How arithmetic generates the logic of quantum experiments

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Abstract As opposed to the classical logic of *true* and *false,* viewed as an axiomatised theory, ordinary arithmetic conveys the three logical values: *provable*, *negatable* and *logically independent.* This research proposes the hypothesis that Axioms of Arithmetic are the fundamental foundation running arithmetical processes in Nature, upon which physical processes rest. And goes on to show, in detail, that under these axioms, quantum mathematics derives and initiates logical independence, agreeing with indeterminacy in quantum experiments. Supporting arguments begin by explaining *logical independence* in arithmetic, in particular, independence of the *square root of minus one.* The method traces all sources of information entering arithmetic, needed to write mathematics of the free particle. Wave packets, prior to measurement, are found to be the only part of theory logically independent of axioms; the rest of theory is logically dependent. Ingress of logical independence is via uncaused, unprevented *self-reference*, sustaining the wave packet, but implying unitarity. Quantum mathematics based on axiomatised arithmetic is established as foundation for the *3-valued logic* of Hans Reichenbach, which reconciles quantum theory with experimental anomalies such as the Einstein, Podolsky & Rosen paradox.

1 Introduction

Coin-tossing and other experiments in classical statistical physics are deterministic, in the sense that, a complete knowledge of the physical detail would render an outcome perfectly predictable, and that randomness in outcome is attributable to our degree of ignorance. Intuitively, this philosophy suggests that randomness in quantum experiments also, lays similarly in physical detail of which we are ignorant.

But contradicting this intuitive viewpoint, theorems of Kocken and Specker [19], the inequalities of John Bell [4], and the experiments of Alain Aspect and others [1, 2, 31, 24, 26] all strongly support the view that, prior to measurement, identically-prepared quantum systems are truly *physically identical* – ruling out physical detail as the *cause* of variation in measured outcomes. A reasoned inference is that classical concepts of *cause and effect* are inconsistent with quantum processes in Nature.

In a new approach, ordinary arithmetic – denoted ARITHMETIC[1] – is acknowledged as foundation on which quantum theory rests. ARITHMETIC is then given formal, logical treatment and a quantum theory results invoking a *causeology*, a degree more complex than simple *cause.* Underpinning this is the fact, well-known to Mathematical Logic, that ARITHMETIC'S formula are incapable of simple classification, as either true or otherwise false, but require an additional category that is neither.

The formal treatment, mentioned, consists of ARITHMETIC as a *theory of propositions.* This is an approach referred to by Stabler as a 'postulational approach' [27]. Simply put, abandoned is the notion that ARITHMETIC is a *property* of scalars: real, complex or any other, all presumed to automatically exist. Instead, axioms listed in Table 1 – denoted AXIOMS[2] – are adopted *a priori.* Then from these, all scalars result as incidental objects. The thesis of this paper is that this formal ARITHMETIC

Recommended reading *An Introduction to Mathematical Thought* Stabler [27]

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[1] ARITHMETIC denotes this particular arithmetic, as opposed to any other.
[2] AXIOMS denotes axioms for this particular arithmetic.

AXIOMS of ARITHMETIC

	ADDITIVE GROUP	
A0	$\forall\beta\forall\gamma\exists\alpha:\ \alpha = \beta + \gamma$	CLOSURE
A1	$\exists 0\forall\alpha:\ \alpha + 0 = \alpha$	IDENTITY 0
A2	$\forall\alpha\exists\beta:\ \alpha + \beta = 0$	INVERSE
A3	$\forall\alpha\forall\beta\forall\gamma:\ (\alpha+\beta)+\gamma = \alpha + (\beta+\gamma)$	ASSOCIATIVITY
A4	$\forall\alpha\forall\beta:\ \alpha+\beta = \beta+\alpha$	COMMUTATIVITY
	MULTIPLICATIVE GROUP	
M0	$\forall\beta\forall\gamma\exists\alpha:\ \alpha = \beta \times \gamma$	CLOSURE
M1	$\exists 1\forall\alpha:\ \alpha \times 1 = \alpha$	IDENTITY 1
M2	$\forall\alpha\exists\beta:\ \alpha \times \beta = 1 \wedge \alpha \neq 0$	INVERSE
M3	$\forall\alpha\forall\beta\forall\gamma:\ (\alpha\times\beta)\times\gamma = \alpha \times (\beta\times\gamma)$	ASSOCIATIVITY
M4	$\forall\alpha\forall\beta:\ \alpha\times\beta = \beta\times\alpha$	COMMUTATIVITY
D	$\forall\alpha\forall\beta\forall\gamma:\ \alpha\times(\beta+\gamma) = (\alpha\times\beta)+(\alpha\times\gamma)$	DISTRIBUTIVITY
	EXCLUSION OF ALL MODULO ADDITION	
	$1 \neq 0$	MODULO 1
	$1+1 \neq 0$	MODULO 2
	$1+1+1 \neq 0$	MODULO 3
	$\vdots \qquad \vdots$	$\vdots \qquad \vdots$
	$1+\cdots+1 \neq 0$	MODULO $\vdots$

Table 1 AXIOMS of ARITHMETIC. These are written as sentences in *first-order logic.* They comprise the field axioms with added axioms that exclude modulo addition. Variables: $\alpha,\beta,\gamma,0,1$ represent objects the axiom-set acts upon. Semantic interpretations of objects complying with AXIOMS are known as *scalars.* The fact ARITHMETIC is intrinsically existential is clearly seen in the general use of the 'there exists' quantifier: $\exists$.

is *logically isomorphic* to 'arithmetical processes in Nature' and is key to gaining logical agreement between quantum theory and experiment.

The new approach elevates quantum theory from a theory of equations to a *theory of existence.* This is so because, collectively, AXIOMS assert *existence* of ARITHMETICAL objects we call *scalars,* sets of which make up structures we call *fields.* Crucially, two *modes of existence* arise, distinct in their logical qualities. Both modes are invoked because each is sanctioned (differently) by AXIOMS. To explain: the various fields of scalars fall under different existence modes: the rational field being special amongst all others. Existence of every *infinite-field* is consistent with AXIOMS, but only in the special case of the rational field, do AXIOMS also ***prove*** the field's existence.

In this paper, scalars and fields are concepts from linear algebra. It would be misleading to visualise scalars as zero rank tensors from relativity or fields from quantum field theory.

consistent with = satisfied by

And so, along with the familiar *quantitative* information of quantum theory, *existential* information is also conveyed. Interpretationally*,* *consistent* and *provable* existences, in Theory, are seen respectively as *permitted*[3] and *caused* existences, in Nature: permitted existence being neither caused nor prevented. Permission and cause together constitute the earlier mentioned *causeology.*

Key, are concepts of *logical dependence* and *logical independence*[4]*,* and then of *consistency* and *inconsisten*cy[5]. Informally, these may be understood as concepts of provability, and truth. Section 5 explains theorems, dictating relationships connecting these. Briefly, taking any two mathematical formulae (in the same language), precisely one of the following relates them.

1. Each implies the other; such formulae are *dependent and consistent.*
2. Each implies the negation of the other; these are *dependent but inconsistent.*
3. Neither implies the other, nor does either imply the other's negation; these are *independent and consistent.*

Importantly therefore, a consistent theory may comprise both dependent and independent formulae. Indeed, single formulae can contain individual variables, some conveying logical dependence, others, logical independence, the mix relying ultimately on the origins of information entering the theory. By way of this mix, rather than the classical logic comprising values true[6] and false, a 3-valued logic

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[3] Permitted and caused existence may be seen as *possible* and *necessary* values, in a *modal logic.*

[4] Synonymous with *mathematical undecidability.*

[5] Two formulae are *inconsistent* when they contradict. If they do not, they are *consistent.*

[6] 'True' is an informal term for 'consistent with AXIOMS'.

of *provable-true*, *nonprovable-true* and *provable-not-true* is transmitted around the theory, the middle one of these being identifiable with *indeterminate*.

In order to read this causeological information, in her *consistent* theory, the quantum theorist needs a practical method of distinguishing provable elements from non-provable ones. Model Theory, a branch of Mathematical Logic, furnishes powerful tools that address exactly this problem. These are the Soundness Theorem and its converse, the Completeness Theorem. These theorems apply to all *first-order theories*[7] of which group theories and field theories are examples. In particular, ARITHMETIC is a field theory. Soundness ensures that any provable formula is guaranteed true, whether variables are interpreted as complex scalars, real scalars, rational scalars, or scalars of any other infinite field. In the converse sense, completeness ensures that any formula, true in every infinite field, is guaranteed provable. Soundness and completeness, together, exclude an *excluded-middle* of formulae, neither provable nor negatable – these are the formulae having *disagreeing* truth-tables. Section 5 provides the detail.

And so, what is the specific relationship between quantum theory and ARITHMETIC? Leaving aside orthogonality of 3-space, all information in quantum mathematics is wholly ARITHMETICAL. I am not suggesting ARITHMETIC proves the physics, only that information in physics is written down as formulae in ARITHMETIC. This includes operators: they are essentially, linear combinations of ARITHMETICAL terms. Even orthogonality in spectral vector spaces is information within ARITHMETIC! Standard theory is always unitary. Through much of theory, this unitarity is redundant information. When all this redundant information is removed, much of quantum mathematics is ARITHMETIC consisting of logically dependent theorems, derivable from AXIOMS – with the exception of formulae involving orthogonal vector spaces of dimension $\geq$ 3. There are just two circumstances for this.

1. In symmetries, such as $\mathsf{su}(2)$. See Section 11.
2. In wave packets. See Section 16

In 1944, Hans Reichenbach published his book detailing a 3-valued logic that resolves 'causal anomalies' of quantum experiments [22]. His findings were later appraised in a paper by Hilary Putnam [21]. This non-classical logic has the feature that 'false' is not the same as 'not true', and consists of values: true, false and *indeterminate.* Reichenbach arrived at the qualities for his indeterminate middle through detailed analysis and elimination, with the aim of designing a logic isomorphic to the epistemology of quantum experiments. His logic is an adaptation of the 3-valued logic of Jan Łukasiewicz, which Reichenbach gives certain truth tables, conjunctions, disjunctions, tautology etc [13].

From his 3-valued logic, Reichenbach derived a consistent epistemology for *prepared* and *measured* states – typically the question of what we may know about the state of a photon immediately before measurement. From his 3-valued logic, he successfully derived *complimentary* propositions – if statement A is either true or false, statement B is indeterminate, and vice verse, as is the case with measurements of complimentary pairs such as momentum and position. And his logic also resolves the problem of *action at a distance*, a paradox identified by Einstein, Podolsky & Rosen [10,17].

Nevertheless, Reichenbach was not generally taken up. His logic is a construction *designed* to fit experiment. It had no basis in physical first principles. Furthermore, Reichenbach is not in alignment with the mainstream quantum logics, originating with Birkhoff and von Neumann [5] and based on the quantum postulates in Hilbert space quantum theory. Acceptance of these mainstream logics has tended to discount Reichenbach's. That said, Hardegree argues that these logics are not in opposition: while Birkhoff and von Neumann's logic reflects current theory, Reichenbach's reflects an alternative formulation, awaiting discovery [16]. The work of this paper provides Reichenbach with foundation in ARITHMETIC and unitarity.

Private thought: if Birkhoff and von Neumann's logic models conventional theory, is validity of their logic not questionable, since theory, their basic premise, contradicts experiment?

The research of this present paper stems originally from a thought-experiment, posed by the author, contemplating the problem of making a machine that would simulate the Universe! Ignoring the small matter of immense complexity, this idea throws up restrictions constraining the *logical-form* of Physical Law in Nature. Regarding Physical Law as information in a program of instructions that operates the machine, in order to truly replicate the Universe, a seminal factor is the topology

[7] First-order is a term in logic not to be confused with the term describing approximation.

of logical connections linking Physical Law with the Universe. This involves how and where this information originates. I identify three broad possibilities:

1. Physical Law is simply 'THERE', pre-Universe;
2. Physical law has foundation in even deeper information, which in turn has foundation in yet deeper – and so on...;
3. The Universe is the generating source of information, that then governs the Universe.

Favouring 3, I took the view that, within Nature, we should expect to find *circularity* or *self-reference*. That is, conditions in Nature where flow of information is cyclic. For example, where the outcome of A has dependency on B and the outcome of B has dependency on A. Symbolically:

$$\exists \mathsf{A} : \ \mathsf{A} = f\left(\mathsf{B}\right) \qquad \text{AND} \qquad \exists \mathsf{B} : \ \mathsf{B} = F\left(\mathsf{A}\right). \tag{1}$$

This conclusion prompted an enquiry into theorems of Kurt Gödel, where self-reference plays an instrumental part and led to a search for logical independence in quantum theory. Interestingly, although a presence of self-reference is not a premise of this research, the original idea is born out by its discovery, when in Section 17, occurrence of self-reference is confirmed, revealing that textbook physics, as it stands today, unwittingly relies on self-reference of the type in (1).

In the first two decades of the twentieth century, mathematical logicians, notably Whitehead and Russell, were working to set pure mathematics on a consistent logical foundation [32], but their endeavours led to paradoxes they could not resolve. Attempting to recover the situation, in 1920, David Hilbert announced his *Hilbert's Programme* [33]. But Hilbert's hopes were wrecked by publication of Gödel's First Incompleteness Theorem in 1931, proving that no such foundation is possible and that arithmetic is *incomplete*, unavoidably incorporating formulae that are *mathematically undecidable (logically independent)* [12,14].

In 1936, Alan Turing took concepts of undecidability into the domain of the physical world by showing computational machines, including mechanical ones, unavoidably suffer the 'Halting Problem': that no algorithm can decide whether a given program will ever halt [30]. In his 1982 paper, Gregory Chaitin gives a proof that Gödel's Theorem may be deduced from Turing's [9]. More recently, by extension of Turing, Svozil argued that undecidability is in physics [28]. In 2009, Elemér Rosinger published his paper: "Self-Referential Definition of Orthogonality" [23]. In recent work by Tomasz Paterek et. al., logical independence is formalised algebraically, in orthogonal pairs of Pauli operators, and a link is demonstrated, connecting this independence with quantum randomness in experiments, measuring polarisation of photons, previously prepared, orthogonal to the measurement [20]. More recently still, Gergely Székely has shown that faster-than-light particles are consistent with, but logically independent of special relativity [29].

2 Language

The material of this paper spans both formal ARITHMETIC and mathematical physics. These do not share the same language; indeed the language of the former is far smaller. For example, there is no definition for the symbol: 4. And so extension of the former into the latter needs all manner of statements typified by: $4 = 1{+}1{+}1{+}1$. In the interest of accessibility, these 'higher' definitions are omitted and left to the intuition of the reader. Definitions for some high level expressions, such as: $\exp x = 1 + x + \cdots$ are given in Section 8.

In places, formulae use logical connectives: *not* ($\neg$), *and* ($\wedge$) , or ($\vee$), *implies* ($\Rightarrow$), *if-and-only-if* ($\Leftrightarrow$): the quantifiers: *there-exists* ($\exists$), *for-all* ($\forall$): and turnstile symbols: *derives* ($\vdash$) and *models* ($\models$).

Part I: Pure ARITHMETIC

3 Foundations of ARITHMETIC

Before there can be hope of logical isomorphism linking quantum theory with Nature, we must first view ARITHMETIC as Nature views it. That is, as ARITHMETIC works automatically as a machine of its own accord, without interference from the mathematician. Its foundation must be that of Nature's.

Alternative foundations for ARITHMETIC are known. In the *definitional approach,* scalars and their properties exist, a priori. This is the familiar everyday approach, assumed and taken for granted, as a matter of course, in Applied Mathematics and Mathematical Physics. But once aware of an alternative, this definitional approach must be seen as an arbitrary choice. A different foundation, known to Mathematical Logicians, is the *postulational approach* [27]. In this, AXIOMS are adopted, a priori. Quantitatively, definitional and postulational approaches amount to the same, but their *logical forms* are radically distinct.

In formulating the definitional approach, firstly, the natural numbers and their properties are assumed. From there, their rules of addition and multiplication, Associativity, commutativity and distributivity, are extended to new *objects* by defining: negative, rational, irrational, then complex numbers [11, Vol 1, Chap 22]. These *rules* are perceived as properties *belonging to* numbers: significance, meaning and 'reality' are placed on *numbers,* with rules perceived as subordinate and incidental. The reverse is the case in the postulational approach; apriority is switched from ARITHMETICAL objects to ARITHMETICAL rules. These rules are listed as AXIOMS: those in Table 1 [27, Chap. 6]. A scalar (number) is *then* any mathematical object whose existence is consistent with these AXIOMS. In the postulational approach AXIOMS are the foundation and scalars are incidental.

In place of the conventional concept, where ARITHMETIC is a machine of *computation*, in the postulational approach, ARITHMETIC is a theory asserting *existence,* of scalars obeying particular equalities: some, logically dependent of AXIOMS, all others logically independent. With this postulational ARITHMETIC in place, the mathematician is not at liberty to interfere by declaring that certain scalars are real and others complex; such a declaration would corrupt information in the theory.

AXIOMS (of Table 1) are a collection of propositions prescribing ARITHMETIC. Each conveys exclusive information, logically independent of the others [27]. Essentially, they are the *field axioms* appended with additional axioms that exclude modulo addition. The field axioms themselves are a union of axioms for the Additive Group and Multiplicative Group, with a single axiom for distributivity.

Mathematical structures consistent with AXIOMS are the *infinite-fields.* They satisfy AXIOMS collectively, and by doing so, are said to *model* them. There may be many such fields, three of which are the complex plane $\mathbb{C}$, the real line $\mathbb{R}$ and the rational field $\mathbb{Q}$. A *scalar* is a particular element of a field. These are numbers we typically add and multiply and use as entries in arrays such as vectors or matrices.

4 Modes of existence in ARITHMETIC

The following five formulae are examples illustrating the different logical standings, possible under AXIOMS. Each is a proposition asserting the existence of some instance of a variable α, complying with an equality specifying a particular numerical value. Note: these formulae do not assert equality, they assert *existence.*

$$\exists\alpha : \ \alpha = 3 \tag{2}$$
$$\exists\alpha : \ \alpha^2 = 4 \tag{3}$$
$$\exists\alpha : \ \alpha^2 = 2 \tag{4}$$
$$\exists\alpha : \ \alpha^2 = -1 \tag{5}$$
$$\exists\alpha : \ \alpha^{-1} = 0 \tag{6}$$

Of these, AXIOMS prove only (2) and (3) (see below). Also, they prove the negation of only (6); in point of fact (6) is inconsistent with axiom FM2. Consequently, the remaining formulae, (4) and (5), are neither proved nor negated, but are logically independent of AXIOMS and both these, as well as their negations, are consistent with AXIOMS.

Accordingly, instances of α in (2) and (3) are accepted as scalars, consistent with AXIOMS, proved to *necessarily* exist. The instance of α in (6) is inconsistent with AXIOMS and rejected as *necessarily* non-existent. And instances of α in (4) and (5) are consistent with AXIOMS and accepted as scalars whose existences are *possible*.

Substitution between propositions
For substitution to be valid, an existential quantifier of one proposition must be matched with a universal quantifier of the other: In this example these are highlighted by underlining:

$$\begin{aligned} \forall\beta\underline{\forall\gamma}\exists\alpha :\ & \alpha = \beta + \gamma \\ \forall\beta\underline{\exists\gamma} :\ & \gamma = \beta + \beta \end{aligned}$$

This is the technique used in this paper.

4.1 Proof of (2): $\exists\alpha\,(\alpha = 3)$

$$\forall\beta\forall\gamma\exists\alpha :\ \alpha = \beta + \gamma \qquad \text{Axiom A0} \tag{7}$$
$$\forall\beta\forall\varepsilon\exists\gamma :\ \gamma = \beta + \varepsilon \qquad \text{Axiom A0} \tag{8}$$
$$\forall\beta\exists\gamma :\ \gamma = \beta + \beta \qquad \text{contraction of (8)} \tag{9}$$
$$\forall\beta\exists\alpha :\ \alpha = \beta + \beta + \beta \qquad \text{Substitute (9), (7))} \tag{10}$$
$$\exists\beta :\ \beta = 1 \qquad \text{by Axiom M1} \tag{11}$$
$$\exists\alpha :\ \alpha = 1 + 1 + 1 \qquad \text{Substitute (11), (10)).} \tag{12}$$

4.2 Proof of (3): $\exists\alpha\left(\alpha^2 = 4\right)$

$$\begin{array}{lll}
\forall\alpha :\ \alpha\times\alpha = \alpha\times\alpha & \text{identity rule} & (13) \\
\forall\beta\exists\alpha :\ \alpha = \beta + \beta & \text{Axiom A0} & (14) \\
\forall\beta\exists\alpha :\ \alpha\times\alpha = (\beta+\beta)\times(\beta+\beta) & \text{Substitute (14), (13)} & \\
\forall\beta\exists\alpha :\ \alpha\times\alpha = \beta\times(\beta+\beta) + \beta\times(\beta+\beta) & \text{Axiom D} & \\
\forall\beta\exists\alpha :\ \alpha\times\alpha = (\beta\times\beta)+(\beta\times\beta)+(\beta\times\beta)+(\beta\times\beta) & \text{Axiom D} & (15) \\
\exists\beta :\ \beta = 1 & \text{Axiom M1} & (16) \\
\exists\alpha :\ \alpha\times\alpha = (1\times1)+(1\times1)+(1\times1)+(1\times1) & \text{Substitute (16), (15)} & \\
\exists\alpha :\ \alpha\times\alpha = 1+1+1+1 & \text{Axioms M0, M1} &
\end{array}$$

In the cases of propositions (2) and (3), logical dependence is proved by their direct derivation from AXIOMS and likewise for the negation of (6). However, logical independence of (4) and (5) is not provable by any direct derivation because AXIOMS are devoid of such information; in essence, this is the whole point of the discussion. What *does* identify a logically independent proposition is its consistency table evaluated against individual fields. Enter the Soundness and Completeness Theorems.

5 Model Theory

Model theory is a branch of Mathematical Logic applying to all first-order theories and therefore, to ARITHMETIC [7,8]. Our interest is in two standard theorems: the Soundness Theorem and its converse, the Completeness Theorem. Each, in the converse sense of the other, these theorems guarantee a correspondence binding provability with consistency in ARITHMETIC. Together, the combined action of both, excepts an *excluded middle*, isolating the set of all non-provable, non-negotiable propositions – logically independent of AXIOMS.

Briefly: any given (first-order) axiom-set is *modelled* by particular mathematical structures. That is to say, there are certain structures, consistent with each and every axiom of that axiom-set. In the case of ARITHMETIC, these modelling structures are the various *infinite fields,* consisting of *scalars.* The logically independent propositions are identified by proving disagreement between AXIOM'S *models.*

5.1 Standard Theorems

Theorem 1 *The Soundness Theorem:*

$$\Sigma \vdash \mathcal{S} \Rightarrow \forall\mathcal{M}^{\Sigma}\left(\mathcal{M}^{\Sigma} \models \mathcal{S}\right). \tag{17}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ and Σ derives sentence $\mathcal{S}$, then every structure $\mathcal{M}^{\Sigma}$ models $\mathcal{S}$.
Alternatively**:** *If a sentence is a theorem, provable under an axiom-set, then that sentence is true for every model of that axiom-set.*

A *sentence* is formula, such as:

$$\forall\alpha\forall\beta\,(\alpha + \beta = \beta + \alpha)$$

where there is no occurrence of any variable not *bound* by a quantifier.

Theorem 2 *The Completeness Theorem:*

$$\Sigma \vdash \mathcal{S} \Leftarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{18}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ and every structure $\mathcal{M}^{\Sigma}$ models sentence $\mathcal{S}$, then *Σ derives sentence $\mathcal{S}$.*
Alternatively**:** *If a sentence is true for every model of an axiom-set, then that sentence is a theorem, provable under that axiom-set.*

Jointly, (17) and (18) result in the 2-way implication:

Theorem 3 *The set of all provable sentences:*

$$\Sigma \vdash \mathcal{S} \Leftrightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{19}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, then axiom-set Σ derives sentence $\mathcal{S}$, if-and-only-if, all structures $\mathcal{M}^{\Sigma}$ model sentence $\mathcal{S}$.
Alternatively**:** *A sentence is provable under an axiom-set, if-and-only-if, that sentence is true for all models of that axiom-set.*

Also, for every sentence $\mathcal{S}$ there is a sentence $\neg\,\mathcal{S}$; hence, in addition to (19), jointly, (17) and (18) also guarantee a second 2-way implication:

Theorem 4 *The set of all negatable sentences:*

$$\Sigma \vdash \neg\,\mathcal{S} \Leftrightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\,\mathcal{S} \right). \tag{20}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, then axiom-set Σ derives the negation of sentence $\mathcal{S}$, if-and-only-if, all structures $\mathcal{M}^{\Sigma}$ model the negation of $\mathcal{S}$.
Alternatively**:** *A sentence is disprovable under an axiom-set, if-and-only-if, that sentence is false for all models of that axiom-set.*

In summary, while (19) isolates all sentences provable from axiom–set Σ, (20) isolates all sentences negatable by this axiom-set. Significantly, together they except sentences excluded by both. In the left hand sides of (19) and (20), there is no indication of any excluded set of sentences that are *neither provable, nor disprovable.* And so, it is of particular interest that the right hand sides of (19) and (20) together imply the existence of sentences that correspond precisely to this condition. These are those remaining sentences, excluded by the right hand sides of both (19) and (20) defined by this conditionality:

$$\neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right) \wedge \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\,\mathcal{S} \right). \tag{21}$$

The aim now is to match this with its corresponding left side. We firstly deduce (22) and (23), the negations of (19) and (20):

$$\neg\,(\Sigma \vdash \mathcal{S}) \Leftrightarrow \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right); \tag{22}$$

$$\neg\,(\Sigma \vdash \neg\,\mathcal{S}) \Leftrightarrow \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\,\mathcal{S} \right); \tag{23}$$

and combine these, so as to construct:

$$\neg\,(\Sigma \vdash \mathcal{S}) \wedge \neg\,(\Sigma \vdash \neg\,\mathcal{S}) \Leftrightarrow \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right) \wedge \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\,\mathcal{S} \right). \tag{24}$$

This limits sentences that are neither provable nor negatable, to those that are neither true nor false across all structures that model the Axioms. For theories whose axioms are modelled by more than one single structure, where $\mathcal{M}_1^{\Sigma}$ and $\mathcal{M}_2^{\Sigma}$ are distinct, we deduce (25):

Theorem 5 *Logical independence:*

$$\neg\,(\Sigma \vdash \mathcal{S}) \wedge \neg\,(\Sigma \vdash \neg\,\mathcal{S}) \Leftrightarrow \exists \mathcal{M}_1^{\Sigma} \left(\mathcal{M}_1^{\Sigma} \models \mathcal{S} \right) \wedge \exists \mathcal{M}_2^{\Sigma} \left(\mathcal{M}_2^{\Sigma} \models \neg\,\mathcal{S} \right). \tag{25}$$

Axiom-set Σ derives neither sentence $\mathcal{S}$ nor its negation, if-and-only-if, there exist structures $\mathcal{M}_1^{\Sigma}$ and *$\mathcal{M}_2^{\Sigma}$ which each model axiom-set Σ, such that $\mathcal{M}_1^{\Sigma}$ models $\mathcal{S}$, and $\mathcal{M}_2^{\Sigma}$ models the negation of $\mathcal{S}$.*
Alternatively**:** *A sentence is true for some but not all models of an axiom-set, if-and-only-if, that sentence is undecidable under that axiom-set.*

This is covered in the section on logical independence by Edward Stabler, in his 1948 book. [27].

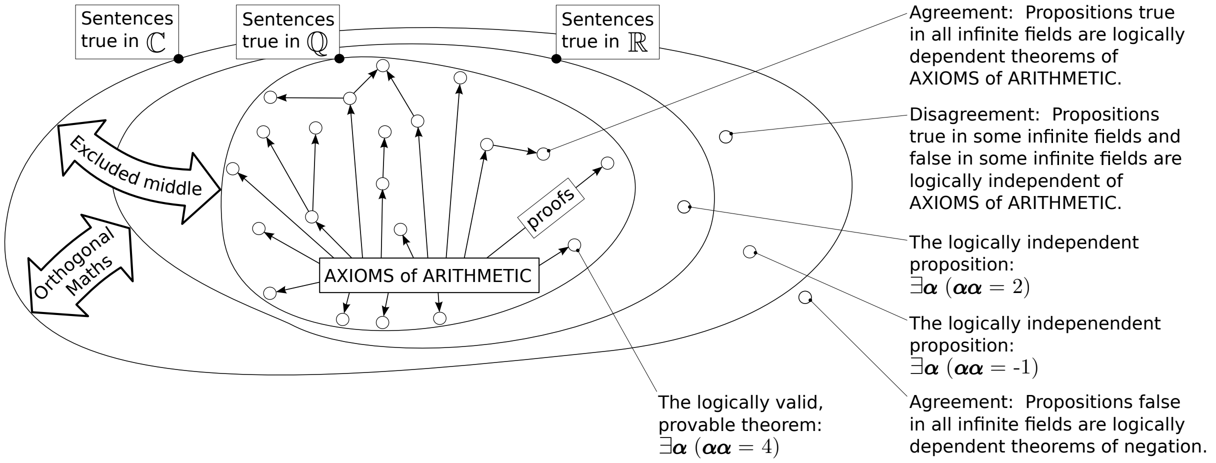

Figure 1 Truth-space under AXIOMS of ARITHMETIC, for all propositions (small circles) asserting existence of particular numbers. The innermost nesting is of propositions true (consistent with AXIOMS) in all fields. The Completeness Theorem guarantees these are *logically dependent* theorems. The exterior set comprises propositions false (inconsistent with AXIOMS) in all fields; these are the only propositions inconsistent with AXIOMS. The Completeness Theorem guarantees these are *logically dependent* negations. Soundness plus Completeness Theorems guarantee the excluded middle consists of *logically independent*, mathematically undecidable propositions.

6 Model Theory acting on ARITHMETIC

This section introduces two, tests by inspection, confirming whether a proposition asserting existence is logically dependent, or otherwise, logically independent of AXIOMS. In the *Independence Test*, Theorem 5 is applied in the context of ARITHMETIC. The structures that model AXIOMS are the infinite-fields, but there are potentially very many of these. Fortunately, in order to prove a proposition's independence, finding disagreement between only two suffices. The two fields of critical interest in quantum theory are the complex field $\mathbb{C}$ and the rational field $\mathbb{Q}$. And for the sake of extending insight, the real field $\mathbb{R}$ is included in the discussion.

Independence Test *For any given sentence S, whose variables are interpreted as scalars of the complex plane $\mathbb{C}$, the real line $\mathbb{R}$, and the rational field $\mathbb{Q}$: S is confirmed independent of* AXIOMS *if S is true in some field and false in another.*

This reduces to a check for *disagreement* within a truth-table.

Dependence Test *Any proposition asserting existence is a logical consequence of* AXIOMS*,* if-and-only-if*, that existence is true in the rational field $\mathbb{Q}$.*

$$\text{AXIOMS} \vdash \mathcal{S} \Longleftrightarrow \mathbb{Q} \vDash \mathcal{S}. \tag{26}$$

Proof

Existence of any given rational is provable from AXIOMS, by direct derivation.
By axiom M1, existence of the number is provable.
It follows, by axiom A0, existence of all positive integers $\mathbb{Z}^+$ is provable.
It follows, by axiom A2, existence of all negative integers $\mathbb{Z}^-$ is probable.
It follows, by axiom M1, existence of reciprocals of all integers is provable.
It follows, by axiom A0, existence of all sums of reciprocals is provable.
Also, by axiom A1, existence of the number 0 *is provable.*
Hence:

$$\mathbb{Q} \vDash \mathcal{S} \Rightarrow \text{AXIOMS} \vdash \mathcal{S} \tag{27}$$

And by Soundness, for any infinite field $\mathbb{F}$:

$$\text{AXIOMS} \vdash \mathcal{S} \Rightarrow \mathbb{F} \vDash \mathcal{S}$$

implying

$$\text{AXIOMS} \vdash \mathcal{S} \Rightarrow \mathbb{Q} \vDash \mathcal{S}. \tag{28}$$

Together, (27) and (28) imply (27).

7 Test by inspection for dependent or independent existence

7.1 Existence of particular scalars

All the truth-tables below concern propositions asserting existence of *particular* scalars. Each table consists of a formula and values T or F denoting the formula's truth or falsity as its variables are interpreted as members of $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$. The technique then is inspection for two of the possible outcomes:

- ▷ true in the rational field, indicating the formula is a theorem,
- ▷ disagreement, indicating the formula is logically independent of AXIOMS.

Table 2 deals with the simplest illustrations: formulae (3), (4), (5) and (6) from page 5.

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha :\ \alpha = 3$	T	T	T
$\exists\alpha :\ \alpha \times \alpha = 4$	T	T	T
$\exists\alpha :\ \alpha \times \alpha = 2$	T	T	F
$\exists\alpha :\ \alpha \times \alpha = -1$	T	F	F
$\exists\alpha :\ \alpha^{-1} = 0$	F	F	F

Table 2 Truth-tables for simple existential propositions. In these T and F denote true and false. Disagreement confirms independence; true in $\mathbb{Q}$ confirms dependence.

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha :\ \alpha = \xi^{\mathbb{Q}}$	T	T	T
$\exists\alpha :\ \alpha = \zeta^{\mathbb{R}}$	T	T	F
$\exists\alpha :\ \alpha = \eta^{\mathbb{C}}$	T	F	F

Table 3 Examples that are more general. Truth values showing dependence of an arbitrary, particular, rational scalar $\xi^{\mathbb{Q}}$, independence of the real scalar $\zeta^{\mathbb{R}}$ and independence of the complex scalar $\eta^{\mathbb{C}}$.

7.2 Existence of functions

This next example illustrates logical ambiguity of formulae in applied mathematics. A function in applied mathematics can spurn different first-order propositions, some of which might be theorems and others independent. This results by interchanging quantifiers. Note the crucial difference in validity between $\forall x \exists y \left(y = x^2\right)$ and $\forall y \exists x \left(y = x^2\right)$. The first of these, quantified by $\forall x \exists y$, is true for the rational field and therefore is a theorem of AXIOMS. The second, quantified by $\forall y \exists x$ is independent of AXIOMS, as confirmed by its disagreeing truth-table.

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y :\ y = x^2$	T	T	T
$\forall y \exists x :\ y = x^2$	T	F	F

Table 4 Truth-tables concerning existence of x and y in the function $y = x^2$. When reading truth-tables, variables of the same sort are interpreted as members of the same model: in this case, the same field.

7.3 Existence of finite polynomials versus transcendental functions

Table 5 compares a *finite* polynomial $p\left(x^{\mathbb{Q}}\right)$ with a transcendental function $\exp\left(x^{\mathbb{Q}}\right)$, both with rational, and therefore, logically dependent arguments. The polynomial, a finite sum of rationals, is itself necessarily rational, but in contrast, the exponential:

$$\exp\left(x^{\mathbb{Q}}\right) \equiv \lim_{n\to\infty}\left[1 + x^{\mathbb{Q}} + \frac{\left(x^{\mathbb{Q}}\right)^2}{2} + \cdots + \frac{\left(x^{\mathbb{Q}}\right)^n}{n!}\right],$$

is a never-ending sum that maps the rational argument to a generally irrational value. By the Dependence Test, $p\left(x^{\mathbb{Q}}\right)$ exists by theorem, but by the Independence Test $\exp\left(x^{\mathbb{Q}}\right)$ exists independent of AXIOMS. The exponential function, therefore, introduces information not present in the AXIOMS.

	$y \in \mathbb{C}$	$y \in \mathbb{R}$	$y \in \mathbb{Q}$
$\exists y:\ y = p\left(x^{\mathbb{Q}}\right)$	T	T	T
$\exists y:\ y = \exp\left(x^{\mathbb{Q}}\right)$	T	T	F

Table 5 Truth-table for finite polynomial: $\exists y\left(y = p\left(x^{\mathbb{Q}}\right)\right)$ and the transcendental function: $\exists y\left(y = \exp\left(x^{\mathbb{Q}}\right)\right)$.

7.4 Dependence via Independence

Some scalars, existing independent of AXIOMS, can combine ARITHMETICALLY to yield new scalars having *dependent* existence. To illustrate, consider the propositions: $\exists\alpha\,(\alpha = 3 + 4\mathrm{i})$ and $\exists\beta\,(\beta = 3 - 4\mathrm{i})$. Both these are independent of AXIOMS, but the product of these scalars is the rational scalar 25, which exists by theorem. The multiplication of conjugates, therefore, annihilates information held in the individuals of the pair.

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha:\ \alpha = 3 + 4i$	T	F	F

	$\beta \in \mathbb{C}$	$\beta \in \mathbb{R}$	$\beta \in \mathbb{Q}$
$\exists\beta:\ \beta = 3 - 4i$	T	F	F

	$\beta \in \mathbb{C}$	$\beta \in \mathbb{R}$	$\beta \in \mathbb{Q}$
$\exists\gamma:\ \gamma = \alpha\beta$	T	T	T

Table 6 Truth-tables showing the *independent* existence of a complex-conjugate pair and *dependent* existence of their ARITHMETICAL product.

This next example throws up a rather peculiar concept. The proposition $\exists y\left(y^2 = -\left(x^{\mathbb{Q}}\right.\right.$ is independent of AXIOMS. Nonetheless, its limiting case tends toward a theorem. There is the suggestion here that possibly in the real world of Heisenberg's uncertainty, which is a finite discrepancy, propositions could conceivably become theorems before the limit is reached, in the perfect mathematical sense.

	$y \in \mathbb{C}$	$y \in \mathbb{R}$	$y \in \mathbb{Q}$
$\exists y:\ y^2 = -\left(x^{\mathbb{Q}}\right)^2$	T	F	F
$\exists y:\ \lim_{x\to 0}\left[y^2 = -\left(x^{\mathbb{Q}}\right)^2\right]$	T	$\mathsf{F} \to \mathsf{T}$	$\mathsf{F} \to \mathsf{T}$

Table 7 A proposition approaching theorem status.

8 Analysis and limits in ARITHMETIC

In later sections, when considering ARITHMETIC infected with unitarity, we shall be very interested in identifying certain logically independent formulae from amongst a general environment of theorems. It will be helpful to avoid distractions of logical independence, having nothing to do with the argument. To that end, this section prepares a logically clean environment of theorems.

Any first-order proposition is a theorem if-and-only-if there exists a proof that *terminates* [7, Unit 5, p17][6, p183]. And so no formula in ARITHMETIC, requiring a never-ending derivation of infinitely many steps, can be a theorem. From the viewpoint of provability in ARITHMETIC, this means certain definitions in analysis need special consideration. Of particular interest are the exponential function, the derivative and the integral: all constructs entailing limiting processes that never 'halt'. Hence, through these constructs, quantum mathematics inherits a muddied logical environment.

Here, alternative definitions are given, relying on constructs that *do* terminate. The general idea is truncation of *infinite* series to 'long' series that are *finite,* specifically designed to spawn errors, only when they are physically imperceptible. Along with these definitions, insistence that polynomial coefficients be *rational* ensures all functions are finite polynomials, derivable purely from AXIOMS of Table 1. Conveniently, the theory's notation reads unchanged.

Combined application of AXIOMS M0 and A0 derives the sequence of theorems

$$\forall\alpha_0,\ldots,\forall\alpha_n\forall x\exists P_n: \quad P_n(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2+\ldots+\alpha_n x^n \tag{29}$$

asserting existence for all finite polynomials of every degree $n=1,2,\ldots$. Now the fact that (29) is a theorem should not be understood to imply, for example, that: $\forall x\exists P\,(P(x)=2+ix)$ is automatically proved. This would need additional theorems stating existence for numbers 2 and i. Existence of 2 can be proved, but there is no theorem for existence of i. To be certain that some *particular* polynomial exists by theorem, we must firstly be sure that all its coefficients exist by theorem. So, because every rational scalar exists in logical consequence of AXIOMS, assignment of particular rationals $\alpha_0^{\mathbb{Q}},\alpha_1^{\mathbb{Q}},\ldots,$ to variables $\alpha_0,\alpha_1,\ldots$ results in all finite polynomials with rational coefficients, existing in logical consequence of AXIOMS:

$$\forall x\exists P_n: \quad P_n(x)=\alpha_0^{\mathbb{Q}}+\alpha_1^{\mathbb{Q}}x+\alpha_2^{\mathbb{Q}}x^2+\ldots+\alpha_n^{\mathbb{Q}}x^n \tag{30}$$

The α_i in (29) are logical objects. They are valueless, bound (dummy) variables. In contrast, the $\alpha_i^{\mathbb{Q}}$, in (30), are particular values. As an illustration of bound variables: in writing the equation: $\alpha+\beta=\beta+\alpha$, an algebraic property is implied and the informal use of *bound variables* is invoked. The formal version explicitly shows use of the $\forall$ quantifier, thus:

$$\forall\alpha\forall\beta\,(\alpha+\beta=\beta+\alpha)\,.$$

Quantifiers $\forall\alpha$ and $\forall\beta$ apply to every occurrence of α and β within the brackets. A formula where every variable is bound is known as a *sentence.* An example of a formula that is not a sentence is:

$$\forall\beta\exists\alpha\,(\alpha=\beta+\vartheta)\ .$$

In this, ϑ is not bound but is a *free variable.* It is free to be substituted by a particular value.

8.1 The transcendental function

Through the above stratagem of employing sequences of theorems, partial sums $P_0,P_1,\ldots,$ can be constructed, converging on any given transcendental function Φ whose coefficients are known rationals: $\alpha_0^{\mathbb{Q}},\alpha_1^{\mathbb{Q}},\ldots$. With no limitation on accuracy, by choosing partial sum P_N of conveniently large degree N, this method approximates any given transcendental function, to a logically dependent finite polynomial.

Definition 1 The transcendental function:

$$\Phi(x)=P_N(x)=\alpha_0^{\mathbb{Q}}+\alpha_1^{\mathbb{Q}}x+\alpha_2^{\mathbb{Q}}x^2+\ldots+\alpha_N^{\mathbb{Q}}x^N$$

where N is sufficiently large that the discrepancy: $P_{N+1}(x)-P_N(x)=\alpha_{N+1}^{\mathbb{Q}}x^{N+1}$ becomes physically imperceptible for all x.

8.2 The exponential

Choice of rational values: $\alpha_0^{\mathbb{Q}}=1$, $\alpha_1^{\mathbb{Q}}=1$, $\alpha_2^{\mathbb{Q}}=1/2$, $\ldots$, $\alpha_n=1/n!$ specifies the sequence of theorems asserting existence of partial sums $E_0,E_1,\ldots,$ exemplified by:

$$\forall x\exists E_n: \quad E_n(x)=1+x+\frac{x^2}{2}+\ldots+\frac{x^n}{n!}.$$

Definition 2 The exponential function:

$$\exp(x) = E_N(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^N}{N!} \tag{31}$$

where N is sufficiently large that the discrepancy: $E_{N+1}(x) - E_N(x) = \frac{x^{N+1}}{(N+1)!}$ becomes physically imperceptible for all x.

8.3 The derivative

For a transcendental Φ from Definition 1, the sequence of theorems asserting existence of ratios $D_0^x[\Phi(x)], D_1^x[\Phi(x)], \ldots,$ exemplified in:

$$\forall x \forall \Phi \exists D_n^x : \quad D_n^x[\Phi(x)] = \frac{\Phi(x + \epsilon_n) - \Phi(x)}{\epsilon_n}$$

converges on the derivative, for $\epsilon_n > \epsilon_{n+1} > 0$.

Definition 3 The derivative D_N:

$$\frac{d}{dx}\Phi(x) = D_N^x[\Phi(x)] = \frac{\Phi(x + \epsilon_N) - \Phi(x)}{\epsilon_N}$$

with $\epsilon_N = (1/2)^N$ and N sufficiently large that the discrepancy: $D_{N+1}^x[\Phi(x)] - D_N^x[\Phi(x)]$ becomes physically imperceptible for all x.

8.4 The Integral

Definition 4 The integral

If

$$\forall x : \; f(x) = \frac{d}{dx}\Phi(x)$$

then

$$\int_{\mathsf{x}_1}^{\mathsf{x}_2} f(\mathsf{x})\, d\mathsf{x} = \Phi(\mathsf{x}_2) - \Phi(\mathsf{x}_1)$$

where x is the bound variable on the integral and x_1 and x_2 are particular values.

Part II: Arithmetic infected with unitarity

9 How ARITHMETIC accommodates quantum theory

I have been discussing ARITHMETIC as a mathematical system whose informational content is precisely that of AXIOMS in Table 1, free of any ingress of information originating in other areas of mathematics, physics or elsewhere. Moving away from this situation of isolation, I now consider ARITHMETIC as an environment *altered* by quantum theory. I shall demonstrate that quantum theory carries information, materialising in ARITHMETIC as structures conveying the *middle, indeterminate* value of ARITHMETIC'S 3-valued logic, carried in the factor $i = \sqrt{-1}$.

To demonstrate this, an experiment is conducted in *derivability*, tracing sources of information that must enter ARITHMETIC before the various formulae of quantum mathematics may be written. The scenario for the experiment is as follows. Postulates of quantum theory are dismissed and play no part whatever. Instead, to begin, ARITHMETIC is initialised by positing AXIOMS. Adoption of these AXIOMS represents the emplacement of a definite set of information asserting a definite set of theorems. The line pursued then, aims to replicate quantum mathematics, written as theorems, derived purely from AXIOMS, but with the expectation that a point will be discovered, revealing ingress of an extra item of information, that raises ARITHMETIC'S informational content to a state exceeding that of AXIOMS.

Evidence of gain and loss of information may be of interest to entropy theorists.

Results of this experiment show that only a single item of extra information is needed and its ingress, as it enters arithmetic, goes unopposed, as no contradiction with AXIOMS ensues. The extra item of information concerns superpositions in a wave packet. In the pure, isolated ARITHMETIC, AXIOMS *cause* a pair of superpositions to exist for a pair of complimentary variables, such as momentum–position. But these are not the wave-like superpositions familiar in quantum mechanics; AXIOMS cannot *cause* those. The ones *caused* are very general in character; they are general linear combinations of 'somewhat general' basis vectors. This general form is the condition of the caused superpositions, viewed as two non-interacting *individuals.* But acting as a coexistent *pair*, ***cross-substitution*** between one and the other is unpreventable, though not caused. And through this mechanism, existence of definite, wave-like, quantum superpositions occurs without cause – at the expense of a spontaneously born, implied, *unitar*y ontology.

Cause – Where I speak of AXIOMS either *causing* or *preventing* a condition, my meaning is that AXIOMS prove or negate it.

The cross-substitution constitutes a logical circularity or self-reference. The *fact* of this self-reference is new information, manifesting in ARITHMETIC as unitary structures, and more visibly, as *the square-root of minus one* – the three then: self-reference, unitarity and the square root, all logically independent of AXIOMS.

The fact that information can enter a theory 'inadvertantly' and then impose implications on that theory is strange, but that is the essence of the discovery.

The whole mechanism swings around the scalar product, as I now explain. The scalar product is axiomatised not in AXIOMS of ARITHMETIC, but in Linear Algebra. And yet, profoundly, the scalar product is free to form inadvertantly, in ARITHMETIC, without axiomatisation from Linear Algebra. To see this, it is helpful to notice a weakness in thinking of function vector-spaces, using extrapolated ideas of classical vector spaces, such as 3-space; the logical form of 3-space is different and unrepresentative. In the case of 3-space, unit vectors are assumed to exist, *a priori*; mathematically, they are non-existent without definition (or axiom). But in the case of function spaces, basis vectors exist already as combinations in ARITHMETIC. And so, a theory of 3-space has informational content of AXIOMS *plus* definitions of the unit vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$, whereas function spaces require information from AXIOMS only. When scalar products form in theories of 3-space, orthogonality is axiomatic in definitions of $\mathbf{x}, \mathbf{y}, \mathbf{z}$. But in scalar products forming between function spaces, orthogonality is possible through accident.

And so, concisely, the overall scenario is thus. Information enters quantum theory as two items. Firstly, there is the package of information constituting the AXIOMS. This causes existence of function vector spaces, free of a scalar product; we know these as Banach spaces. Independently, the second item enters: the fact of circularity. This allows Banach spaces to couple into pairs and cross-substitute. In successful instances, pairs skew into 'dual-spaces', as a result of implicit unitary scalings – these are the Hilbert spaces. The Banach spaces are essentially rational, whereas the Hilbert spaces are essentially imaginary. As we have seen in previous sections, rational and imaginary structures are logically distinct. Expression of both enables a quantum theory to convey its full logic.

Note: non-unitary Banach spaces adequately and faithfully represent quantum eigenspaces. Hilbert space is not needed for computation of quantum eigenvalues.

In the region of the theory where the scalar product exists, (inadvertantly!), ARITHMETIC guarantees the whole unitary package: orthogonality, preserved probability amplitude, self-adjoint operators, Hilbert space and complex scalars. To mark its containment within an otherwise rational theory, I call this the ***unitary fragment***.

10 The evidence

In order to prove my claim, that ARITHMETIC generates the logic of quantum experiments, I show that ARITHMETIC, essential to quantum mathematics, is responsible for a 3-valued logic, agreeing with the 3-valued logic of Reichenbach.

Evidence shows this logic *is* present in standard quantum theory, but hidden; that the *indeterminate* value of Reichenbach's logic is there, but obscured. In deriving the quantum-ARITHMETIC, evidence shows that actually unitarity arises freely, as component part of the logical mechanism. But believing it necessary, the quantum theorist has ***imposed*** blanket unitarity, covering the whole theory, that obliterates the logic.

Whilst there is no argument in favour of imposing unitarity, there is conclusive evidence against. Sections below consider derivation for the free-particle. From first principles, assuming Homogeneity of Space, working through the Canonical Commutation Relation, representation by operators, eigensolutions, their superpositions, finally wave packets are derived. Through the whole of this, unitarity is found to be *redundant* except in the wave packet, where it arises unpreventably, without cause, as part of ARITHMETIC. The fact wave packets existence prior to measurement shows agreement between this 'unitary-logic' and Reichenbach.

The following list is an outline of the evidence.

1. **Sections 5 and 6** confirm existence of imaginary-i in ARITHMETIC, as logically independent of AXIOMS.
   - ▷ Soundness and Completeness Theorems, in combined action, exclude its logical dependence.
2. **Section 11** shows that imaginary-i originates outside ARITHMETIC, in Linear Algebra.
   - ▷ Generally, for 3+ dimensions, orthogonality implies existence of imaginary-i.
3. **Section 12** confirms that Homogeneity of Space is non-unitary.
   - ▷ Specifically, the (unitary) Canonical Commutation Relation is shown incorrect algebra if the *whole information* of homogeneity is to be conveyed.
   - ▷ Instead a non-unitary super-algebra of the Canonical Relation, not requiring existence of imaginary-i, is established as the genuine commutator.
4. **Section 12.** Deposing complex arithmetic, ARITHMETIC under AXIOMS is confirmed correct foundation for wave mechanics.
   - ▷ Specifically, assuming the most general homogeneity as foundation for wave mechanics, there is no demand for any existence of imaginary-i.
   - ▷ Hence, Proposition (5) is not an axiom.
5. **Section 13** shows wave packets, written as formulae in ARITHMETIC under AXIOMS, are *consistent* with general homogeneity.
6. **Sections 16** and **17** show that a wave packet is a purely ARITHMETICAL object and profoundly, comprises information of two logical qualities. Specifically:
   - ▷ superpositions, *caused and implie*d by AXIOMS,
   - ▷ unitarity, *neither causable nor preventable* by AXIOMS, inadvertantly arising out of self-reference, consistent with AXIOMS.

11 How imaginary-*i* originates in Linear Algebra

The following is a proof adapted from the work of W E Baylis, J Huschilt and Jiansu Wei [3]. It shows that the square-root of minus one arises in logical consequence of orthogonality in any vector space of 3+ dimensions[8].

[8] The notion of dimensionality is complicated by, for instance, the fact that the su2 Lie algebra is 3-dimensional but has for its basis, 2-dimensional matrices.

Assume existence of a vector space with independent basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, ... and further assume orthogonality embodied in their products, thus:

$$\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = \mathbf{0} \tag{32}$$
$$\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 = \mathbf{0} \tag{33}$$
$$\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3 = \mathbf{0} \tag{34}$$
$$\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_3 = \mathbf{1} \tag{35}$$

where $\mathbf{0}$ and $\mathbf{1}$ are linear operators such that $\mathbf{e}_i + \mathbf{0} = \mathbf{e}_i$ and $\mathbf{e}_i\mathbf{1} = \mathbf{e}_i$.

By (32) and (35):

$$\begin{aligned} \mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_1 \\ \Rightarrow \mathbf{e}_1\mathbf{e}_3\mathbf{e}_3\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_1 \\ \Rightarrow \mathbf{e}_1\mathbf{e}_3 &= -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \ . \end{aligned} \tag{36}$$

And similarly, by (33) and (35),

$$\begin{aligned} \mathbf{e}_3\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_1 &= -\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 \ . \end{aligned} \tag{37}$$

Adding (36) and (37) gives:

$$\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3 = -\left(\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right) \ .$$

And substituting (34) results in:

$$\begin{aligned} \mathbf{0} &= \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ \Rightarrow \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= -\mathbf{1} \\ \Rightarrow \left(\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\right)^2 &= (-1)\,\mathbf{1} \qquad (38) \\ \Rightarrow \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 &= \pm i\mathbf{1} \ . \qquad (39) \end{aligned}$$

Remark 1 Introduction of $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_6, \ldots$, adds nothing more of interest.

Remark 2 In this proof, there is no reliance on non-commutativity. Though less obvious, its validity for the commuting case can be visualised through the limiting case, as $\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i \to \mathbf{0}$

12 Tracing the origins of unitarity – eliminating homogeneity of space

The *Canonical Commutation Relation* (55) embodies core algebra at the heart of wave mechanics. It's purported significance is representation of the homogeneity of space, and this is accepted by quantum theorists as unitary. In the section here, I re-examine the Canonical Relation's derivation to provide evidence that, contrary to this accepted view, homogeneity is *not* generally unitary. And in consequence show that the Canonical Commutation Relation does not simply and exactly represent homogeneity but contains other information also.

Imposing homogeneity on a system is identical to imposing null effect under arbitrary translation of reference frame. The principle we invoke is *form invariance*: this is the the concept from relativity that a symmetry transformation leaves formulae fixed in *form*, though *values* may alter [25]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position:

$$\mathbf{x}\left|f_x\right\rangle = x\left|f_x\right\rangle . \tag{40}$$

With form held fixed as the reference system is displaced, variation in the position operator $\mathbf{x}$ determines a group relation representing the homogeneity symmetry. Under arbitrarily small displacements, this group relation corresponds to homogeneity's Lie algebra.

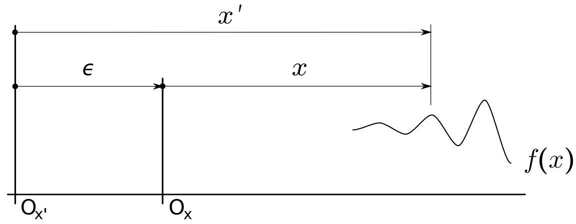

Figure 2 Passive translation of a function Two reference systems, O_x and $\mathsf{O}_{\mathsf{x}'}$, arbitrarily displaced by ϵ, act individually as reference for position of a function f. If space along x is homogeneous, physics concerning this function is described by formulae, of the same form, both for O_x and $\mathsf{O}_{\mathsf{x}'}$. **Note:** The function and reference frames are not epistemic; f is non-observable and O_x and $\mathsf{O}_{\mathsf{x}'}$ are not observers.

To maintain the form of (40), under translation, the basis $|f_x\rangle$ is cleverly managed. While translation transforms the basis from $|f_x\rangle$ to $|f_{x-\epsilon}\rangle$, a similarity transformation is applied also, chosen to revert $|f_{x-\epsilon}\rangle$ back to $|f_x\rangle$. This works only for *real* ϵ, due to the nature of the similarity transform, and if conceived as *rational,* existence of ϵ is attributable to AXIOMS. Then happily, for a continuum of translations there exists a corresponding continuum of similarity transformations. Acting together, they hold $|f_x\rangle$ static, with translation and similarity transformation, both, very nicely, parameterised by the same single parameter ϵ. The scheme of transformations is depicted in (41). The bottom left hand formula is the resulting group relation.

$$
\begin{tikzcd}
\mathbf{x}\,|f_x\rangle = x\,|f_x\rangle \arrow[r, "\text{translation}", "O_\mathsf{x}\to O_{\mathsf{x}'}"'] \arrow[d] & \mathbf{x}\,|f_{x-\epsilon}\rangle = (x+\epsilon)\,|f_{x-\epsilon}\rangle \arrow[d, "|f_{x-\epsilon}\rangle\to|f_x\rangle"', "\text{similarity}"] \\
\left(\mathsf{SxS}^{-1}-\epsilon\mathbf{1}\right)|f_x\rangle = \mathbf{x}\,|f_x\rangle & \left(\mathsf{SxS}^{-1}-\epsilon\mathbf{1}\right)|f_x\rangle = x\,|f_x\rangle \arrow[l]
\end{tikzcd}
\tag{41}
$$

Now, in textbook theory, S is understood to be *necessarily* unitary, for the reason that any theory must preserve invariance of the scalar product [17,18]. And so, because it is needed for probabilistic reasons, unitarity is imposed, axiomatically, at this point in theory, as additional information, restricting the similarity transformation and the homogeneity symmetry.

Use of quantifiers: Quantifier notation eliminates ambiguity suffered in ordinary equations. To illustrate: the equation $y = x^2$ doesn't express which of $\forall y \exists x \left(y = x^2\right)$ or $\forall x \exists y \left(y = x^2\right)$ is intended. Quantified bound variables express algebraic information and may not be substituted by particular values; if they were, information specifying algebra would be lost.

As an experiment, we proceed in this paper, by treating unitarity as a purely separate issue from homogeneity and allow S it's widest generality so that homogeneity's *whole information* is genuinely conveyed through the theory. The experiment begins with the eigen-equation for position (40) being rewritten, in notation of first-order logic, as the *eigenformula* in proposition (42). Where $\mathbf{x}$ is a rational linear operator: that is, a linear combination of rational variables, and f and x as rational numbers, (42) can be accepted as a provable theorem of AXIOMS.

All informal assumptions are to be shed; the Dirac notation is dropped to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or forming scalar products; none of these is implied. The only information present is the homogeneity symmetry and the set of AXIOMS listed in Table 1. Information held in the fact that the eigenformula *is* an eigenformula is a filter accepting only *persistent* information from AXIOMS.

Significance of eigenformulae A possible interpretation for eigenformulae is they are propositions that support stable, persistent existence for certain functions. (42) can be considered as the mapping:

$$\frac{\mathbf{x}}{x} f\left(x\right) \mapsto f\left(x\right).$$

Action of the operator $\mathbf{x}/x$ on $f\left(x\right)$ returns $f\left(x\right)$ itself.

Consider the eigenformula for position x of function f, corresponding to reference frame O_x

$$\exists \mathbf{x} \exists x \exists f : \quad \mathbf{x} f\left(x\right) = x f\left(x\right) \tag{42}$$

Translation: Homogeneity demands existence of an equally relevant, further reference frame $\mathsf{O}_{\mathsf{x}'}$ displaced arbitrarily through ϵ. **This includes *complex* displacements.** Under this displacement the principle of relativity guarantees a formula for $\mathsf{O}_{\mathsf{x}'}$ of the same form as that for O_x, in (42), thus:

$$\exists \mathbf{x} \exists x' \exists f' : \quad \mathbf{x} f'\left(x'\right) = x' f'\left(x'\right) \tag{43}$$

The translation transforms position, thus:

$$\forall\epsilon\forall x\exists x' : \quad x \mapsto x' = x + \epsilon \tag{44}$$

and the function, so:

$$\forall\epsilon\forall x\forall f\exists f'\exists x' : \quad f(x) \mapsto f'(x') = f(x-\epsilon) \tag{45}$$

Substituting these into (43):

$$\exists\epsilon\exists\mathbf{x}\exists x\exists f : \quad \mathbf{x}f(x-\epsilon) = (x+\epsilon)f(x-\epsilon) . \tag{46}$$

Similarity: The similarity transformation is now applied. Operator S is a member of the general linear group $\mathsf{GL}(n)$. For vector space, dimension n, there exists an operator $\mathsf{S} \in \mathsf{GL}(n)$, transforming the set of basis vectors. Members of the *one-parameter subgroup* $\mathsf{S}_{(\epsilon)} \subset \mathsf{S} \in \mathsf{GL}(n)$ perform all transformations that correspondingly match the full set of translations. We may write:

$$\forall f\forall x\exists\epsilon\exists\mathsf{S} : \quad \mathsf{S}^{-1}_{(\epsilon)}f(x) = f(x-\epsilon) . \tag{47}$$

Substituting this into (46) the similarity transformation is thus formed:

$$\exists\epsilon\exists\mathbf{x}\exists x\exists f\exists\mathsf{S} : \quad \mathsf{S}_{(\epsilon)}\mathbf{x}\mathsf{S}^{-1}_{(\epsilon)}f(x) = (x+\epsilon)f(x) .$$

Introducing the trivial eigenvalue equation: $\forall f\forall x : \epsilon\mathbf{1}f(x) = \epsilon f(x)$ and subtracting:

$$\exists\epsilon\exists\mathbf{x}\exists x\exists f\exists\mathsf{S} : \quad \left(\mathsf{S}_{(\epsilon)}\mathbf{x}\mathsf{S}^{-1}_{(\epsilon)} - \epsilon\mathbf{1}\right)f(x) = xf(x) . \tag{48}$$

Now comparing the original position eigenvalue equation (42) against (48), we deduce the group relation for homogeneity:

There may seem apparent risk that $\exists f$ in (42) might not match (equal) the $\exists f$ in (48), but note: the translation and similarity were *designed* for the purpose of matching one f with the other
.

$$\exists\epsilon\exists\mathbf{x}\exists x\exists f\exists\mathsf{S} : \quad \mathbf{x}f(x) = \left(\mathsf{S}_{(\epsilon)}\mathbf{x}\mathsf{S}^{-1}_{(\epsilon)} - \epsilon\mathbf{1}\right)f(x) . \tag{49}$$

From this group relation, the commutator for the *Lie algebra* is now computed. Because $\mathsf{S}_{(\epsilon)}$ is a one-parameter subgroup of $\mathsf{GL}(n)$, there exists a unique linear operator $\mathbf{g}$ [15, p 37. but see footnote][9] such that:

$$\forall\mathsf{S}\exists\epsilon\exists\mathbf{g} : \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\epsilon\mathbf{g}} \tag{50}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor η is extracted, thus: $\forall\mathbf{g}\forall\eta\exists\mathbf{k} : \mathbf{g} = \eta\mathbf{k}$, implying:

$$\forall\eta\forall\mathsf{S}\exists\epsilon\exists\mathbf{k} : \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\eta\epsilon\mathbf{k}} \tag{51}$$

$$\forall\eta\forall\mathsf{S}\exists\epsilon\exists\mathbf{k} : \quad \mathsf{S}^{-1}_{(\epsilon)} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\eta\epsilon\mathbf{k}} \tag{52}$$

Substitution of (51) and (52) into (49) gives:

$$\begin{aligned}
&\forall\eta\exists\epsilon\exists x\exists f\exists\mathbf{x}\exists\mathbf{k} : && \exp(+\eta\epsilon\mathbf{k})\,\mathbf{x}\exp(-\eta\epsilon\mathbf{k})\,f(x) = [\mathbf{x}+\epsilon\mathbf{1}]\,f(x)\\
\Rightarrow\ &\forall\eta\exists\epsilon\exists x\exists f\exists\mathbf{x}\exists\mathbf{k} : && \left[\mathbf{1}+\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\mathbf{x}\left[\mathbf{1}-\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]f(x) = [\mathbf{x}+\epsilon\mathbf{1}]\,f(x)\\
\Rightarrow\ &\forall\eta\exists\epsilon\exists x\exists f\exists\mathbf{x}\exists\mathbf{k} : && \left[\mathbf{x}+\eta\epsilon\mathbf{k}\mathbf{x}+\mathcal{O}\left(\epsilon^2\right)\right]\left[\mathbf{1}-\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]f(x) = [\mathbf{x}+\epsilon\mathbf{1}]\,f(x)\\
\Rightarrow\ &\forall\eta\exists\epsilon\exists x\exists f\exists\mathbf{x}\exists\mathbf{k} : && \left[\mathbf{x}+\eta\epsilon\mathbf{k}\mathbf{x}-\eta\epsilon\mathbf{x}\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]f(x) = [\mathbf{x}+\epsilon\mathbf{1}]\,f(x)\\
\Rightarrow\ &\forall\eta\exists\epsilon\exists x\exists f\exists\mathbf{x}\exists\mathbf{k} : && [\mathbf{k}\mathbf{x}-\mathbf{x}\mathbf{k}]\,f(x) = \left[\eta^{-1}\mathbf{1}-\mathcal{O}(\epsilon)\right]f(x)
\end{aligned}$$

After freeing ϵ from its $\exists\epsilon$ quantifier, at the limit, as $\epsilon \to 0$, we have:

$$\forall\eta\exists x\exists f\exists\mathbf{x}\exists\mathbf{k} : \quad [\mathbf{k},\mathbf{x}]\,f(x) = \eta^{-1}\mathbf{1}f(x) \tag{53}$$

And by a similar, symmetric proof for the same commutator, but with eigenfunctions $g(k)$, of $\mathbf{k}$:

$$\forall\eta\exists x\exists f\exists\mathbf{x}\exists\mathbf{k} : \quad [\mathbf{k},\mathbf{x}]\,g(k) = \eta^{-1}\mathbf{1}g(k) . \tag{54}$$

Importantly, we see (53) and (54) is $\forall\eta$, rather than the special case of $\eta^{-1} = -i$, in sub-algebra we know as the Canonical Commutation Relation:

$$[\mathbf{k},\mathbf{x}] = -i\mathbf{1} \quad \text{or} \quad [\mathbf{p},\mathbf{x}] = -i\hbar\mathbf{1} \tag{55}$$

And in conclusion, the above establishes that the homogeneity symmetry, of itself, is not unitary.

[9] This assertion requires further argument; the citation asserts finite dimensional S only.

Note: *Exponentials, derivatives and integrals take definition from Section 8.*

13 Tracing the origins of unitarity – eliminating representation

From here, theory moves away from the purely abstract objects, $\mathbf{k}$ and $\mathbf{x}$, by introducing explicit operators, satisfying the commutator, and furnishing an algebra we know as a 'representation'. The operators are *linear operators* and are essentially arithmetical objects. Crucially however, there exists more than one representation and the different representations convey different information-sets extracted from the original information of the homogeneity symmetry.

In standard theory, the commutator that operators satisfy is the Canonical Relation (55). This has two unitary representations, each a sub-algebra of homogeneity, whose operators are self-adjoint. They are known as the p*osition-space representation* and *wavenumber-space representation.* Respectively:

$$\begin{aligned} &\mathbf{k}f(x) = -i\tfrac{d}{dx}f(x) \qquad \mathbf{x}f(x) = xf(x) \qquad \mathbf{1}f(x) = \tfrac{dx}{dx}f(x)\,; \\ &\mathbf{x}g(k) = i\tfrac{d}{dk}g(k) \qquad \mathbf{k}g(k) = kg(k) \qquad \mathbf{1}g(k) = \tfrac{dk}{dk}g(k) \end{aligned} \tag{56}$$

The new theory parallels this standard approach, but conveys the *whole information* of the homogeneity symmetry by employing operators satisfying the commutators (53) and (54). These new operators are not self-adjoint and furnish a non-unitary algebra. They are deduced as follows. The identity for the derivative of a product, $(uv)' \equiv uv' + u'v$, is invoked, in turn for each of the products: $xf(x)$ and $kg(k)$. Written out formally they form the pair of theorems, deriving purely from AXIOMS:

$$\begin{aligned} \forall\eta\forall x\forall f\exists\frac{d}{dx}: &\qquad \eta\frac{d}{dx}[xf(x)] = \eta\frac{dx}{dx}f(x) + \eta x\frac{d}{dx}f(x) \\ \forall\eta\forall k\forall g\exists\frac{d}{dk}: &\quad -\eta\frac{d}{dk}([kg(k)]) = -\eta\frac{dk}{dk}g(k) - \eta k\frac{d}{dk}g(k) \end{aligned} \tag{57}$$

Rearranged and written as commutators, these become:

$$\forall\eta\forall x\forall f\exists\frac{d}{dx}: \quad \left[\eta\frac{d}{dx}, x\right]f(x) = \eta\frac{dx}{dx}f(x) \tag{58}$$

$$\forall\eta\forall k\forall g\exists\frac{d}{dk}: \quad \left[k, \eta\frac{d}{dk}\right]g(k) = -\eta\frac{dk}{dk}g(k) \tag{59}$$

These prove (53) and (54); providing the two non-unitary representations:

$$\begin{aligned} &\forall\eta\forall f\forall x\exists\tfrac{d}{dx}: \quad \mathbf{k}f(x) = +\eta^{-1}\tfrac{d}{dx}f(x) \qquad \mathbf{x}f(x) = xf(x) \qquad \mathbf{1}f(x) = \tfrac{dx}{dx}f(x) \\ &\forall\eta\forall g\forall k\exists\tfrac{d}{dk}: \quad \mathbf{x}g(k) = -\eta^{-1}\tfrac{d}{dk}g(k) \qquad \mathbf{k}g(k) = kg(k) \qquad \mathbf{1}g(k) = \tfrac{dk}{dk}g(k) \end{aligned} \tag{60}$$

14 Eigenformulae and persistent functions

Assume existence of particular functions $f(x)$ and $g(k)$, that survive under the action of these operators:

$$\forall\eta\forall x\exists f: \quad +\eta^{-1}\tfrac{d}{dx}f(x) \mapsto f(x) \qquad xf(x) \mapsto xf(x) \qquad \tfrac{dx}{dx}f(x) \mapsto f(x) \tag{61}$$

$$\forall\eta\forall k\exists g: \quad -\eta^{-1}\tfrac{d}{dk}g(k) \mapsto g(k) \qquad kg(k) \mapsto kg(k) \qquad \tfrac{dk}{dk}g(k) \mapsto g(k) \tag{62}$$

Then under sustained, repeated action of these operators, existence of $f(x)$ and $g(k)$ is *stable* and *persistent.* The fact of any such stability is expressed in the pair of eigenformulae:

$$\forall\eta\forall x\exists f: \quad +\frac{1}{\eta}\frac{d}{dx}f(x) = f(x) \tag{63}$$

$$\forall\eta\forall k\exists g: \quad -\frac{1}{\eta}\frac{d}{dk}g(k) = g(k) \tag{64}$$

The reason there are just two eigenformulae, rather than six, is that only the first pair of mappings in (61) and (62), restrict $f(x)$ and $g(k)$; the second and third pairs carry no additional information, that is not redundant.

15 Tracing the origins of unitarity – eliminating persistent functions

The line of argument now moves to the objective aim of the paper – tracing the logic connecting AXIOMS with stable functions, consistent with homogeneity. I now start afresh, with AXIOMS as the *only* rules in force. I shall derive information caused by AXIOMS, that is coincidentally, stably consistent with homogeneity. To start the process, in the manner of Section 8, through repeated application of AXIOMS of Table 1, I derive existence of the finite polynomial:

$$\forall y \exists E_N :\ E_N\left(y\right) = 1 + y + \frac{y^2}{2} + \ldots + \frac{y^N}{N!} \tag{65}$$

where N is sufficiently large that the discrepancy: $E_{N+1}\left(y\right) - E_N\left(y\right) = \frac{y^{N+1}}{(N+1)!}$ becomes physically imperceptible for all y. Then using AXIOM M0, I may write:

$$\begin{aligned} &\forall \eta \forall k \forall x \exists y : y = +\eta k x \\ &\forall \eta \forall k \forall x \exists y : y = -\eta k x \end{aligned}$$

Substituting these in turn, into (65):

$$\begin{aligned} &\forall \eta \forall k \forall x \exists E_N :\ E_N\left(+\eta k x\right) = 1 + \eta k x + \frac{\left(\eta k x\right)^2}{2} + \ldots + \frac{\left(\eta k x\right)^N}{N!} \\ &\forall \eta \forall k \forall x \exists E_N :\ E_N\left(-\eta k x\right) = 1 + \left(-\eta\right) k + \frac{\left(-\eta k x\right)^2}{2} + \ldots + \frac{\left(-\eta k x\right)^N}{N!} \end{aligned}$$

Simply relabelling $E_N\left(+\eta k x\right) \rightarrow f\left(x\right)$ and $E_N\left(-\eta k x\right) \rightarrow g\left(k\right)$:

$$\begin{aligned} &\forall \eta \forall k \forall x \exists f :\ f\left(x\right) = 1 + \eta k x + \frac{\left(\eta k x\right)^2}{2} + \ldots + \frac{\left(\eta k x\right)^N}{N!} \\ &\forall \eta \forall k \forall x \exists g :\ g\left(k\right) = 1 + \left(-\eta\right) k + \frac{\left(-\eta k x\right)^2}{2} + \ldots + \frac{\left(-\eta k x\right)^N}{N!} \end{aligned}$$

And utilising Definition 2 of Section 8:

$$\forall \eta \forall k \forall x \exists f :\ f\left(x\right) = \exp\left(+\eta k x\right) \tag{66}$$

$$\forall \eta \forall k \forall x \exists g :\ g\left(k\right) = \exp\left(-\eta k x\right) \tag{67}$$

Now utilising Definition 3 of Section 8, for the derivative:

$$\forall \eta \forall k \forall x \exists f :\ +\frac{1}{\eta}\frac{d}{dx} f\left(x\right) = +\frac{1}{\eta}\frac{d}{dx} \exp\left(+\eta k x\right) \tag{68}$$

$$\forall \eta \forall k \forall x \exists g :\ -\frac{1}{\eta}\frac{d}{dk} g\left(k\right) = -\frac{1}{\eta}\frac{d}{dk} \exp\left(-\eta k x\right) \tag{69}$$

$$\forall \eta \forall k \forall x \exists f :\ +\frac{1}{\eta}\frac{d}{dx} f\left(x\right) = \exp\left(+\eta k x\right) \tag{70}$$

$$\forall \eta \forall k \forall x \exists g :\ -\frac{1}{\eta}\frac{d}{dk} g\left(k\right) = \exp\left(-\eta k x\right) \tag{71}$$

$$\forall \eta \forall k \forall x \exists f :\ +\frac{1}{\eta}\frac{d}{dx} f\left(x\right) = f\left(x\right) \tag{72}$$

$$\forall \eta \forall k \forall x \exists g :\ -\frac{1}{\eta}\frac{d}{dk} g\left(k\right) = f\left(x\right) \tag{73}$$

Thus proving (63) and (64) are derivable as theorems of AXIOMS.

16 Tracing the origins of unitarity – eliminating superpositions

Rational scalars
From here onward, unbound, particular, scalars emerge, notated using sanserif font, thus: $\mathsf{k}, \mathsf{k}', \mathsf{k}_1\mathsf{k}_2,\ \mathsf{x}, \mathsf{x}', \mathsf{x}_1, \mathsf{x}_2, \ldots$
In Section 8, the same variables are denoted: $k^{\mathbb{Q}}, x^{\mathbb{Q}}$ etc., but that notation is inconvenient here. All these scalars are rational, and therefore exist by theorem, introducing no logically independent information. A further point is that the Riemann integral *does* exist on a rational domain.

Freeing the bound variable k of its quantifier $\forall k$ in (72), and x of its quantifier $\forall x$ in (73); then writing these as particular rational scalars, k and x, notated in sanserif font, we have the two theorems:

$$\forall\eta\forall x\exists f_{\mathsf{k}}:\ +\frac{1}{\eta\mathsf{k}}\frac{d}{dx}f_{\mathsf{k}}(x) = \exp(+\eta\mathsf{k}x) \tag{74}$$

$$\forall\eta\forall k\exists g_{\mathsf{x}}:\ -\frac{1}{\eta\mathsf{x}}\frac{d}{dk}g_{\mathsf{x}}(k) = \exp(-\eta k\mathsf{x}) \tag{75}$$

Introducing particular rational scalars k_1,k_2,$\mathsf{k}_3\ldots$ and x_1,x_2,$\mathsf{x}_3\ldots$, then invoking AXIOMS, linear combinations of may be constructed which also exist by theorem. Writing the illustrative example where only 2 dimensions are non-zero:

$$\forall\eta\forall x\forall a_{\mathsf{k}_1}\forall a_{\mathsf{k}_2}\exists f_{\mathsf{k}_1}\exists f_{\mathsf{k}_2}:\quad \frac{1}{\eta}\frac{d}{dx}\left[+\frac{a_{\mathsf{k}_1}}{\mathsf{k}_1}f_{\mathsf{k}_1}(x)+\frac{a_{\mathsf{k}_2}}{\mathsf{k}_2}f_{\mathsf{k}_2}(x)\right]$$
$$= a_{\mathsf{k}_1}\exp(+\eta\mathsf{k}_1x)+a_{\mathsf{k}_2}\exp(+\eta\mathsf{k}_2x) \tag{76}$$

$$\forall\eta\forall k\forall b_{\mathsf{x}_1}\forall b_{\mathsf{x}_2}\exists g_{\mathsf{x}_1}\exists g_{\mathsf{x}_2}:\quad \frac{1}{\eta}\frac{d}{dk}\left[-\frac{b_{\mathsf{x}_1}}{\mathsf{x}_1}g_{\mathsf{x}_1}(k)-\frac{b_{\mathsf{x}_2}}{\mathsf{x}_2}g_{\mathsf{x}_2}(k)\right]$$
$$= b_{\mathsf{x}_1}\exp(-\eta k\mathsf{x}_1)+b_{\mathsf{x}_2}\exp(-\eta k\mathsf{x}_2) \tag{77}$$

And for J non-zero dimensions:

$$\forall\eta\forall x\forall a_{\mathsf{k}_1},\ldots,\forall a_{\mathsf{k}_J}\exists f_{\mathsf{k}_1},\ldots,\exists f_{\mathsf{k}_J}:\ \frac{1}{\eta}\frac{d}{dx}\left[\sum_{j=1}^{J}\frac{a_{\mathsf{k}_n}}{\mathsf{k}_n}f_{\mathsf{k}_n}(x)\right]=\sum_{j=1}^{J}a_{\mathsf{k}_n}\exp(+\eta\mathsf{k}_nx)$$

$$\forall\eta\forall k\forall b_{\mathsf{x}_1},\ldots,\forall b_{\mathsf{x}_J}\exists g_{\mathsf{x}_1},\ldots,\exists g_{\mathsf{x}_J}:\ -\frac{1}{\eta}\frac{d}{dk}\left[\sum_{j=1}^{J}\frac{b_{\mathsf{x}_n}}{\mathsf{x}_n}g_{\mathsf{x}_n}(k)\right]=\sum_{j=1}^{J}b_{\mathsf{x}_N}\exp(-\eta k\mathsf{x}_n)$$

And so, for the entire continuous case:

Notation: $\int_{d\mathsf{k}} f(\mathsf{k},x) \equiv \int\limits_{\text{all}\,\mathbb{Q}} f(\mathsf{k},x)\,d\mathsf{k}$

$$\forall\eta\forall x\forall a\exists f:\quad \frac{1}{\eta}\frac{d}{dx}\left[\int_{d\mathsf{k}}\frac{a(\mathsf{k})}{\mathsf{k}}f(\mathsf{k},x)\right]=\int_{d\mathsf{k}}a(\mathsf{k})\exp(+\eta\mathsf{k}x) \tag{78}$$

$$\forall\eta\forall k\forall b\exists g:\quad -\frac{1}{\eta}\frac{d}{dk}\left[\int_{d\mathsf{x}}\frac{b(\mathsf{x})}{\mathsf{x}}g(k,\mathsf{x})\right]=\int_{d\mathsf{x}}b(\mathsf{x})\exp(-\eta k\mathsf{x}) \tag{79}$$

But this continuous case is not so straightforward; (78) and (79) contradict AXIOM M2. Specifically, existences of the sums (integrals) on the right require:

$$\forall\eta\forall x\forall a\exists\Psi:\quad \Psi(x)=\int_{d\mathsf{k}}a(\mathsf{k})\exp(+\eta\mathsf{k}x) \tag{80}$$

$$\forall\eta\forall k\forall b\exists\Phi:\quad \Phi(k)=\int_{d\mathsf{x}}b(\mathsf{x})\exp(-\eta k\mathsf{x}) \tag{81}$$

but examples exist that break the $\forall a\exists\Psi$ and $\forall b\exists\Phi$ quantifier combinations. This is because the integral is over an unbounded domain, and in the case of these examples, there are only unbounded, non-converging sums. To illustrate, $a(\mathsf{k})=1$ or $b(\mathsf{x})=1$ furnish integrals that never converge, implying non-existence of Ψ or Φ. Essentially, this type of integral sum, with $\forall a$ or $\forall b$, cannot be defined in a way for which there is a derivation that terminates. And with no prospect of derivability, (80) and (81) are *inconsistent* with AXIOMS.

▷ **Result** This *inconsistency* is fundamental for quantum theory; it prohibits all wave functions unless square integrable. As a result, the well known principle requiring square integrability is reducible to a result, implied by AXIOMS.

We proceed with the pair of weaker, existential assertions, which *are* theorems:

$$\forall\eta\forall x\exists a\exists\Psi:\quad \Psi(x)=\int_{d\mathsf{k}}a(\mathsf{k})\exp(+\eta\mathsf{k}x) \tag{82}$$

$$\forall\eta\forall k\exists b\exists\Phi:\quad \Phi(k)=\int_{d\mathsf{x}}b(\mathsf{x})\exp(-\eta k\mathsf{x}) \tag{83}$$

In these, existential quantifiers $\exists a$ and $\exists b$ replace universal quantifiers $\forall a$ and $\forall b$. As *both* (82) and (83) are theorems, it follows they are mutually consistent, with $\Phi(k)$ and $\Psi(x)$ *coexistent*.

17 Tracing the origins of unitarity – Self-reference

I now explore the possibility of theorems (82) and (83) accepting information, circularly, from one another, through a mechanism where $\Phi(k)$ feeds $a(\mathsf{k})$ and $\Psi(x)$ feeds $b(\mathsf{x})$. There is no *cause* implying this self-reference; the idea is that nothing prevents it.

Elemér E Rosinger and Gusti van Zyl also connect self-reference, orthogonality and independence [23].

To proceed, the strategy followed will be to posit a hypothesis that such self-reference does occur, then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

Circularity Hypothesis:

$$\forall a \exists \Phi : \; a = \Phi; \tag{84}$$

$$\forall b \exists \Psi : \; b = \Psi. \tag{85}$$

If these assumptions are substituted into (82) and (83) we get:

$$\forall \eta \forall x \exists \Phi \exists \Psi : \quad \Psi(x) = \int_{\mathsf{dk}} \Phi(\mathsf{k}) \exp(+\eta \mathsf{k} x) \tag{86}$$

$$\forall \eta \forall k \exists \Psi \exists \Phi : \quad \Phi(k) = \int_{\mathsf{dx}} \Psi(\mathsf{x}) \exp(-\eta k \mathsf{x}) \tag{87}$$

and that allows cross-substitution of Φ and Ψ, invoking a simultaneous pair of propositions, which together, will force particular values on η. Before the pair can be considered as simultaneous, in order to preserve validity, the repeated $\forall \eta$ quantifier must be lost, leaving the particularised (bold) $\boldsymbol{\eta}$. Substituting (87) into (86) and (86) into (87), we get:

$$\forall x \exists \Psi : \quad \Psi(x) = \int_{\mathsf{dk}} \left(\int_{\mathsf{dx}} \Psi(\mathsf{x}) \exp(-\boldsymbol{\eta} \mathsf{kx}) \right) \exp(+\boldsymbol{\eta} \mathsf{k} x) \tag{88}$$

$$\forall k \exists \Phi : \quad \Phi(k) = \int_{\mathsf{dx}} \left(\int_{\mathsf{dk}} \Phi(\mathsf{k}) \exp(+\boldsymbol{\eta} \mathsf{kx}) \right) \exp(-\boldsymbol{\eta} k \mathsf{x}) \tag{89}$$

Simultaneous propositions
Illustrating. Taking the two propositions:

$$\forall x : \; \mathsf{y} = \mathsf{a} x + \mathsf{b}$$
$$\forall x : \; \mathsf{y} = \mathsf{c} x + \mathsf{d}$$

If these are to be solved simultaneously, the *repeated* $\forall x$ must lost, with instances of x from each formulae, being particularised first. Their joint solution then:

$$\mathsf{ax} + \mathsf{b} = \mathsf{cx} + \mathsf{d}$$

where x is the *particular value* variable.

Tidying up, with reversed ordering of integrals:

$$\forall x \exists \Psi : \quad \Psi(x) = \int_{\mathsf{dx}} \Psi(\mathsf{x}) \int_{\mathsf{dk}} \exp[\boldsymbol{\eta}(x - \mathsf{x}) \mathsf{k}] \tag{90}$$

$$\forall k \exists \Phi : \quad \Phi(k) = \int_{\mathsf{dk}} \Phi(\mathsf{k}) \int_{\mathsf{dx}} \exp[-\boldsymbol{\eta}(k - \mathsf{k}) \mathsf{x}] \tag{91}$$

The integrals over the exponentials, exist only when $\boldsymbol{\eta}$ is pure imaginary.

18 Unitarity

Currently, and up to this point, no imaginary scalars exist because there is no source of such information in the theory. Existence of imaginary-i must be hypothesised, logically independently of AXIOMS.

Existence Hypothesis – for the square-root of minus one:

$$\exists i : \; i^2 = -1$$

Setting the *particular* number $\mathsf{i} = \sqrt{-1}$ and also $\boldsymbol{\eta} = \mathsf{is}$, where s is rational, we may write the following pair of formulae – consistent with axioms, but logically independent of them:

$$\forall x \exists \Psi : \quad \Psi(x) = \int_{\mathsf{dx}} \Psi(\mathsf{x}) \int_{\mathsf{dk}} \exp[+\mathsf{is}(x - \mathsf{x}) \mathsf{k}] \tag{92}$$

$$\forall k \exists \Phi : \quad \Phi(k) = \int_{\mathsf{dk}} \Phi(\mathsf{k}) \int_{\mathsf{dx}} \exp[-\mathsf{is}(k - \mathsf{k}) \mathsf{x}] \tag{93}$$

Conclusions

This research set out to discover logical artefacts in mathematical physics that derive and initiate indeterminacy, agreeing with quantum experiment. The main finding tells how indeterminate information is constituted and where, in quantum theory, it is present. But in arriving at these deductions, a hypothesis is proposed concerning arithmetic's place in Nature, demanding we regard arithmetic in physics as a formal, axiomatised theory.

The original question inspiring this research asked whether logical circularity, or *self-reference,* might possibly be present in Nature. And this speculation was reinforced, knowing self-reference is a feature in the proof of Kurt Gödel's Incompleteness Theorems, which guarantee the existence of *non-provable,* but *true* statements in arithmetic. In the language of Mathematical Logic, these statements are *logically independent* of arithmetic's axioms, being neither provable nor negatable. One well-known example is the statement asserting existence of the square root of minus one. And so, given the insistent presence of this number in quantum theory, this statement was taken as entry-point for investigation.

This is the thesis. A hypothesis is posed assuming the proposition: *Axioms of arithmetic exist in Nature*. This viewpoint is the reverse of the ordinary notion that fields of scalars are fundamental in Nature, with arithmetic being an abstraction, encoding their rules of combination. The difference is subtle but profound; instead, *axioms of arithmetic* collectively assert existence for fields of scalars. Arithmetic results, influencing physical processes, including logically independent statements. And these we interpret as logical anomalies in experiments. To gain *logical isomorphism* between quantum theory and experiment, quantum mathematics must view arithmetic as an axiomatised theory, also – as Nature views it.

Treating arithmetic as an axiomatised theory, this paper finds that formulae representing wave packets (prior to measurement) are logically distinct from all other formulae in quantum mathematics. Only these are *essentially* unitary; only these *require* existence of imaginary-i; only these rely on self-reference and only these are logically independent of axioms.

A wave packet consists of a pair of *mutually consistent* superpositions of complimentary variables, such as wave-number and position. The wave packet is unitary, but the superpositions, as individuals, are not. Considering an individual superposition as unitary makes no sense because both must coexist as a pair. To be unitary, a superposition must feed off information offered by its complimentary partner. From each superposition in the pair, there is a flow of information, satisfying a void, deficient in the other. As a sustainable entity, existence of the whole wave packet is dependent on self-reference. At first, the exchanging information might be indefinite, but after repeated cycles of self-reference, information settles toward something definite with square-integrability guaranteed. The self-reference does not contradict axioms, but is consistent with them and therefore is not prevented. It is possible through coincident coexistence of the superpositions.

This artefact of self-reference divides quantum mathematics into two logical partitions: that part of theory, *logically dependent* on axioms, separate from wave packets which are *logically independent:* on the one side, theory *provable* from axioms, and on the other, theory *consistent* with axioms, but not provable – this partitioned theory being interpretable as a ***causeology*** that ***causes*** observables, but ***permits*** different spectral outcomes to result from identically prepared experiments.

This theory posits axioms of arithmetic as profound and fundamental foundation in Nature. It does not tell us the origins of these, but philosophical questions reduce neatly to them. Axioms assert a theory of *existence.* They explain: *persistent, stable existence*; *caused, deterministic existence*; *uncaused, indeterminate existence.* The theory tells us observables are always real because provable existence is always *rational* and it tells us that amplitudes are on the complex plane because wave packets exist unprovably. It dispenses with all possible existence of unbounded superpositions. And uncaused existence is suppressed in classical physics, because there, no scalar product is invoked in arithmetic.

Finally it acts as foundation for the 3-valued logic of Hans Reichenbach which he showed resolves the EPR paradox, the logic of complimentarity and the logic of states, prior to measurement.

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