ON A CLASS OF REDUCIBLE TRINOMIALS

Abstract. In this short note we give an expression for some numbers n such that the polynomial $x^{2p} - nx^p + 1$ is reducible.

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Trinomial reducibility was generally treated by Schinzel[S] but he only mentions in passing in his first theorem the case of a degree 2 factor. Filaseta et al[F] give some easy criteria for reducibility of some trinomials. Bremner[B1] determines all trinomials $x^n + Ax^m + 1$ with irreducible cubic factor. In a more recent work[B2], Bremner and Ulas find reducibility criteria for several trinomials; in particular they show in theorem 4.4 the conditions for factorization of $x^6 + Ax^3 + B$ into three factors of degree 2.

Not many polynomials where only one coefficient is variable allow for a general treatment of their irreducibility. Let $P(m, A) = x^{2m} - Ax^m + 1$, then trivially $A = 2$ makes P reducible for all m. Also, by substitution, if $P(m, A)$ is reducible then so is $P(km, A), k > 1$. The interesting cases are therefore the polynomials $P(p, A)$, with odd prime p.

In summary we have

Theorem 1. *The polynomial $P(p, A) = x^{2p} - Ax^p + 1$ is reducible if*

$$(1) \qquad A = \sum_{0 \le i \le (p-1)/2} (-1)^{\left(i + \frac{p-1}{2}\right)} \frac{p}{2i+1} \binom{i + \frac{p-1}{2}}{2i} k^{2i+1}, \quad k = 1, 2, 3 \ldots$$

In particular,

$$(2) \qquad x^{2p} - Ax^p + 1 = (x^2 - kx + 1) \cdot Q(k, p),$$

with Q a palindromic polynomial having coefficients from a subset of the values of the Lucas-type sequence defined by $a_{i+2} = ka_{i+1} - a_i$, $a_1 = 0$, $a_2 = 1$.

An example would be $p = 5$, $k = 3$:

$$x^{10} - 123x^5 + 1 = (x^2 - 3x + 1)(x^8 + 3x^7 + 8x^6 + 21x^5 + 55x^4 + 21x^3 + 8x^2 + 3x + 1).$$

Integer sequences a_i that satisfy linear recurrences with constant coefficients have the property that $c_i a_i + c_{i+1} a_{i+1} + \ldots + c_{i+h} a_{i+h} = 0$ for some h, and this can be used to construct two polynomials which when multiplied vanish at nearly all coefficients. This paper is concerned with $h = 2$.

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Let a_i a Lucas-type sequence defined by $a_{i+2} = ka_{i+1} - a_i$, or $a_i - ka_{i+1} + a_{i+2} = 0$. Let $s_i, 1 \leq i \leq 2p+1$ a finite palindromic sequence with $s_i = s_{2p+2-i}$ and for $i \leq p+1$, $s_i = a_i$. The term $s_i - ks_{i+1} + s_{i+2}$ is zero for all i except when $i = p+1$, and the process of computing the term $s_i - ks_{i+1} + s_{i+2}$ over all $1 \leq i \leq 2p-1$ is equivalent to multiplying $\sum_i s_i x^{i-1}$ (the polynomial in x with coefficients s_i) by the polynomial $x^2 - kx + 1$, the resulting nonzero terms being x^{2p}, x^p, and 1.

It remains to derive the coefficient $A(k,p)$ of x^p which equals $ks_{p+2} - 2s_{p+1}$. This is equivalent to

$$\begin{aligned} A(k,p) &= ka_{p+1} - 2a_p \\ &= a_{p+2} - a_p \\ &= [z^p]\frac{1-z^2}{1-kz+z^2} \\ &= [z^p]\left(\frac{1-z^2}{1+z^2} \cdot \frac{1}{1-\frac{z}{1+z^2}k}\right) \\ &= \sum_{0<i\leq p} d_{i,p} k^{p-i-1}, \end{aligned}$$

with d the elements of the Riordan array $\mathcal{R}\left(\frac{1-z^2}{1+z^2}, \frac{z}{1+z^2}\right)$, from which the proposition follows (see for example Merlini[M]). As example, $x^{10} - Ax^5 + 1$ is reducible if A is of form $k^5 - 5k^3 + 5k$.

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