Pre-print version manuscript No.
(will be inserted by the editor)

E.P.J. de Haas (Paul)

Towards a 4-D extension of the quantum helicity rotator with a hyperbolic rotation angle of gravitational nature.

First version: July 19, 2013; this version August 30, 2013

Abstract In this paper we present an anachronistic pre-YM and pre-GR attempt to formulate an alternative mathematical physics language in order to treat the problem of the electron in twentieth century physics. We start the construction of our alternative to the Minkowski-Laue consensus by putting spin in the metric. This allows us to simplify Lorentz transformations as metric transformations with invariant coordinates. Using the developed formalism on the Pauli-Dirac level,we expand the quantum helicity operators into helicity rotators and then extend them from the usual 3-D expressions to 4-D variants. We connect the resulting 4-D Dirac-Weyl hyperbolic rotators to mathematical expressions that are very similar to their analogues in the pre General Relativity attempts towards a relativistic theory of gravity. This relative match motivates us to interpret the 4-D hyperbolic rotation angle as possibly gravitational in nature. At the end we apply the 4D hyperbolic rotator to the Dirac equation and investigate how it might change this equation and the related Lagrangian. We are curious to what extend the result enters the realm of quantum gravity and thus might be beyond pre-GR relativistic theories of gravity of Abraham, Nordström, Mie and Einstein.

Keywords Electron · Quaternions · Lorentz transformation · Helicity · Quantum gravity

PACS 01.65.+g · 03.30.+p · 11.30.Cp · 02.40.Gh · 04.60.-m

1 The problem of the electron

The problem of the electron in twentieth century physics was the starting point of the research that lead to this paper. In the early-relativistic or pre-Einstein period, Lorentz, Abraham and Poincaré worked on the electromagnetic electron theory in

E.P.J. de Haas
Nijmegen, The Netherlands
E-mail: haas2u[at]gmail.com

an attempt to make it compatible with classical mechanics [1], [2]. In their approach Abraham, Lorentz and Poincaré used the hypothesis that *all forces should behave as if they were of electromagnetic origin during a global translation of the system. [...] If all forces, including inertial forces, transformed like electromagnetic forces; [...] in order to respect the relativity principle all forces had to transform like electromagnetic forces; that is, according to a representation of the Lorentz group* [1], [3]. But the three couldn't produce a problem free electron theory. Due to the work of Einstein, Minkowski and Laue, the problem of the electron evolved into its modern, relativistic form as the problem of the non-zero divergence of the electron's self stress energy density tensor [4], [5], [6], [7].The key sentence in Laue's 1911 paradigmatic paper on relativistic mechanics, confirming the opinion of Abraham, Lorentz and Poincaré, stated: *Planck and Einstein have already expressed that all ponderomotoric forces should behave under a Lorentz transformation in an equal manner as in electrodynamics. Thus it should be possible in all areas of physics to put the force density together with the power density into a four force density.* This leading principle lead to what we propose to call the Minskowski-Laue consensus, the fixation of the already growing consensus in the relativistic avant-garde in the first decade of the nineteenth century regarding relativistic dynamics in EM and mechanics. The general expression for the Minkowski Laue relativistic mechanics is $\mathscr{F}^{\nu} = -\partial_{\mu} T^{\mu\nu}_{mech}$, with $T^{\mu\nu}_{mech} = V^{\mu} G^{\nu}$. To this day the problem of the electron is formulated in the paradigmatic math-phys language of the Minkowski-Laue consensus.

According to von Laue, the electron in free space is a (quasi-)static system, so the divergence of its stress-energy density tensor should be zero. But in Minkowski's relativistic electro-magnetics, the divergence of the stress-energy density tensor of the electromagnetic field is zero only in charge-free space and equals the electromagnetic Lorentz four-force when charges are present. Electromagnetically the electron in free space is a charge in its own field and should feel its own em-four-force, which is not zero and not balanced by a reaction-force. So the electron in free space acts a net em-force and em-power on itself, leading to and infinite four force. This conclusion is refuted by experiments on the real electron, which, in free space and in the classical limit, behaves as a Laue closed system. Up till now, two strategies have been developed to find a way out of this conflict. The first is to add a mechanical tensor to the electro-magnetic field-tensor and to declare the divergence of the sum to be zero. This could be called the Poincaré-Laue strategy or the compensation-method. The second strategy is to suggest changes in the electro-magnetic stress-energy tensor, or in the connected EM four momentum, in such a way that the problem can be solved within the frame of electromagnetics. Both strategies have never solved the fundamental problem in such a way that a consensus was established and the foundational discussions ceased. Instead, the formulation of the problem of the electron seems to have been frozen in time, without a solution coming in sight [8], [9], [10].

Originally, the electron problem was seen as a discrepancy between the relativistic theory, SR and GR, on the one hand and Maxwell-Lorentz EM theory on the other hand and motivated early pre-QM unification programs. But with the introduction of intrinsic electron spin, the problem of the electron attracted quantum physicist. Theorists like Thomas, Frenkel, de Broglie and Kramers tried to deal with intrinsic electron spin in the formalism of the Minskowski-Laue consensus in order to fuse the concept of intrinsic spin with the Minskowski-Laue math-phys language [11], [12], [13], [14]. In the works of Kramers and de Broglie, two sets of languages are invoked to deal with the relativistic problem of the spinning electron, the Minkowski-Laue consensus and Pauli-Dirac spin QM but they remained incapable of fusing the concepts and the math-phys of these two approaches. In 1938 Dirac himself, the father of the relativistic quantum theory of the electron, returned to the Lorentz model of the electron in an attempt to find an opening regarding the self-energy problem as it reappeared in quantum mechanics [15]. In the years thereafter, Dirac continued to try to solve the electron's self-energy problem by going back to the pre-quantum theory of relativistic electrodynamics [16]. According to Dirac, the problem of the electron was related to our understanding of empty space: *We can see now that we may very well have an æther, subject to quantum mechanics and conforming to relativity, [...]. We must make some profound alterations in our theoretical ideas of the vacuum. It is no longer a trivial state, but needs elaborate mathematics for its description. [17] It (the new æther) will probably have to be modified by the introduction of spin variables before a satisfactory quantum theory of electrons can be obtained from it [18].*

This is the point were, in retrospective, we hook on, Dirac's suggestion of introducing spin variables into the vacuum/metric/æther as a necessary step forward in our understanding of the electron. This meant that in dealing with the problem of relativistic dynamics regarding the problem of the (spinning) electron, we ignored what was to come afterwards, the Yang-Mills theories of the weak force and the strong force. Our goal was to create a math-phys language for the electron problem that replaced the Minkowski-Laue consensus and contained pre-YM Pauli-Dirac QM. The tricky side catch of our approach was that Einstein used Laue's closed system condition, so essentially the Minkowski-Laue consensus or the relativistic formulation of the conservation of energy, momentum and angular momentum, as a basis for his theory of gravity ([19], postulate 1 on p. 1250; [7], p. 57). In order to be able to remain pre-GR as long as possible, we simplified the gravity side of the problem by only trying to connect our yet to develop math-phys language to the attempts of the pre-GR theorists of relativistic gravity, Abraham [20], Nordström [21], [22], Mie [23], [24] and of course Einstein himself. As such, our approach is pre-YM and pre-GR, but without criticizing these well established, experimentally verified monumental theories of physics. We deliberately choose such an anachronistic approach in order to simplify the environment in which to deal with the problem of the electron. The hope was that our approach would nevertheless lead to some useful new insights.

2 Using quaternions to put 'spin' into the metric

This paragraph recaptures the biquaternion definitions developed in previous paper, in which the matrix representation was not yet used [25]. Quaternions can be represented by the basis $(\hat{1}, \hat{\text{I}}, \hat{\text{J}}, \hat{\text{K}})$. This basis has the properties $\hat{\text{I}}\hat{\text{I}} = \hat{\text{J}}\hat{\text{J}} = \hat{\text{K}}\hat{\text{K}} = -\hat{1}$; $\hat{\text{I}}\hat{\text{J}} = -\hat{\text{J}}\hat{\text{I}} = \hat{\text{K}}$; $\hat{\text{J}}\hat{\text{K}} = -\hat{\text{K}}\hat{\text{J}} = \hat{\text{I}}$; $\hat{\text{K}}\hat{\text{I}} = -\hat{\text{I}}\hat{\text{K}} = \hat{\text{J}}$. A quaternion number in its summation representation is given by $A = a_0\hat{1} + a_1\hat{\text{I}} + a_2\hat{\text{J}} + a_3\hat{\text{K}}$, in which the a_μ are real numbers. Bi-quaternions or complex quaternions are given by $C = A + \mathbf{i}B$ in which the $c_\mu = a_\mu + \mathbf{i}b_\mu$ are complex numbers and the a_μ and b_μ are real numbers. The complex conjugate of a bi-quaternion C is given by $\widetilde{C} = A - \mathbf{i}B$. A set of four numbers, real or complex, is given by

$$C_\mu = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}, \tag{1}$$

or by $C^\mu = [c_0, c_1, c_2, c_3]$, a set of four numbers $\in \mathbb{C}$. The quaternion basis can be given as a set $\hat{\mathrm{K}}^\mu = [\hat{1}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}]$ and then a biquaternion C can also be written as $C = C^\mu \hat{\mathrm{K}}_\mu = c_0\hat{1} + c_1\hat{\mathrm{I}} + c_2\hat{\mathrm{J}} + c_3\hat{\mathrm{K}}$. We apply this to the space-time four vector of relativistic bi-quaternion 4-space R with the four numbers $R^\mu = [\mathbf{i}ct, r_1, r_2, r_3] = [r_0, r_1, r_2, r_3]$, so with $r_1, r_2, r_3 \in \mathbb{R}$ and $r_0 \in \mathbb{C}$. Then we have $R = R^\mu \hat{\mathrm{K}}_\mu = r_0\hat{1} + r_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = r_0\hat{1} + \mathbf{r}\cdot\hat{\mathbf{K}}$. We use the threevector analogue of $R^\mu \hat{\mathrm{K}}_\mu$ when we write $\mathbf{r}\cdot\hat{\mathbf{K}}$. In this notation we define the complex conjugate of a four vector as $R^T = -r_0\hat{1} + \mathbf{r}\cdot\hat{\mathbf{K}}$ and the quaternion conjugate of a four vector as $R^P = r_0\hat{1} - \mathbf{r}\cdot\hat{\mathbf{K}}$.

Quaternions can be represented by 2x2 matrices. Several representations are possible and our choice is given by

$$R = \begin{bmatrix} r_0 + \mathbf{i}r_1 & r_2 + \mathbf{i}r_3 \\ -r_2 + \mathbf{i}r_3 & r_0 - \mathbf{i}r_1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix}. \tag{2}$$

Then we can write R as

$$R = r_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{3}$$

This gives us the quaternions as the following matrices:

$$\hat{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{\mathrm{I}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \hat{\mathrm{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \hat{\mathrm{K}} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{4}$$

We can compare these to the Pauli spin matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. If we exchange the σ_x and the σ_z, we have $\hat{\mathbf{K}} = \mathbf{i}\boldsymbol{\sigma}$ and $\hat{\mathrm{K}}^\mu = (\sigma_0, \mathbf{i}\boldsymbol{\sigma})$. The reason to use the quaternion matrices and not the Pauli matrices lies in the way the Lorentz transformations can be abridged using the quaternion matrices in combination with hyperbolic relativity.

Multiplication of two vectors A and B follows matrix multiplication. So we have

$$C = AB = \begin{bmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}. \tag{5}$$

Of course, vectors A, B and C can be expressed with their a_μ, b_μ, c_μ coordinates and if we use them, after some elementary but elaborate calculations and rearrangements we arrive at the following result:

$$\begin{aligned} c_0 &= a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ c_1 &= a_2b_3 - a_3b_2 + a_0b_1 + a_1b_0 \\ c_2 &= a_3b_1 - a_1b_3 + a_0b_2 + a_2b_0 \\ c_3 &= a_1b_2 - a_2b_1 + a_0b_3 + a_3b_0 \end{aligned} \tag{6}$$

In short, if we use the classic Euclidean dot and cross products of Euclidean three-vectors, this gives for the coordinates $c_0 = a_0b_0 - \mathbf{a}\cdot\mathbf{b}$ and $\mathbf{c} = \mathbf{a}\times\mathbf{b} + a_0\mathbf{b} + \mathbf{a}b_0$. And in the quaternion notation we get

$$C = AB = (a_0b_0 - \mathbf{a}\cdot\mathbf{b})\hat{1} + (\mathbf{a}\times\mathbf{b} + a_0\mathbf{b} + \mathbf{a}b_0)\cdot\hat{\mathbf{K}} \tag{7}$$

From this it immediately follows that $A^TA = (-a_0a_0 - \mathbf{a}\cdot\mathbf{a})\hat{1}$, and, with $a_0 = \mathbf{i}ca_t$, we get $A^TA = (c^2a_t^2 - \mathbf{a}^2)\hat{1}$. This can be used to define the main quadratic form of the metric as $dR^TdR = (c^2dt^2 - d\mathbf{r}^2)\hat{1} = c^2d\tau^2\hat{1} = -ds^2\hat{1}$, with $ds = \mathbf{i}cd\tau$. It may be clear that our definitions lead to many analogies with Hestenes' space time algebra [26]. But our matrix representation, which remains crucial all along, follows a quaternion basis and not the Pauli basis, to mention a difference.

3 The Lorentz transformation as a twist of the metric

A normal Lorentz transformation between two reference frames connected by a relative velocity v in the $x-$ or $\hat{\mathrm{I}}$-direction, with the usual $\gamma = 1/\sqrt{1-v^2/c^2}$, $\beta = v/c$ and $r_0 = \mathbf{i}ct$, can be expressed as

$$\begin{bmatrix} r_0' \\ r_1' \end{bmatrix} = \begin{bmatrix} \gamma & -\mathbf{i}\beta\gamma \\ \mathbf{i}\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \mathbf{i}\beta\gamma r_1 \\ \gamma r_1 + \mathbf{i}\beta\gamma r_0 \end{bmatrix}. \tag{8}$$

We want to connect this to our matrix representation of R as in Eq.(2) which gives

$$R_{00}' = r_0' + \mathbf{i}r_1' = \gamma r_0 - \mathbf{i}\beta\gamma r_1 + \mathbf{i}\gamma r_1 - \beta\gamma r_0 \tag{9}$$
$$R_{11}' = r_0' - \mathbf{i}r_1' = \gamma r_0 - \mathbf{i}\beta\gamma r_1 - \mathbf{i}\gamma r_1 + \beta\gamma r_0. \tag{10}$$

Now we want to introduce the rapidity and thus hyperbolic Special Relativity in order to integrate Lorentz transformations into our matrix metric. For this we only need elementary rapidity definitions as they were already formulated by Varičak in 1912 [27]. If we use the rapidity ψ as $e^\psi = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, the previous transformations can be rewritten as

$$R_{00}' = r_0' + \mathbf{i}r_1' = (\gamma - \beta\gamma)(r_0 + \mathbf{i}r_1) = R_{00}e^{-\psi} \tag{11}$$
$$R_{11}' = r_0' - \mathbf{i}r_1' = (\gamma + \beta\gamma)(r_0 - \mathbf{i}r_1) = R_{11}e^{\psi}. \tag{12}$$

As a result we have

$$R^L = \begin{bmatrix} R_{00}' & R_{01}' \\ R_{10}' & R_{11}' \end{bmatrix} = \begin{bmatrix} R_{00}e^{-\psi} & R_{01} \\ R_{10} & R_{11}e^{\psi} \end{bmatrix} = U^{-1}RU^{-1}. \tag{13}$$

In the expression $R^L = U^{-1}RU^{-1}$ we used the matrix U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{14}$$

But this means that we can write the result of a Lorentz transformation on R with a Lorentz velocity in the $\hat{\mathrm{I}}$-direction between the two reference systems as

$$R^L = r_0 \begin{bmatrix} e^{-\psi} & 0 \\ 0 & e^{\psi} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & -\mathbf{i}e^{\psi} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{15}$$

This can be written as

$$R^L = r_0 U^{-1}\hat{1}U^{-1} + r_1 U^{-1}\hat{\mathrm{I}}U^{-1} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = r_0\hat{1}^L + r_1\hat{\mathrm{I}}^L + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}}. \quad (16)$$

But because we started with Eq.(8), we now have two equivalent options to express the result of a Lorentz transformation, either as a coordinate transformation or as a basis transformation: $R^L = r'_0\hat{1} + r'_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = r_0\hat{1}^L + r_1\hat{\mathrm{I}}^L + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}}$.

This result only works for Lorentz transformation between v_x-, v_1- or $\hat{\mathrm{I}}$-aligned reference systems. Reference systems which do not have their relative Lorentz velocity aligned in the $\hat{\mathrm{I}}$-direction will have to be rotated into such an alignment before the Lorentz transformation in the form $R^L = U^{-1}RU^{-1}$ is applied. With this requirement we restrict ourselves to a limited realm of applications. The interesting thing about the $e^{\Psi} = \gamma + \beta\gamma$ term is that it represents a relativistic Doppler-correction applied to the frequency ν of light-signals exchanged between two inertial reference systems, $\nu/\nu_0 = e^{\Psi}$ [27].

4 The Maxwell-Lorentz structures in our math-phys environment

If we want to apply the previous to relativistic electrodynamics and to quantum physics, we need to develop the mathematical language further. We start with a particle with a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1-v^2/c^2})^{-1}$. We define the coordinate velocity four vector as $V = V^{\mu}\hat{\mathrm{K}}_{\mu} = \mathbf{i}c\hat{1} + \mathbf{v}\cdot\hat{\mathbf{K}} = v_0\hat{1} + \mathbf{v}\cdot\hat{\mathbf{K}}$. The proper velocity four vector on the other hand will be defined using the proper time $\tau = t_0$ as $U = U^{\mu}\hat{\mathrm{K}}_{\mu} = \frac{d}{d\tau}R^{\mu}\hat{\mathrm{K}}_{\mu} = \gamma V^{\mu}\hat{\mathrm{K}}_{\mu} = u_0\hat{1} + \mathbf{u}\cdot\hat{\mathbf{K}}$. The four vector partial derivative $\partial = \partial^{\mu}\hat{\mathrm{K}}_{\mu}$ will be defined using the coordinate four set $\partial^{\mu} = (-\mathbf{i}\frac{1}{c}\partial_t, \boldsymbol{\nabla}) = (\partial_0, \partial_1, \partial_2, \partial_3)$. The electrodynamic potential four vector $A = A^{\mu}\hat{\mathrm{K}}_{\mu}$ will be defined by the coordinate four set $A^{\mu} = (\mathbf{i}\frac{1}{c}\phi, \mathbf{A}) = (A_0, A_1, A_2, A_3)$. The electric four current density vector $J = J^{\mu}\hat{\mathrm{K}}_{\mu}$ will be defined by the coordinate four set $J^{\mu} = (\mathbf{i}c\rho_e, \mathbf{J}) = (J_0, J_1, J_2, J_3)$, with ρ_e as the electric charge density. The electric four current with a charge q will be also be written as J_{μ} and the context will indicate which one is used. And although we defined these four vectors using the coordinate column notation, we will mostly use the matrix or summation notation, as for example in Eqn.(2).

I we apply the multiplication rules of our four vectors as matrices to the electromagnetic field with four derivative ∂ and four potential A we get $B = \partial^T A$ and then insert $\partial_0 = -\mathbf{i}\frac{1}{c}\partial_t$ and $A_0 = \mathbf{i}\frac{1}{c}\phi$ we get

$$B = \partial^T A = (-\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\mathbf{A})\hat{1} + (\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{i}\frac{1}{c}(-\partial_t\mathbf{A} - \boldsymbol{\nabla}\phi))\cdot\hat{\mathbf{K}}. \quad (17)$$

If we apply the Lorenz gauge $\mathbb{B}_0 = -\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\mathbf{A} = 0$ and the usual EM definitions of the fields in terms of the potentials we get $B = (\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E})\cdot\hat{\mathbf{K}} = \overrightarrow{\mathbb{B}}\cdot\hat{\mathbf{K}}$ and we can write B as

$$B = \mathbb{B}_1\hat{\mathrm{I}} + \mathbb{B}_2\hat{\mathrm{J}} + \mathbb{B}_3\hat{\mathrm{K}} = \overrightarrow{\mathbb{B}}\cdot\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{i}\mathbb{B}_1 & \mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 \\ -\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 & -\mathbf{i}\mathbb{B}_1 \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}. \quad (18)$$

For the Lorentz transformation of B we can go back to the 1908 paper by Minkowski [5], where he wrote the transformation in a form equivalent to

$$\begin{bmatrix}\mathbb{B}_1'\\ \mathbb{B}_2'\\ \mathbb{B}_3'\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \gamma & \mathbf{i}\beta\gamma\\ 0 & -\mathbf{i}\beta\gamma & \gamma\end{bmatrix}\begin{bmatrix}\mathbb{B}_1\\ \mathbb{B}_2\\ \mathbb{B}_3\end{bmatrix}=\begin{bmatrix}\mathbb{B}_1\\ \gamma\mathbb{B}_2+\mathbf{i}\beta\gamma\mathbb{B}_3\\ \gamma\mathbb{B}_3-\mathbf{i}\beta\gamma\mathbb{B}_2\end{bmatrix} \tag{19}$$

So we have $B_{01}'=\mathbb{B}_2'+\mathbf{i}\mathbb{B}_3'=\gamma\mathbb{B}_2+\mathbf{i}\beta\gamma\mathbb{B}_3+\mathbf{i}\gamma\mathbb{B}_3+\beta\gamma\mathbb{B}_2$ and $B_{10}'=-\mathbb{B}_2'+\mathbf{i}\mathbb{B}_3'=-\gamma\mathbb{B}_2-\mathbf{i}\beta\gamma\mathbb{B}_3+\mathbf{i}\gamma\mathbb{B}_3+\beta\gamma\mathbb{B}_2$. If we use the rapidity ψ as $e^{\psi}=\cosh\psi+\sinh\psi=\gamma+\beta\gamma$, this can be rewritten as $B_{01}'=\mathbb{B}_2'+\mathbf{i}\mathbb{B}_3'=(\gamma+\beta\gamma)(\mathbb{B}_2+\mathbf{i}\mathbb{B}_3)=B_{01}e^{\psi}$ and $B_{10}'=-\mathbb{B}_2'+\mathbf{i}\mathbb{B}_3'=(\gamma-\beta\gamma)(-\mathbb{B}_2+\mathbf{i}\mathbb{B}_3)=B_{10}e^{-\psi}$, which leads to

$$B^L=\begin{bmatrix}B_{00} & B_{01}e^{\psi}\\ B_{10}e^{-\psi} & B_{11}\end{bmatrix}=\begin{bmatrix}e^{\frac{\psi}{2}} & 0\\ 0 & e^{-\frac{\psi}{2}}\end{bmatrix}\begin{bmatrix}B_{00} & B_{01}\\ B_{10} & B_{11}\end{bmatrix}\begin{bmatrix}e^{-\frac{\psi}{2}} & 0\\ 0 & e^{\frac{\psi}{2}}\end{bmatrix}=UBU^{-1}. \tag{20}$$

So just as with a four vector, the Lorentz transformation of the EM field coordinates can also be written as a transformation of the basis, while leaving the field coordinates invariant: $B^L=\mathbb{B}_1U\hat{\mathrm{I}}U^{-1}+\mathbb{B}_2U\hat{\mathrm{J}}U^{-1}+\mathbb{B}_3U\hat{\mathrm{K}}U^{-1}=\mathbb{B}_1\hat{\mathrm{I}}+\mathbb{B}_2\hat{\mathrm{J}}^L+\mathbb{B}_3\hat{\mathrm{K}}^L$.

For the EM-field 'six-vector' matrix B as a product of two four vectors we can write $B^L=UBU^{-1}=U(\partial^TA)U^{-1}=U(\partial^T)UU^{-1}AU^{-1}=(\partial^TA)^L=(\partial^T)^{-L}A^L$. This can be generalized to the Lorentz transformation of the product $C=AB$. For a coherent relativistic math-phys language we need the following rule for the Lorentz transformation of the product C of two fourvectors A and B who individually transform as $A^L=U^{-1}AU^{-1}$ and $B^L=U^{-1}BU^{-1}$:

$$C^L=(AB)^L=A^{-L}B^L=UAUU^{-1}BU^{-1}=UABU^{-1}=UCU^{-1}. \tag{21}$$

This implies that $(AB)^L\neq A^LB^L$ if A and B are fourvectors. As a result, it is easy to prove that the quadratic $A^TA=c^2a_\tau^2\hat{\imath}$ is Lorentz invariant.

Using the four density current J, the Maxwell equations in our language are $\partial B=\mu_0J$ and the Lorentz force law, with a four force density $\mathscr{F}$, as $JB=\mathscr{F}$. Maxwell's inhomogeneous wave equations can be written as $(-\partial^T\partial)B=-\mu_0\partial^TJ$ and with the Lorentz invariant quadratic derivative $-\partial^T\partial=(\boldsymbol{\nabla}^2-\frac{1}{c^2}\partial_t^2)\hat{\imath}$ we get the homogeneous wave equations of the EM field in free space in the familiar form as $(-\partial^T\partial)B=\boldsymbol{\nabla}^2B-\frac{1}{c^2}\partial_t^2B=0$. Given the definition of the four vectors, our matrix multiplication contains the Maxwell-Lorentz structure.

As for the electromagnetic energy of a pure EM field, we have the two products BB and B^TB, with $B^T=(\partial^TA)^T=\partial A^T=(\mathbf{B}+\mathbf{i}\frac{1}{c}\mathbf{E})\cdot\hat{\mathbf{K}}=\overrightarrow{\mathbb{B}}^*\cdot\hat{\mathbf{K}}$ in which we used $\overrightarrow{\mathbb{B}}^*$ instead of $\widetilde{\overrightarrow{\mathbb{B}}}$ as the complex conjugate of $\overrightarrow{\mathbb{B}}$. We then can calcutate the results, which gives $BB=(\mathbf{B}^2-\frac{1}{c^2}\mathbf{E}^2-2\mathbf{i}\frac{1}{c}\mathbf{B}\cdot\mathbf{E})\hat{\imath}=\overrightarrow{\mathbb{B}}\cdot\overrightarrow{\mathbb{B}}\,\hat{\imath}$ for the first. The Lorentz invariance follows from $B^L=UBU^{-1}$ and the fact that BB result in a scalar value, so $B^LB^L=UBU^{-1}UBU^{-1}=UBBU^{-1}=\overrightarrow{\mathbb{B}}\cdot\overrightarrow{\mathbb{B}}U\hat{\imath}U^{-1}=\overrightarrow{\mathbb{B}}\cdot\overrightarrow{\mathbb{B}}\,\hat{\imath}=BB$. We also have the interesting product $\partial(\frac{1}{\mu_0}BB)$, the four divergence of this Lorentz invariant EM energy related product. Using the Maxwell equations $\partial B=\mu_0J$ and the Lorentz force density law $JB=\mathscr{F}$, we get $\partial(\frac{1}{\mu_0}BB)=\frac{1}{\mu_0}\partial BB=JB=\mathscr{F}$ (factor 2 question; $\partial(\frac{1}{2\mu_0}BB)=\mathscr{F}$?).

For the second EM energy related product $B^T B$ we get

$$B^T B = -\overrightarrow{\mathbb{B}}^* \cdot \overrightarrow{\mathbb{B}}\,\hat{1} + (\overrightarrow{\mathbb{B}}^* \times \overrightarrow{\mathbb{B}}) \cdot \hat{\mathbf{K}} = -(\mathbf{B}^2 + \frac{1}{c^2}\mathbf{E}^2)\hat{1} + 2\mathbf{i}\frac{1}{c}(\mathbf{E} \times \mathbf{B}) \cdot \hat{\mathbf{K}} =$$
$$\mathbf{i}\frac{2\mu_0}{c}(\mathbf{i}cu_{EM}\hat{1} + \mathbf{S} \cdot \hat{\mathbf{K}}) = \mathbf{i}2\mu_0 c(\mathbf{i}\frac{1}{c}u_{EM}\hat{1} + \mathbf{g} \cdot \hat{\mathbf{K}}), \quad (22)$$

with the Poynting vector as $\mu_0 \mathbf{S} = \mathbf{E} \times \mathbf{B}$, the EM momentum density $c^2\mathbf{g} = \mathbf{S}$ and the EM energy as $2\mu_0 u_{EM} = \mathbf{B}^2 + \frac{1}{c^2}\mathbf{E}^2$. Thus we get the usual EM four momentum density G and the four energy current density S as $G = \frac{1}{c^2}S = \frac{-\mathbf{i}}{2\mu_0 c}B^T B$, a result that again mimics STA [26]. For the Lorentz transformation of $B^T B$, as a four vector it should be

$$G^L = U^{-1}GU^{-1} = \frac{-\mathbf{i}}{2\mu_0 c}U^{-1}B^T BU^{-1} =$$
$$\frac{-\mathbf{i}}{2\mu_0 c}U^{-1}B^T U^{-1}UBU^{-1} = \frac{-\mathbf{i}}{2\mu_0 c}U^{-1}B^T U^{-1}B^L, \quad (23)$$

but this gives the problematic $U^{-1}B^T U^{-1}$ which doesn't seem to give $(B^T)^L$. So here we enter the discussion, in our math-phys language and already in the vacuum context, of the genuine EM momentum density four vector, a discussion strongly related to the correct formulation of the EM energy density tensor. Problematic also is the product $\partial(\frac{1}{\mu_0}BB^T)$, for which we get $\partial(\frac{1}{\mu_0}BB^T) = \frac{1}{\mu_0}\partial BB^T = JB^T \neq \mathscr{F}_{Lorentz}$. We get the signs wrong, also with $\partial(\frac{1}{\mu_0}B^T B)$. We could try $\partial^T G$ but that leads to $\partial^T B^T B$ which is also problematic. The fact that in our math-phys environment things do not run smoothly and unproblematic as regards to the relativistic Poynting vector and the EM four momentum density, especially when charges are included, indicates that fundamental problems in physics have a tendency to stay around, independent of the formalism. We believe that the capacity of not-solving but merely rephrasing these more that a century old problems in our own math-phys is a good sign.

5 Relativistic mechanics

In Special Relativity, Laue's condition for a conserved energy-momentum is $\partial^\nu T_{\nu\mu} = 0$. In our language we have the $\partial^T P = 0$ condition as a starting point of our alternative relativistic mechanics. Given the symmetry condition $\mathbf{p} = m_i \mathbf{v}$, the momentum four vector will be $P = m_i V$ and the condition $\partial^T P = 0$ leads to

$$\partial^T P = (-\frac{1}{c^2}\partial_t U_i - \nabla \cdot \mathbf{p})\hat{1} + (\nabla \times \mathbf{p} + \mathbf{i}\frac{1}{c}(\partial_t \mathbf{p} + \nabla U_i)) \cdot \hat{\mathbf{K}} = 0. \quad (24)$$

so to three subconditions $\frac{1}{c^2}\partial_t U_i + \nabla \cdot \mathbf{p} = 0$, $\nabla \times \mathbf{p} = 0$ and $\partial_t \mathbf{p} = -\nabla U_i$. The first one is the continuity equation, the second means that we have zero vorticity and the third that the related force field can be connected to a potential energy. We can take the time derivative of the second condition, giving the conserved force field condition $\nabla \times \mathbf{F} = 0$. So the condition $\partial^T P = 0$ can represent a central force.

In the Laue condition $\partial^\nu T_{\nu\mu} = 0$ the stress-energy density tensor is $T_{\nu\mu} = V_\nu G_\mu$. In our math-phys language we would get the not exact analog $T = V^T G$ and $\partial T = 0$, but that would imply a full homogeneous Maxwell-Lorentz structure with the product $\partial V^T G = 0$. Our stress energy density 'tensor' T gives

$$T = V^T G = (u_i - \mathbf{v}\cdot\mathbf{g})\hat{1} + (\mathbf{v}\times\mathbf{g} + \mathbf{i}c(-\mathbf{g} + \frac{1}{c^2}u_i\mathbf{v}))\cdot\hat{\mathbf{K}}. \tag{25}$$

This tensor analog contains all the elements of $T_{\nu\mu} = V_\nu G_\mu$, with the difference that the cross product $\mathbf{v}\times\mathbf{g}$ appears directly in our $T = V^T G$ whereas ony half of it is in the usual tensor and the anti-symmetric tensor product is needed to get the full cross product. In the case of a symmetric situation $\mathbf{v}$ has the same direction as $\mathbf{g}$, resulting in the three equations $T = (u_i - \mathbf{v}\cdot\mathbf{g})\hat{1} = u_0\hat{1}$, $\mathbf{v}\times\mathbf{g} = 0$ and $\mathbf{g} = \frac{1}{c^2}u_i\mathbf{v}$. The first of these equations equals the scalar Lagrangian density, the trace of the Laue mechanical stress-energy density tensor. Then the divergence of the symmetric T gives the four force density $\mathscr{F} = -\partial T$ as $\mathscr{F} = -\partial u_0$. Only if $\mathbf{v}$ doesn't have the same direction as $\mathbf{g}$ will there be an anti-symmetric component present that is analog to the structure of the Maxwell-Lorentz electromagnetic field. In our math-phys language, the compactified Minkowski-Laue equation's content is spread out over several products and equations existing at different layers of complexity. Interestingly, through the equation $\mathscr{F}_\mu = -\partial_\mu u_0$, Abraham, Nordström and Mie unsuccessfully tried to construct their relativistic theories of gravity.

6 Pauli spin QM and the Lorentz transformation of spinors

The basic Klein-Gordon wave equation in Quantum Mechanics can be written in our math-phys environment as $-\partial^T\partial\Psi = (\nabla^2 - \frac{1}{c^2}\partial_t^2)\hat{1}\Psi = 0$ with a two column spinor instead of the scalar spinor of Schrödinger- and Klein-Gordon QM. But it results in two identical equations, so a degenerate situation in which the two valued spinor equation can be reduced to the original single one.This Klein-Gordon Equation can be linked to the quadratic energy-momentum condition $P^T P = (\frac{1}{c^2}U_i^2 - p^2)\hat{1} = \frac{1}{c^2}U_0^2\hat{1} = -E^2\hat{1}$, with $E = \mathbf{i}\frac{1}{c}U_0$. With the operator convention $\hat{P} = -\mathbf{i}\hbar\partial$ we can switch from the energy equation to the wave equation $\hat{P}^T\hat{P}\Psi = -E^2\hat{1}\Psi$. We can make this canonical by applying the replacement $P \to P + qA$ and $\hat{P} \to \hat{P} + qA$ or $\partial \to D = \partial + \mathbf{i}\frac{q}{\hbar}A$. This results in the canonical Klein-Gordon wave equation in a biquaternion metric, that includes the Pauli-spin EM-field interaction term, as $D^T D\Psi = \frac{E^2}{\hbar^2}\hat{1}\Psi$. The $D^T D\Psi$ part can be expanded as

$$D^T D\Psi = \partial^T\partial\Psi + \mathbf{i}\frac{q}{\hbar}\partial^T A\Psi + \mathbf{i}\frac{q}{\hbar}A^T\partial\Psi - \frac{q^2}{\hbar^2}A^T A\Psi. \tag{26}$$

Now, the first and the last terms give scalar quadratics but the two middle terms must be examined more carefully. By writing out the two matrix products and applying the standard differentiation rule to the scalars in these matrixes, one can show that $\partial^T A\Psi + A^T\partial\Psi = B\Psi + 2(\frac{1}{c^2}\phi\partial_t + \mathbf{A}\cdot\nabla)\hat{1}\Psi$. This gives us for $D^T D\Psi = \frac{E^2}{\hbar^2}\hat{1}\Psi$ the equation

$$\partial^\mu\partial_\mu\hat{1}\Psi - \frac{q^2}{\hbar^2}A^\mu A_\mu\hat{1}\Psi - 2\mathbf{i}\frac{q}{\hbar}A^\mu\partial_\mu\hat{1}\Psi = -\frac{E^2}{\hbar^2}\hat{1}\Psi + \mathbf{i}\frac{q}{\hbar}B\Psi, \tag{27}$$

with $\partial^T \partial = -\partial^\mu \partial_\mu \hat{1}, A^T A = -A^\mu A_\mu \hat{1}$ and $A^\mu \partial_\mu = (\frac{1}{c^2}\phi \partial_t + \mathbf{A}\cdot\nabla)$. The only non-degenerate part in this equation is $\mathbf{i}\frac{q}{\hbar}B\Psi$. In our units we have the Bohr magneton $\mu_B = \frac{e\hbar}{2m_0}$ and if we multiply the equation by $\frac{\hbar^2}{2m_0}$ we get the non-degenerate term as $\mathbf{i}\mu_B B\Psi$. This can be written as

$$\mathbf{i}\mu_B B\Psi = \mathbf{i}\mu_B \overrightarrow{\mathbb{B}} \cdot \hat{\mathbf{K}}\Psi = -\mu_B \overrightarrow{\mathbb{B}} \cdot \boldsymbol{\sigma}\Psi = -\mu_B \mathbf{B}\cdot\boldsymbol{\sigma}\Psi + \mathbf{i}\mu_B \frac{1}{c}\mathbf{E}\cdot\boldsymbol{\sigma}\Psi, \tag{28}$$

with $\sigma_x \rightleftarrows \sigma_z$. So by putting spin in the metric we get a canonical Klein Gordon equation that includes Pauli-spin EM-field interaction terms. If we replace the degenerate quadratic scalar part by the canonical Schrödinger equation and ignore the spin electric field interaction term, we get the Pauli equation.

With the equation $D^T D\Psi = \frac{E^2}{\hbar^2}\hat{1}\Psi$ we are able to treat Pauli spin relativistically, provided that the spinor Ψ Lorentz transforms as $\Psi^L = U\Psi$ or

$$\Psi^L = U\Psi = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \begin{bmatrix} \Psi_0 \\ \Psi_1 \end{bmatrix} = \begin{bmatrix} \Psi_0 e^{\frac{\psi}{2}} \\ \Psi_1 e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{29}$$

The condition $\Psi^L = U\Psi$ assures the invariance of the equation $D^T D\Psi = \frac{E^2}{\hbar^2}\hat{1}\Psi$. If $\overline{\Psi}$ is the transpose complexe conjugate of Ψ, then $\overline{\Psi^L} = \overline{U\Psi} = (\widetilde{\Psi}_0 e^{-\frac{\psi}{2}}, \widetilde{\Psi}_1 e^{\frac{\psi}{2}})$ so

$$\overline{\Psi^L}\Psi^L = \left[\widetilde{\Psi}_0 e^{-\frac{\psi}{2}}, \widetilde{\Psi}_1 e^{\frac{\psi}{2}}\right] \begin{bmatrix} \Psi_0 e^{\frac{\psi}{2}} \\ \Psi_1 e^{-\frac{\psi}{2}} \end{bmatrix} = \widetilde{\Psi}_0 \Psi_0 + \widetilde{\Psi}_1 \Psi_1 = \overline{\Psi}\Psi. \tag{30}$$

This assures the Lorentz invariance of the QM probability density. It can be given as $\overline{\Psi^L}\Psi^L = \overline{\Psi}U^{-1}U\Psi$ which implies that $\overline{\Psi}^L = \overline{\Psi}U^{-1}$.

7 The Dirac level

Up to the Pauli level we reproduce elements basic to QM. But the relativistic wave equation of the electron exists on the Dirac level, so we aren't there yet if we want a math-phys language for the analysis of the problem of the electron including pre-YM quantum and pre-GR gravity aspects. Dirac linearized the quadratic relativistic Klein-Gordon wave equation by going to four by four matrices instead of the two by two Pauli matrices. In our math-phys language we define α_μ through

$$P^\mu \alpha_\mu = \begin{bmatrix} P & P \\ -P^T & P^T \end{bmatrix} = p_0 \begin{bmatrix} \hat{1} & \hat{1} \\ \hat{1} & -\hat{1} \end{bmatrix} + \mathbf{p}\cdot \begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & \hat{\mathbf{K}} \end{bmatrix} =$$

$$p_0 \begin{bmatrix} \hat{1} & \hat{1} \\ \hat{1} & -\hat{1} \end{bmatrix} + p_1 \begin{bmatrix} \hat{\mathrm{I}} & \hat{\mathrm{I}} \\ -\hat{\mathrm{I}} & \hat{\mathrm{I}} \end{bmatrix} + p_2 \begin{bmatrix} \hat{\mathrm{J}} & \hat{\mathrm{J}} \\ -\hat{\mathrm{J}} & \hat{\mathrm{J}} \end{bmatrix} + p_3 \begin{bmatrix} \hat{\mathrm{K}} & \hat{\mathrm{K}} \\ -\hat{\mathrm{K}} & \hat{\mathrm{K}} \end{bmatrix} = p_0\alpha_0 + \mathbf{p}\cdot\boldsymbol{\alpha} \tag{31}$$

We have $P^\mu \alpha_\mu = p_0\alpha_0 + p_1\alpha_1 + p_2\alpha_2 + p_3\alpha_3$. We can split this into $P^\mu \alpha_\mu = P^\mu \beta_\mu + P^\mu \gamma_\mu$ and then get

$$P^\mu \beta_\mu = \begin{bmatrix} P & 0 \\ 0 & P^T \end{bmatrix} = p_0 \begin{bmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix} + \mathbf{p}\cdot \begin{bmatrix} \hat{\mathbf{K}} & 0 \\ 0 & \hat{\mathbf{K}} \end{bmatrix} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta} =$$

$$p_0 \begin{bmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix} + p_1 \begin{bmatrix} \hat{\mathrm{I}} & 0 \\ 0 & \hat{\mathrm{I}} \end{bmatrix} + p_2 \begin{bmatrix} \hat{\mathrm{J}} & 0 \\ 0 & \hat{\mathrm{J}} \end{bmatrix} + p_3 \begin{bmatrix} \hat{\mathrm{K}} & 0 \\ 0 & \hat{\mathrm{K}} \end{bmatrix} \tag{32}$$

with $P^{\mu}\beta_{\mu} = p_0\beta_0 + p_1\beta_1 + p_2\beta_2 + p_3\beta_3$, and

$$\not{P} = P^{\mu}\gamma_{\mu} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = p_0 \begin{bmatrix} 0 & \hat{1} \\ \hat{1} & 0 \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} 0 & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & 0 \end{bmatrix} = p_0\gamma_0 + \mathbf{p}\cdot\boldsymbol{\gamma} =$$
$$p_0 \begin{bmatrix} 0 & \hat{1} \\ \hat{1} & 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 & \hat{\mathrm{I}} \\ -\hat{\mathrm{I}} & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & \hat{\mathrm{J}} \\ -\hat{\mathrm{J}} & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & \hat{\mathrm{K}} \\ -\hat{\mathrm{K}} & 0 \end{bmatrix} \tag{33}$$

with $\not{P} = P^{\mu}\gamma_{\mu} = p_0\gamma_0 + p_1\gamma_1 + p_2\gamma_2 + p_3\gamma_3$. If we use $\hat{\mathbf{K}} = \mathbf{i}\boldsymbol{\sigma}$ we have

$$\boldsymbol{\gamma} = \begin{bmatrix} 0 & \mathbf{i}\boldsymbol{\sigma} \\ -\mathbf{i}\boldsymbol{\sigma} & 0 \end{bmatrix} \tag{34}$$

with as only difference to the standard notation the exchange of γ_1 with γ_3. With these definitions we have obtained a Clifford Algebra four set with the γ_{μ}'s, a Weyl set. Another Clifford four set, a Dirac set, can be obtained with $(\beta_0, \boldsymbol{\gamma})$. Clifford Algebra three sets can be made with the $\boldsymbol{\alpha}$'s and with the $\boldsymbol{\beta}$'s. The $\boldsymbol{\beta}$'s form the Dirac spin set. We define the unit matrix on the Dirac level as $\not{1}$.

Using these definitions of the matrices on the Dirac level, we can define the Dirac-spinor wave equations. The wave equations in the Dirac environment have to be reducible to the Klein Gordon energy condition $P^T P = -E^2\hat{1}$ with $E = \mathbf{i}\frac{U_0}{c}$. The Dirac equation and the Weyl equations match this demand.The Weyl or chiral equation stems from the quadratic $\not{P}\not{P} = E^2\not{1}$.

$$\not{P}\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = \begin{bmatrix} -PP^T & 0 \\ 0 & -P^T P \end{bmatrix} = E^2 \begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix} = E^2\not{1} \tag{35}$$

This leads to the two options for the Weyl equations

$$\hat{\not{P}}\Psi = E\not{1}\Psi \tag{36}$$
$$\hat{\not{P}}\Psi = -E\not{1}\Psi \tag{37}$$

if we use $\hat{\not{P}} = -\mathbf{i}\hbar\not{\partial}$ and a four column spinor Ψ. The Dirac equation stems from the quadratic $(p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\gamma})^2 = E^2\not{1}$.

$$\begin{bmatrix} p_0\hat{1} & \mathbf{p}\cdot\hat{\mathbf{K}} \\ -\mathbf{p}\cdot\hat{\mathbf{K}} & -p_0\hat{1} \end{bmatrix} \begin{bmatrix} p_0\hat{1} & \mathbf{p}\cdot\hat{\mathbf{K}} \\ -\mathbf{p}\cdot\hat{\mathbf{K}} & -p_0\hat{1} \end{bmatrix} = \begin{bmatrix} (p_0^2 + \mathbf{p}^2)\hat{1} & 0 \\ 0 & (p_0^2 + \mathbf{p}^2)\hat{1} \end{bmatrix} = E^2\not{1} \tag{38}$$

This leads to the two options for the Dirac equations

$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\gamma})\Psi = E\not{1}\Psi \tag{39}$$
$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\gamma})\Psi = -E\not{1}\Psi \tag{40}$$

if we use $\hat{P} = -\mathbf{i}\hbar\partial$ and a four column spinor Ψ. In a previous section we showed that with the canonical $D^T D\Psi = \frac{E^2}{\hbar^2}\hat{1}\Psi$ we could treat Pauli spin relativistically in a Klein-Gordon condition. Based on that result, it is not to difficult to show that on the Dirac level we are also able to deal with $\not{D}\not{D}\Psi = \frac{E^2}{\hbar^2}\not{1}\Psi$. This results in a extra term $\mathbf{B}\cdot\boldsymbol{\beta}\Psi$, the magnetic field Dirac spin interaction term. This indicates that in our language, $\boldsymbol{\beta}$ is indeed the Dirac spin vector.

By imitating Dirac's jump from Pauli-spin to a double version of it, but at the same time remaining in our spin-metric environment, we made it plausible that our math-phys language has in principle the capacity to include the Dirac-Weyl quantum environment. In the line of our anachronistic project, pre-YM and pre-GR, only relativistic pre-GR gravity was not yet included in our math-phys language. While working on a translation of the helicity formalism and the spin half representation of the Lorentz group into our biquaternion spin-metric language, it seemed only natural to extend the helicity formalism from 3D to a 4D version. Unexpectedly, the necessary 4D rotation angle, as the 4D analogy of rapidity, seemed of gravitational nature. This surprising possibility regarding the 4D hyperbolic quantum rotator's angle was the motivation to write this paper.

8 Helicity, 3D hyperbolic rotations and Lorentz transformations

We have the Taylor expansion of e^{Ax} as

$$e^{Ax} = 1 + Ax + \frac{1}{2!}A^2x^2 + \frac{1}{3!}A^3x^3 + \frac{1}{4!}A^4x^4 + \frac{1}{5!}A^5x^5 + \dots \tag{41}$$

If $A^2 = 1$ but $A \neq 1$, this can be written as

$$e^{Ax} = 1 + Ax + \frac{1}{2!}x^2 + A\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + A\frac{1}{5!}x^5 + \dots \tag{42}$$

and

$$e^{Ax} = (1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots) + A(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots) = \cosh x + A\sinh x. \tag{43}$$

Of course, this only works if the norm of A can be split in two equal parts and results in a scalar outcome, so when the norm of A is a perfect quadratic. And if, with $A^2 = 1$, we calculate $(A+1)e^{Ax}$ we get an invariant product $e^{-Ax}(A+1)e^{Ax} = e^{-x}(A+1)e^{x} = A+1$.

We can apply this to Helicity, on the Pauli level first and then on the Dirac level. We have $(\mathbf{p}\cdot\hat{\mathbf{K}})(\mathbf{p}\cdot\hat{\mathbf{K}}) = -p^2\hat{1} = (\mathbf{i}p)^2\hat{1}$ or

$$H^2 = \left[\frac{\mathbf{p}\cdot\hat{\mathbf{K}}}{\mathbf{i}p}\right]\left[\frac{\mathbf{p}\cdot\hat{\mathbf{K}}}{\mathbf{i}p}\right] = \hat{1}, \tag{44}$$

which means that we have a perfect quadratic and can apply the above to write

$$\hat{1}e^{H\psi} = \hat{1}\cosh\psi + H\sinh\psi = \hat{1}\cosh\psi + \left[\frac{\mathbf{p}\cdot\hat{\mathbf{K}}}{\mathbf{i}p}\right]\sinh\psi. \tag{45}$$

We also have $(\mathbf{p}\cdot\hat{\mathbf{K}})e^{H\psi} = (\mathbf{p}\cdot\hat{\mathbf{K}})\cosh\psi + \mathbf{i}p\hat{1}\sinh\psi$ and $e^{-H\psi}(H+\hat{1})e^{H\psi} = e^{-\psi}(H+\hat{1})e^{\psi} = (H+\hat{1})$. About the effect of the hyperbolic rotation angle ψ, if we look at the following expression

$$He^{H\psi} = H\cosh\psi + \hat{1}\sinh\psi = \cosh\psi(H + \hat{1}\tanh\psi), \tag{46}$$

we see that if ψ goes to ∞, $\tanh\psi$ goes to 1 and $\cosh\psi$ goes from 1 to ∞.

On the Dirac level, helicity can be defined as Λ with

$$\Lambda^2 = \left[\frac{\mathbf{p}\cdot\boldsymbol{\beta}}{\mathbf{i}p}\right]\left[\frac{\mathbf{p}\cdot\boldsymbol{\beta}}{\mathbf{i}p}\right] = \not{1}, \tag{47}$$

Again we can define a hyperbolic rotator as

$$\not{1}e^{\Lambda\psi} = \not{1}\cosh\psi + \Lambda\sinh\psi = \not{1}\cosh\psi + \left[\frac{\mathbf{p}\cdot\boldsymbol{\beta}}{\mathbf{i}p}\right]\sinh\psi. \tag{48}$$

As before we have a hyperbolic rotation invariant $\Lambda + \not{1}$ and we have $\Lambda e^{\Lambda\psi} = \Lambda\cosh\psi + \not{1}\sinh\psi = \cosh\psi(\Lambda + \not{1}\tanh\psi)$.

Helicity is based upon the momentum 3-D vector $\mathbf{p}$ and the 3-D Pauli spin $\hat{\mathbf{K}}$ or the 3-D Dirac spin $\boldsymbol{\beta}$. What we are doing with these hyperbolic rotators is rotating three momentum $\mathbf{p}$ relative to its norm p, the action of $\tanh\psi$. Actually, we change the projection angle of $\mathbf{p}$ on its norm p. At the same time we scale both up, the action of $\cosh\psi$. Helicity is strongly related to Lorentz transformations. Hyperbolic helicity rotators can be connected to the Lorentz transformation of Pauli spinors. We had for the Lorentz boost of a Pauli two spinor $\Psi^L = U\Psi$. This gives

$$\Psi^L = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}\Psi = \cosh\frac{\psi}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Psi + \sinh\frac{\psi}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\Psi =$$

$$\left(\hat{1}\cosh\frac{\psi}{2} + \frac{\hat{\text{I}}}{\mathbf{i}}\sinh\frac{\psi}{2}\right)\Psi = \left(\hat{1}\cosh\frac{\psi}{2} + \frac{\mathbf{p}\cdot\hat{\mathbf{K}}}{\mathbf{i}p}\sinh\frac{\psi}{2}\right)\Psi = \hat{1}e^{H\frac{\psi}{2}}\Psi \tag{49}$$

In this derivation we end up with the requirement that the particle moves in the direction $\hat{\text{I}}$ of the Lorentz velocity. Only then can we replace $\hat{\text{I}}/\mathbf{i}$ by $(p_1\hat{\text{I}})/(p_1\mathbf{i})$ and then by H. If the particle has a different direction, we have to rotate the whole reference frame including the moving particle until the directions are aligned.

This gives us $\Psi^L = U\Psi = \hat{1}e^{H\frac{\psi}{2}}\Psi$, which implies that $U = \hat{1}e^{H\frac{\psi}{2}}$ and $\overline{\Psi}^L = \overline{\Psi}U^{-1} = \overline{\Psi}\hat{1}e^{-H\frac{\psi}{2}}$. But then we can write for particles that move in the direction of an applied Lorentz boost between two aligned reference systems $P^L = U^{-1}PU^{-1} = e^{-H\frac{\psi}{2}}Pe^{-H\frac{\psi}{2}}$ and for EM fields with the proper alignment, where $H \equiv (\hat{\text{I}}/\mathbf{i})$, $B^L = UBU^{-1} = e^{H\frac{\psi}{2}}Be^{-H\frac{\psi}{2}}$. We see that in our environment, Pauli helicity as a hyperbolic rotator functions as half a Lorentz transformation. Helicity provides the rotator structure and direction, whereas the rapidity contributes the size of the boost. So most of the necessary information contained in the Lorentz transformation is situated in the Pauli helicity operator, and a smaller part is given by the rapidity.

If we go from helicity on the Pauli level to Helicity on the Dirac level, things become complicated. On the Dirac-Weyl level, there is no simple relationship between helicity and the Lorentz transformation of Weyl-Dirac vectors like $\not{P}$ and 4-spinors. So what works on the 3D SU(2) level, connecting a hyperbolic quantum rotator to the Lorentz transformation, doesn't work on the 3D double SU(2) level. In our perception, this had to be related to the fact that Dirac helicity cannot be expanded into a 4D version.

9 Weyl and Dirac hyperbolic rotations

Dirac helicity with $\mathbf{p}\cdot\boldsymbol{\beta}$ cannot be extended to 4-D because the β_μ's don't form a Clifford 4-set. So a perfect quadratic with the β_μ's isn't possible. But we already formulated perfect 4-D quadratics in the context of the Weyl and the Dirac equations, using the two Clifford 4-sets γ_μ and $(\beta_0,\boldsymbol{\gamma})$. Because both are based on perfect quadratics, we can define a new hyperbolic rotator using either the Weyl equation structure or the Dirac equation structure. The Weyl rotator is the easiest, due to its relativistic symmetry $\not{P}\not{P}=E^2\not{1}$, with $E=\mathbf{i}\frac{1}{c}U_0=\mathbf{i}cm_0$. We get the hyperbolic rotator

$$\not{1}e^{\frac{\not{P}}{E}\alpha}=\not{1}\cosh\alpha+\frac{\not{P}}{E}\sinh\alpha, \tag{50}$$

with $E\not{1}e^{\frac{\not{P}}{E}\alpha}=E\not{1}\cosh\alpha+\not{P}\sinh\alpha$, and $\not{P}e^{\frac{\not{P}}{E}\alpha}=\not{P}\cosh\alpha+E\not{1}\sinh\alpha$, which can also be written as $\not{P}e^{\frac{\not{P}}{E}\alpha}=\cosh\alpha(\not{P}+E\not{1}\tanh\alpha)$. Applied to $(\not{P}-E\not{1})$ we get $(\not{P}-E\not{1})e^{\frac{\not{P}}{E}\alpha}=(\not{P}-E\not{1})e^{\alpha}$ with the invariant

$$e^{-\frac{\not{P}}{E}\alpha}(\not{P}-E\not{1})e^{\frac{\not{P}}{E}\alpha}=e^{-\alpha}(\not{P}-E\not{1})e^{\alpha}=(\not{P}-E\not{1}). \tag{51}$$

The $\tanh\alpha$ part implies that we are changing the projection of $\not{P}$ on $E\not{1}$ due to the hyperbolic rotation with angle α. Clearly this rotator is not part of a helicity representation of the Lorentz transformation, because then we would rotate $\mathbf{p}$ relative to p and not $\not{P}$ relative to $E\not{1}$. A Lorentz transformation can be seen as a four vector internal re-balancing, but now we are re-balancing the four vector relative to its norm. This brings us outside known quantum territory. The following equivalent tensor notation emphasizes the rotation aspect

$$\not{P}e^{\frac{\not{P}}{E}\alpha}=\not{P}\cosh\alpha+E\not{1}\sinh\alpha=\begin{bmatrix} E\hat{1}\sinh\alpha & P\cosh\alpha \\ -P^T\cosh\alpha & E\hat{1}\sinh\alpha \end{bmatrix} \tag{52}$$

which, when applied to $(\not{P}+E\not{1})e^{\frac{\not{P}}{E}\alpha}=(\not{P}+E\not{1})e^{\alpha}$, gives

$$(\not{P}+E\not{1})e^{\alpha}=\begin{bmatrix} E\hat{1}e^{\alpha} & Pe^{\alpha} \\ -P^T e^{\alpha} & E\hat{1}e^{\alpha} \end{bmatrix} \tag{53}$$

Then compare this to the Weyl equation with spinor $\Psi=\Upsilon e^{\varphi}$, with aplitude Υ and phase φ, leading to

$$(\widehat{\not{P}}+E\not{1})\Upsilon e^{\varphi}=\begin{bmatrix} E\hat{1}e^{\varphi} & \widehat{P}e^{\varphi} \\ -\widehat{P}^T e^{\varphi} & E\hat{1}e^{\varphi} \end{bmatrix}\begin{bmatrix} \Upsilon_R \\ \Upsilon_L \end{bmatrix}=\begin{bmatrix} E\hat{1}e^{\varphi}\Upsilon_R+Pe^{\varphi}\Upsilon_L \\ -P^T e^{\varphi}\Upsilon_R+E\hat{1}e^{\varphi}\Upsilon_L \end{bmatrix}=0 \tag{54}$$

Things look similar but they aren't. In case of operators we have $(\widehat{\not{P}}+E\not{1})e^{\frac{\not{P}}{E}\alpha}\neq(\widehat{\not{P}}+E\not{1})e^{\alpha}$, a complication that we will study later on, so our hyperbolic rotation angle isn't just a disguised phase shift of the Weyl spinor in the Weyl equation. The Weyl spinor has a scalar phase as starting point whereas our hyperbolic rotator might reduce to a scalar shift of some kind under special circumstances.

In case of the Dirac hyperbolic rotator, we have to replace γ_0 by β_0 in $\not{P}$, giving the rotator effect on $\not{P}$ as

$$\not{P}_\phi = \not{P} e^{\frac{\not{P}}{E}\alpha} = \begin{bmatrix} (E\sinh\alpha + p_0\cosh\alpha)\hat{1} & \mathbf{p}\cdot\hat{\mathbf{K}}\cosh\alpha \\ -\mathbf{p}\cdot\hat{\mathbf{K}}\cosh\alpha & (E\sinh\alpha - p_0\cosh\alpha)\hat{1} \end{bmatrix} \tag{55}$$

$$= \cosh\alpha \begin{bmatrix} (E\tanh\alpha + p_0)\hat{1} & \mathbf{p}\cdot\hat{\mathbf{K}} \\ -\mathbf{p}\cdot\hat{\mathbf{K}} & (E\tanh\alpha - p_0)\hat{1} \end{bmatrix} \tag{56}$$

which makes it clear that our hyperbolic rotator effect also differs from the effect of spinors on the Dirac $\widehat{\not{P}} - E\not{1}$, because left and right spinors affect the Dirac E and $\widehat{p}_0$ in an identical way as a package, like for example $(E + \widehat{p}_0)\hat{1}\Upsilon_R$.

10 The metric Dirac-Weyl hyperbolic rotator

Instead of $\not{P}$ as the quadratic input for the rotator we can also take $d\not{R}$ with $d\not{R}d\not{R} = ds^2\not{1}$ and $ds = \mathbf{i}cd\tau$ and then our hyperbolic rotator, including its effects on $d\not{R}$ and $ds\not{1}$, is

$$\not{1}e^{\frac{d\not{R}}{ds}\alpha} = \not{1}\cosh\alpha + \frac{d\not{R}}{ds}\sinh\alpha \tag{57}$$

$$d\not{R}_\phi = d\not{R}e^{\frac{d\not{R}}{ds}\alpha} = d\not{R}\cosh\alpha + ds\not{1}\sinh\alpha \tag{58}$$

$$ds_\phi\not{1} = ds\not{1}e^{\frac{d\not{R}}{ds}\alpha} = ds\not{1}\cosh\alpha + d\not{R}\sinh\alpha \tag{59}$$

with an hyperbolic rotation invariant $(d\not{R} - ds\not{1})$.

Interesting also is the product $(ds_\phi\not{1})^2 = ds_\phi\not{1}ds_\phi\not{1} = ds\not{1}e^{\frac{d\not{R}}{ds}\alpha}ds\not{1}e^{\frac{d\not{R}}{ds}\alpha}$. After some calculations, using standard hyperbolic trigonometric relations, we arrive at

$$(ds_\phi\not{1})^2 = ds^2\not{1}e^{\frac{d\not{R}}{ds}2\alpha} \approx ds^2(\not{1} + 2\alpha\frac{d\not{R}}{ds}), \tag{60}$$

so if we may assume $\frac{d\not{R}}{ds} \approx \not{1}$ we get $ds_\phi^2 \approx ds^2(1+2\alpha)$. Now, in the real world only one field is thought to be capable of changing ds^2 and that is a gravity field. Assuming such an interpretation, the rotation angle should be related to the gravitational potential as we have for example the GR familiar $\alpha = \phi/c^2$. So suppose $\alpha = \frac{\phi}{c^2} = -\frac{GM}{Rc^2}$. Then we get

$$ds_\phi^2 \approx ds^2(1 - \frac{2GM}{Rc^2}). \tag{61}$$

We can look closer at $\frac{d\not{R}}{ds}$, which gives in the Weyl case

$$\frac{d\not{R}}{ds} = \frac{dr_0}{ds}\gamma_0 + \frac{d\mathbf{r}}{ds}\boldsymbol{\gamma} = \frac{dt}{d\tau}\gamma_0 - \mathbf{i}\frac{d\mathbf{r}}{cd\tau}\boldsymbol{\gamma} = \frac{dt}{d\tau}\gamma_0 - \mathbf{i}\frac{\mathbf{u}}{c}\boldsymbol{\gamma}. \tag{62}$$

and, in the Dirac case

$$\frac{d\not{R}}{ds} = \frac{dr_0}{ds}\beta_0 + \frac{d\mathbf{r}}{ds}\boldsymbol{\gamma} = \frac{dt}{d\tau}\beta_0 - \mathbf{i}\frac{d\mathbf{r}}{cd\tau}\boldsymbol{\gamma} = \frac{dt}{d\tau}\beta_0 - \mathbf{i}\frac{\mathbf{u}}{c}\boldsymbol{\gamma}. \tag{63}$$

So the approximation $\frac{d\not{R}}{ds} \approx \not{1}$ comes down to low proper velocity circumstances, or non-relativistic proper velocities. But this only makes sense in the Dirac case with $\frac{d\not{R}}{ds} \approx \beta_0$, giving

$$(ds_\phi \not{1})^2 \approx ds^2(\not{1} + 2\alpha \frac{dt}{d\tau}\beta_0 - 2\alpha \mathbf{i}\frac{\mathbf{u}}{c}\boldsymbol{\gamma}) \approx ds^2(\not{1} + 2\alpha\beta_0), \tag{64}$$

or

$$(ds_\phi \not{1})^2 \approx ds^2 \begin{bmatrix} (1+2\alpha)\hat{1} & 0 \\ 0 & (1-2\alpha)\hat{1} \end{bmatrix}. \tag{65}$$

This clearly goes beyond the relativistic pre-GR theories or gravity.

But there is an easier approach towards the metric effect of the hyperbolic Dirac-Weyl rotator, one that remains within pre-GR results and is based on

$$ds_\phi \not{1} = ds\not{1} e^{\frac{d\not{R}}{ds}\alpha}. \tag{66}$$

This can be written as

$$\frac{ds_\phi}{ds_0} = e^{\frac{d\not{R}}{ds}\alpha} \approx e^{\alpha}. \tag{67}$$

In terms of the proper time this gives

$$\frac{d\tau_\phi}{d\tau_0} \approx e^{\alpha}. \tag{68}$$

In the approximation we applied, time and proper time are set equal, so this gives us the behavior of clocks under the low velocity approximation of our Dirac hyperbolic rotator. There is only one known field that can change proper time like this and that is a field of gravity. If we set $\alpha = \frac{\phi}{c^2} = -\frac{GM}{Rc^2}$ we have a correspondence between the physical effect of gravity on slowly moving clocks and our theoretical Dirac hyperbolic rotator (see [28], p. 24). We can also relate this to Einstein's pre-GR 1913 equation (3) in [19].

We can apply the same rotator on $E\not{1} = \mathbf{i}\frac{1}{c}U_0\not{1}$, so on the rest energy of a particle, slowly moving in spacetime. That also results in $\frac{U_\phi}{U_0} \approx e^{\alpha}$, with U_ϕ as the rest energy of a particle in a field of gravity compared to the rest energy of the same particle in free space U_0. This is the same result that Nordström arrived at in his 1912 theory of gravity ([21], [7] p. 36). Mie had an almost similar result in his 1912 theory of matter ([23], p. 30), a result which he repeated in 1914 in reaction to Einstein while indicating the difference with Nordström's theory ([24], p. 174).

If we use the proper velocity quadratic as an input for the rotator, with $\not{U} = u_0\beta_0 + \mathbf{u}\cdot\boldsymbol{\gamma} = \mathbf{i}\gamma_L c\beta_0 + \mathbf{u}\cdot\boldsymbol{\gamma}$ and the Lorentz factor γ_L, we have the quadratic $\not{U}\not{U} = -c^2\not{1} = (\mathbf{i}c)^2\not{1}$. The hyperbolic rotator connected to this quadratic, applied to $\mathbf{i}c$ gives

$$\mathbf{i}c_\phi \not{1} = \mathbf{i}c\not{1} e^{\frac{\not{U}}{\mathbf{i}c}\alpha}. \tag{69}$$

leading to a gravitational ajustment of the speed of light as $\frac{c_\phi}{c_0} \approx e^{\alpha}$. Now we run into an interpretation conflict with the previous two rotator results. We have $ds = \mathbf{i}cd\tau = \mathbf{i}cdt_0$ and $E = \mathbf{i}\frac{U_0}{c} = \mathbf{i}m_0c$. The first two results, time dilation and the conversion of gravitational energy into rest energy, were based upon a constant light speed c. But if c is the variable, then proper time and rest mass should be invariants. The interpretation dilemma turns around

$$\frac{c_\phi}{c_0} = \frac{m_0 c_\phi}{m_0 c_0} = \frac{t_0 c_\phi}{t_0 c_0} \approx e^{\alpha}, \tag{70}$$

more specific around the two options

$$\frac{c_\phi}{c_0} = \frac{m_0 c_\phi}{m_0 c_0} \leftrightarrows \frac{m_\phi c_0}{m_0 c_0} = \frac{m_\phi}{m_0}, \tag{71}$$

and the equivalent options

$$\frac{c_\phi}{c_0} = \frac{dt_0 c_\phi}{dt_0 c_0} \leftrightarrows \frac{dt_\phi c_0}{dt_0 c_0} = \frac{dt_\phi}{dt_0}, \tag{72}$$

This dilemma is not new [29], and we do not have to solve it. It is already quite amazing that our hyperbolic rotator, when expanded from the 3-D helicity version into the 4-D Weyl-Dirac version and then interpreted as a gravity rotator, reproduces the same dilemma's as the ones faced by the early theorists of relativistic gravity around 1912, see [7].

Another interesting observation relates to the fact that the experimental static gravitational clock-time dilation factor, which can be expressed, in terms of the clock frequenties, as $\frac{\nu_\phi}{\nu_0} = e^{-\alpha}$, was compensated, in the first experiment directly verifying this gravitational clock frequency dependency [30], by a relativistic Doppler effect, which, using the Lorentz rapidity, gives $\frac{\nu_\phi}{\nu_0} = e^{-\psi}$. This implies that in first approximation, the effect of a static weak field of gravity ϕ can be 'compensated' or 'balanced' by a rapidity ψ, with $\phi = \psi c^2$. This is not pure theory but an experimental fact expressed in a somewhat different way as it is usually presented. Thus, experiments give us a correspondence between the scalar approximations of the 3D helicity rotator and the 4D hyperbolic rotator.

11 The hyperbolic rotation of the Dirac-Weyl equation

In the previous sections we focused upon the effect of the Dirac-Weyl hyperbolic rotator on the energy-momentum vector and the metric vector in the Dirac environment. Now we will apply the rotation to the energy-momentum operators. If we replace $\not{P}$ by $-\mathbf{i}\hbar\not{\partial}$ and assume that the rotation angle stands for the gravitational potential of a static, central weak field of gravity with $\alpha = \frac{\phi}{c^2} = -\frac{GM}{Rc^2}$, then gravity force equations appear. We know that, for the momentum-energy, we have the invariant Eqn.(51). What happens if we apply this to the Dirac-Weyl equation? We start with the form $(\widehat{\not{P}} - E\mathbb{1})\Psi = 0$. With $\widehat{\not{P}} = -\mathbf{i}\hbar\not{\partial}$ we get

$$(\not{\partial} - \mathbf{i}\frac{E}{\hbar}\mathbb{1})\Psi = 0. \tag{73}$$

Then we apply the hyperbolic rotation $e^{\frac{\not{P}}{E}\alpha}$ to get

$$e^{-\frac{\not{P}}{E}\alpha}(\not{\partial} - \mathbf{i}\frac{E}{\hbar}\mathbb{1})e^{\frac{\not{P}}{E}\alpha}\Psi = 0, \tag{74}$$

so

$$e^{-\frac{\not{P}}{E}\alpha}\not{\partial}e^{\frac{\not{P}}{E}\alpha}\Psi - \mathbf{i}\frac{E}{\hbar}\not{1}\Psi = 0, \tag{75}$$

and as a result, we have to examen what we get from $\not{\partial}e^{\frac{\not{P}}{E}\alpha}\Psi$. Using Eqn.(50), this gives

$$\begin{aligned}
\not{\partial}\cosh\alpha\Psi + \not{\partial}\frac{\not{P}}{E}\sinh\alpha\Psi = \\
\cosh\alpha\not{\partial}\Psi + \sinh\alpha(\not{\partial}\alpha)\Psi + \sinh\alpha\frac{1}{E}(\not{\partial}\not{P})\Psi + \\
\cosh\alpha(\not{\partial}\alpha)\frac{\not{P}}{E}\Psi + \sinh\alpha\frac{\not{P}}{E}\not{\partial}\Psi = \qquad (76) \\
e^{\frac{\not{P}}{E}\alpha}\not{\partial}\Psi + \frac{1}{E}(\not{\partial}\alpha)(\not{P}\cosh\alpha + E\not{1}\sinh\alpha)\Psi + \sinh\alpha\frac{1}{E}(\not{\partial}\not{P})\Psi = \qquad (77) \\
e^{\frac{\not{P}}{E}\alpha}\not{\partial}\Psi + (\not{\partial}\alpha)e^{\frac{\not{P}}{E}\alpha}\frac{\not{P}}{E}\Psi + \sinh\alpha\frac{1}{E}(\not{\partial}\not{P})\Psi. \qquad (78)
\end{aligned}$$

In this derivation we simplified a step, skipped a problem, because $\not{\partial}\not{P}\Psi$ is in some cases not simply $(\not{\partial}\not{P})\Psi + \not{P}\not{\partial}\Psi$ due to the non-commutative math, but we still used that as an outcome. So the result in the rest of this section, being based on that assumption is indicative, not definitive. This brings us at

$$e^{-\frac{\not{P}}{E}\alpha}\not{\partial}e^{\frac{\not{P}}{E}\alpha}\Psi = \not{\partial}\Psi + e^{-\frac{\not{P}}{E}\alpha}(\not{\partial}\alpha)e^{\frac{\not{P}}{E}\alpha}\frac{\not{P}}{E}\Psi + e^{-\frac{\not{P}}{E}\alpha}\sinh\alpha\frac{1}{E}(\not{\partial}\not{P})\Psi. \tag{79}$$

which implies for the equation

$$(\not{\partial} - \mathbf{i}\frac{E}{\hbar}\not{1})\Psi + e^{-\frac{\not{P}}{E}\alpha}(\not{\partial}\alpha)e^{\frac{\not{P}}{E}\alpha}\frac{\not{P}}{E}\Psi + e^{-\frac{\not{P}}{E}\alpha}\sinh\alpha\frac{1}{E}(\not{\partial}\not{P})\Psi = 0. \tag{80}$$

Now let us assume that we have the condition $\not{\partial}\not{P} = 0$, a condition that we examined before as $\partial^T P = 0$, see Eq.(24). That were the conditions of a conserved force field. Then we get

$$(\not{\partial} - \mathbf{i}\frac{E}{\hbar}\not{1})\Psi + e^{-\frac{\not{P}}{E}\alpha}(\not{\partial}\alpha)e^{\frac{\not{P}}{E}\alpha}\frac{\not{P}}{E}\Psi = 0, \tag{81}$$

an equation of which we just assume that it can be simplified to

$$(\not{\partial} - \mathbf{i}\frac{E}{\hbar}\not{1})\Psi + (\not{\partial}\alpha)\frac{\not{P}}{E}\Psi = 0. \tag{82}$$

And with $\frac{\not{P}}{E} = -\mathbf{i}\frac{\not{U}}{c}$ we get

$$(\not{\partial} - \mathbf{i}\frac{E}{\hbar}\not{1})\Psi - \mathbf{i}\frac{1}{c}(\not{\partial}\alpha)\not{U}\Psi = 0. \tag{83}$$

So we get an extra condition $\not{\partial}\not{P} = 0$ and an extra term containing $(\not{\partial}\alpha)\not{U}$. If we add $\alpha = \frac{\phi}{c^2}$ and assume it to be a time independent static field we get $c^2\not{\partial}\alpha =$

$\partial\!\!\!/ \phi = \boldsymbol{\gamma}\cdot\nabla\phi = -\boldsymbol{\gamma}\cdot\mathbf{g} = -g\!\!\!/$. The conclusion is that, given our simplifications, we 'rotated' gravity in the Dirac-Weyl equation, resulting in an approximate

$$(\partial\!\!\!/ - \mathbf{i}\frac{E}{\hbar}\mathbb{1})\Psi + \mathbf{i}\frac{1}{c^3}G\!\!\!/ U\!\!\!/\Psi = 0. \tag{84}$$

If we translate the above to a possible Lagrangian density resulting from the hyperbolic rotator, we might start with $\mathscr{L} = \overline{\Psi}e^{-\frac{p\!\!\!/}{E}\alpha}(-\mathbf{i}\hbar\partial\!\!\!/ - E\mathbb{1})e^{\frac{p\!\!\!/}{E}\alpha}\Psi$. If our assumptions leading to Eq.(84) can be applied, the Lagrangian density would become

$$\mathscr{L} = \overline{\Psi}(\widehat{P\!\!\!/} - E\mathbb{1})\Psi + \overline{\Psi}\frac{\hbar}{c^3}G\!\!\!/ U\!\!\!/\Psi \tag{85}$$

with the square of the Planck length $l_p^2 = \frac{\hbar G}{c^3}$ in the last product. But due to the many the assumptions we had to make to get at equations (84) and (85), these results are only indicative. They are indicators of the math-phys structural richness that arises when applying the 4-D quantum rotator of equation (50) to the Dirac-Weyl equations.

12 Conclusion

Initially we hoped that our math-phys language, part of the development of which is presented in this paper, would bring a new spirit in the discussions regarding the problem of the electron as the problematic non-zero divergence of its EM self stress-energy tensor. We tried to replace the Minkowski-Laue paradigm with a different math-phys language for relativistic dynamics, a language that should be beyond symmetric and anti-symmetric issues, that could integrate the full spin matrix formalisms and the Dirac QM treatment of the electron and at the same time include the pre-GR attempts to formulate a relativistic theory of gravity.

Now we are curious to what extend our 4D hyperbolic quantum rotator is beyond the level of the failed pre-GR attempts to formulate a relativistic theory of gravity. We are of course still pre-GR and pre-YM and within the environment of the Dirac equation. But we might have expanded the reach of the Dirac equation by exposing it to our 4D rotator. To what extend have we thus entered the physical realm of quantum gravity? If we interpret the 4D rotator in analogy to the 3D helicity rotator, in which the rapidity contains the info concerning the magnitude of the Lorentz boost and helicity harbors the rest of the info, then the Dirac-Weyl $p\!\!\!/$ part of the rotator should contain the gravity transformation info that is not in the scalar $\alpha = \phi/c^2$. If verifiable, then how more quantum could gravity become?

But of course, our whole math-phys language, including the 4D quantum rotator, is just a math-phys construction without verification. Is it internally consistent? Can it harbor QM and relativistic gravity beyond our anachronistic pre-YM and pre-GR context or will it turn out to be a dead end as soon as we go any further? Is it perhaps just a matter of trying out the idea of the 4D quantum rotator in the standard math-phys language without this papers spin-metric math-phys alternative, by connecting it for example directly to a metric interpretation of the Dirac matrices as proposed by Fock and Iwanenko already in 1929 [31]?

References

1. Darrigol, O.: Electrodynamics from Ampère to Einstein. Oxford University Press, New York (2000), p. 360-366.
2. Miller, A.I.: Albert Einstein's Special Theory of Relativity. Springer-Verlag, Reading Mass. (1981).
3. Abraham, M.: Zur elektromagnetischen Mechanik. Phys. Z. 10, 737-741 (1909)
4. Einstein, A.: Bemerkungen zu der Notiz von Hrn Paul Ehrenfest: 'Die Translation deformierbarer Elektronen und der Flächensatz'. Ann. Phys. 23, 206-208 (1907)
5. Minkowski, H.: "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern". Math. Ann. 68, 472-525 (1910); Nachr. Ges. Wiss. Göttingen, 53-111 (1908)
6. von Laue, M.: Zur Dynamik der Relativitätstheorie. Ann. Phys. 35, 524-542 (1911)
7. Norton, J.D.: Einstein, Nordström and the Early Demise of Scalar, Lorentz-Covariant Theories of Gravitation. Archive for History of Exact Sciences 45, 17-94 (1992)
8. Campos, I., Jiménez, J.L.: Comment on the 4/3 problem in the electromagnetic mass and the Boyer-Rohrlich controversy. Phys. Rev. D 33, 607-610 (1986)
9. Lopéz-Mariño, M.A., Jiménez, J.L.: Analysis of the Abraham-Minkowski controversy by means of two simple examples. Found. Phys. Lett. 17, 1-23 (2004)
10. Griffiths, D.J.: Resource Letter EM-1: Electromagnetic Momentum. Am. J. Phys. 80, 7-18 (2012)
11. Thomas, L.H.: The Kinematics of an Electron with an Axis. Philos. Mag. 3, 1-22 (1927)
12. Frenkel, J.: Die Elektrodynamik des rotierenden Elektrons. Zeit. f. Phys. 37, 243-262 (1926)
13. Kramers, H. A.: Quantentheorie des Elektrons und der Strahlung. In: Hand- und Jahrbuch der Chemischen Physik, EuckenWolf, Leipzig (1938); English translation: Quantum Mechanics. N-H Publishing Company, Amsterdam (1958) Chap. VI, p. 225-232.
14. de Broglie, L.: La Théorie des Particules de Spin 1/2 (Électrons de Dirac). Gauthier-Villars, Paris (1952) p. 40-42.
15. Dirac, P.A.M.: Classical Theory of Radiating Electrons. Proc. R. Soc. Lond. A 167, 148-169 (1938)
16. Dirac, P.A.M.: A new classical theory of electrons. Proc. Roy. Soc. A 209, 291-296 (1951)
17. Dirac, P.A.M.: Is there an Æther? Nature 168, 906-907 (1951)
18. Dirac, P.A.M.: Response to I. Infeld. Nature 169, 702 (1952)
19. Einstein, A.: Zum gegenwärtigen Stande des Gravitationsproblems. Phys. Z. 14, 1249-1262 (1913)
20. Abraham, M.: Zur Theorie der Gravitation; Das Elementargesetz der Gravitation. Phys. Z. 13, 1-5 (1912)
21. Nordström, G.: Relativitätsprinzip und Gravitation. Phys. Z. 13, 1126-1129 (1912)
22. Nordström, G.: Über den Energiesatz in der Gravitationstheorie. Phys. Z. 15, 375-380 (1914)
23. Mie, G.: Grundlagen einer Theorie der Materie. Ann. Phys. 37, 511-534 (1912); Ann. Phys. 39, 1-40 (1912); Ann. Phys. 40, 1-66 (1913)
24. Mie, G.; Bemerkungen zu der Einsteinschen Gravitationstheorie. Phys. Z. 14, 115-122; 169-176 (1914)
25. de Haas, E.P.J.: Biquaternion formulation of relativistic tensor dynamics. Apeiron 15, 358-381(2008)
26. Hestenes, D.: Spacetime physics with geometric algebra. Am. J.Phys. 71, 691-714 (2003)
27. Varičak, V.: Über die nichteuklidische Interpretation der Relativtheorie. Jahresbericht der Deutschen Mathematiker-Vereinigung 21, 103-127 (1912)
28. Rindler, W.: Relativity. Special, General and Cosmological. Oxford University Press, New York (2001) p. 155-156
29. Einstein, A.: Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes. Ann. Phys. 35, 898-908 (1911)
30. Pound, R. V., Rebka Jr. G. A.: Apparent Weight of Photons. Phys. Rev. Lett. 4, 337-341 (1960)
31. Fock, V., Iwanenko, D.: Über eine mögliche geometrische Deutung der relativistischen Quantentheorie. Zeit. f. Phys. 54, 798-802 (1929)