^{(D,f)}) \cong \mathbb{Z} + \mathbb{Z}\theta \subset \mathbb{R}$, the automorphism $h_x$ corresponds to $x = \varepsilon$, where $\varepsilon > 1$ is the Perron-Frobenius eigenvalue of matrix $A$. Lemma 6.6.1 follows. $\square$.

Theorem 6.6.1 is an implication of the following argument.

(i) Let $D \notin \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ be a positive square-free integer; for the sake of simplicity, let $f = 1$, i.e. the endomorphism ring of $\mathcal{E}_{CM}^{(-D,f)}$ coincides with the ring of integers $O_k$ of the imaginary quadratic field $k = \mathbb{Q}(\sqrt{-D})$. In this case $\mathcal{E}_{CM}^{(-D,f)} \cong \mathcal{E}(K)$, where $K = k(j(\mathcal{E}(K)))$ is the Hilbert class field of $k$. It follows from the well-known facts of complex multiplication, that condition $D \notin \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ guarantees that $K \ncong \mathbb{Q}$, i.e. the field $K$ has complex embedding.

(ii) Let $\mathcal{E}_1(K), \dots, \mathcal{E}_h(K)$ be pairwise non-isomorphic curves with $End\ (\mathcal{E}_i(K)) \cong O_k$, where $h$ is the class number of $O_k$. Repeating for each $\mathcal{E}_i(K)$ the argument at the beginning of proof of Theorem 6.6.1, we conclude that $\mu_i e^{2\pi i\theta_i} \in K$.

(iii) But Lemma 6.6.1 says that $\mu_i = \log \varepsilon_i$; thus for each $1 \le i \le h$, one gets

$$(\log \varepsilon_i)\ e^{2\pi i\theta_i} = e^{2\pi i\theta_i + \log\log \varepsilon_i} = \mathcal{J}(\theta_i, \varepsilon_i) \in K.$$

(iv) The transitive action of the ideal class group $Cl\ (k) \cong Gal\ (K|k)$ on the elliptic curves $\mathcal{E}_i(K)$ extends to the algebraic numbers $\mathcal{J}(\theta_i, \varepsilon_i)$; thus $\mathcal{J}(\theta_i, \varepsilon_i) \in K$ are algebraically conjugate.

Theorem 6.6.1 is proved. □

**Guide to the literature.** The complex number is *algebraic* whenever it is the root of a polynomial with integer coefficients; it is notoriously hard to tell if given complex number is algebraic or not. Thanks to Ch. Hermite and C. L. Lindemann the $e$ and $\pi$ are not, but even for $e \pm \pi$ the answer is unknown. The Seventh Hilbert Problem deals with such type of questions, see [Hilbert 1902] [40], p. 456. The famous Gelfond-Schneider Theorem says that $e^{\beta \log \alpha}$ is a transcendental number, whenever $\alpha \not\in \{0, 1\}$ is an algebraic and $\beta$ an irrational algebraic number. An excellent introduction to the theory of transcendental numbers in the book by [Baker 1975] [5]. The noncommutative invariants in transcendence theory were the subject of [79].

## Exercises, problems and conjectures

1. Prove that elliptic curves $\mathcal{E}_\tau$ and $\mathcal{E}_{\tau'}$ are isogenous if and only if

   $$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \quad \text{with} \quad ad - bc > 0.$$

   (Hint: notice that $z \mapsto \alpha z$ is an invertible holomorphic map for each $\alpha \in \mathbb{C} - \{0\}$.)

2. Prove that typically $End\ (\mathcal{E}_\tau) \cong \mathbb{Z}$, i.e. the only endomorphisms of $\mathcal{E}_\tau$ are the multiplication-by-$m$ endomorphisms.

3. Prove that for a countable set of $\tau$

   $$End\ (\mathcal{E}_\tau) \cong \mathbb{Z} + fO_k,$$

   where $k = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field, $O_k$ its ring of integers and $f \geq 1$ is the conductor of a finite index subring of $O_k$; prove that in such a case $\tau \in End\ (\mathcal{E}_\tau)$, i.e. complex modulus itself is an imaginary quadratic number.

4. Show that the noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are stably homomorphic if and only if

$$\theta' = \frac{a\theta + b}{c\theta + d} \quad \text{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \quad \text{with} \quad ad - bc > 0.$$

(Hint: follow and modify the argument of [Rieffel 1981] [88].)

5. **(The rank conjecture)** Prove that the formula

$$rk\ (\mathcal{E}_{CM}^{(-D,f)}) + 1 = c(\mathcal{A}_{RM}^{(D,f)})$$

is true in general, i.e. for all $D \geq 2$ and $f \geq 1$.

6. Verify that the continued fraction

$$[3s+1, \overline{2, 1, 3s, 1, 2, 6s+2}] = \sqrt{(3s+1)^2 + 2s + 1}$$

is a restriction of the continued fraction

$$[x_0, \overline{x_1, 2x_1, x_0, , 2x_1, x_1, 2x_0}] = \sqrt{x_0^2 + 4nx_1 + 2}$$

with $x_0 = n(2x_1^2 + 1) + x_1$ to the case $x_1 = -1$ and $n = s + 1$.

7. Assume that a solution

$$[x_0, \overline{x_1, \ldots, x_{k-1}, x_k, x_{k-1}, \ldots, x_1, 2x_0}]$$

of the diophantine equation in Definition 6.2.2 with the (culminating or almost-culminating) period $P_0 \equiv 7$ **mod** 8 has dimension $d = 1$; prove that for the solution

$$[x_0, \overline{y_1, x_1, \ldots, x_{k-1}, y_{k-1}, x_k, y_{k-1}, x_{k-1}, \ldots, x_1, y_1, 2x_0}]$$

having the period $P_0 + 4$ the dimension remains the same, i.e. $d = 1$. (Hint: use the same argument as in Lemma 6.2.3.)

8. Prove that part (ii) of Theorem 6.3.1 implies its part (i). (Hint: repeat the step by step argument of Section 6.3.2.)

9. Prove Remark 6.4.1, i.e. that if noncommutative torus $\mathcal{A}_{\Theta_0}^{2n}$ has real multiplication, then $\theta_i$ are algebraic integers. (Hint: each endomorphism of $K_0^+(\mathcal{A}_{\Theta_0}^{2n}) \cong \mathbb{Z} + \theta_1\mathbb{Z} + \ldots + \theta_n\mathbb{Z} + \sum_{i=n+1}^{2^{2n-1}} p_i(\theta)\mathbb{Z}$ is multiplication by a real number; thus the endomorphism is described by an integer matrix, which defines a polynomial equation involving $\theta_i$.)

10. **(Langlands conjecture for noncommutative tori)** Prove Conjecture 6.4.1, i.e. that for each finite extension $E$ of the field of rational numbers $\mathbb{Q}$ with the Galois group $Gal\ (E|\mathbb{Q})$ and each irreducible representation
$$\sigma_{n+1} : Gal\ (E|\mathbb{Q}) \to GL_{n+1}(\mathbb{C}),$$
there exists a $2n$-dimensional noncommutative torus with real multiplication $\mathcal{A}_{RM}^{2n}$, such that
$$L(\sigma_{n+1}, s) \equiv L(\mathcal{A}_{RM}^{2n}, s),$$
where $L(\sigma_{n+1}, s)$ is the Artin $L$-function attached to representation $\sigma_{n+1}$ and $L(\mathcal{A}_{RM}^{2n}, s)$ is the $L$-function of the noncommutative torus $\mathcal{A}_{RM}^{2n}$.

11. Prove the identity
$$2\ {}_2F_1(-\pi_1(p),\ \pi_1(p);\ \tfrac{1}{2};\ 1-\lambda) =$$
$$= (-1)^{\pi_1(p)}\ {}_2F_1(\pi_1(p)+1,\ \pi_1(p)+1;\ 1;\ \lambda),$$
where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.