

## Thermodynamic Derivation of Maxwell's Electrodynamic Equations

D-r Sc., prof. V.A.Etkin

The derivation conclusion of Maxwell's equations is given from the first principles of nonequilibrium thermodynamics. It is shown, that these equations represent the modified form of its phenomenological laws and should contain the full derivatives on time from vectors of electric and magnetic induction

Let us consider, for generality, a stationary system being in the external electric  $\mathbf{E}$  and magnetic  $\mathbf{H}$  fields and adequately possessing the electric and magnetic degrees of freedom. The unit volume energy  $E_v$  for such a system is a function of the electric  $\mathbf{D}$  and magnetic  $\mathbf{B}$  induction vectors which in their turn depend on the intensities  $\mathbf{E}$  and  $\mathbf{H}$  of these fields. The thermodynamic relationship between them is [1.2]

$$dE_v = \mathbf{E} \cdot d\mathbf{D} + \mathbf{H} \cdot d\mathbf{B} . \quad (1)$$

The terms on the right side of the equation characterize, respectively, the elementary polarization  $dW_{ev} = \mathbf{E} \cdot d\mathbf{D}$  magnetization  $dW_{mv} = \mathbf{H} \cdot d\mathbf{B}$  works of the system.

Let us assume that in such a system the reciprocal conversions of electric and magnetic field energies run, which powers are

$$N_e = \mathbf{E} \cdot d\mathbf{D}/dt; \quad N_m = \mathbf{H} \cdot d\mathbf{B}/dt. \quad (2)$$

Providing these processes do not change the system energy and entropy (i.e. the energy of the system entirely converts from a one ordered form into the other), the power balance  $N_e = -N_m$  is evident. This directly leads to the following relationship:

$$\mathbf{E} \cdot (d\mathbf{D}/dt) = - \mathbf{H} \cdot (d\mathbf{B}/dt). \quad (3)$$

These simple relationships may be given the form offered by Maxwell. For this let us consider a system consisting of a closed electric circuit with an arbitrary length  $\ell_e$  and variable (in general case) cross section  $f_e$ , which comprises an as well closed magnetic circuit with a length  $\ell_m$  and cross section  $f_m$  variable thru the length [3]. Taking this variability into consideration, (2) should be changed to the integral form:

$$N_e = \int \mathbf{E} \cdot (d\mathbf{D}/dt) dV_e; \quad N_m = \int \mathbf{H} \cdot (d\mathbf{B}/dt) dV_m, \quad (4)$$

The bulk elements may be represented in the form  $dV_e = d\ell_e \cdot df_e$  and  $dV_m = d\ell_m \cdot df_m$ , where  $d\ell_e$ ,  $d\ell_m$  and  $df_e$ ,  $df_m$  – orthogonal vector elements of length and cross section, respectively, of the electric circuit and dielectric. The expressions (4) may be rewritten in terms of energodynamics as follows:

$$N_e = \int \mathbf{E} \cdot d\ell_e \int (d\mathbf{D}/dt) df_e = X_e J_e^D; \quad (5)$$

$$N_m = \int \mathbf{H} \cdot d\ell_m \int (d\mathbf{B}/dt) df_m = X_m J_m^D, \quad (6)$$

where  $J_e^D = \int (d\mathbf{D}/dt) df_e$ ,  $J_m^D = \int (d\mathbf{B}/dt) df_m$  – total fluxes of electric and magnetic displacements, respectively, sometimes named the “linkage fluxes” and traditionally represented by the number of the lines of force linking the cross section of the electric and magnetic circuits, respectively [4];  $X_e = \int \mathbf{E} \cdot d\ell_e$ ,  $X_m = \int \mathbf{H} \cdot d\ell_m$  – the so-called electromotive and magnetomotive forces (emf and

mmf) defined by the circulation of the vectors  $\mathbf{E}$  and  $\mathbf{H}$ , respectively, along the closed electric and magnetic circuits.

Now the electromagnetic field equations may be given the form of Onsager's phenomenological laws [1...3]:

$$J_e^D = L_{ee}X_e + L_{eM}X_M; \quad (7)$$

$$J_M^D = L_{Me}X_e + L_{MM}X_M. \quad (8)$$

Here the terms  $L_{ee}X_e$  and  $L_{MM}X_M$  characterize the relaxation part of the displacement fluxes, which relates to the conduction current  $I_e = L_{ee}X_e$  and its magnetic analog  $I_M = L_{MM}X_M$  and is caused by, respectively, electric and magnetic energy dissipation; the terms  $L_{eM}X_M$  и  $L_{Me}X_e$ , on the contrary, relate to overcoming the "extraneous" forces by the flow, i.e. to conversion of the electric energy into magnetic one and vice versa. Since  $N_e = -N_M$  relationships (5) – (6) may take a more simple form:

$$J_e^D/X_M = -J_M^D/X_e. \quad (9)$$

Comparing this equation with phenomenological laws (7) and (8) one can find that the left-side part of (9) defines the coefficient  $L_{eM}$ , while the right-side part  $-L_{Me}$ . From this the Cazimir's anti-symmetry conditions follow [1...3]:

$$L_{eM} = -L_{Me}. \quad (10)$$

The value and dimensionality of these coefficients depend on the system of units chosen. In the international system of units (SI)  $L_{eM} = -L_{Me} = 1$ , therefore it may be written instead of (10):

$$X_e = -\int (d\mathbf{B}/dt) d\mathbf{f}_M, \quad (11)$$

$$X_M = \int (d\mathbf{D}/dt) d\mathbf{f}_e. \quad (12)$$

The first of these relationships represents Faraday's law (flux rule) reading that emf is equal in value, but opposite in sign to the variation rate of the electric circuit-linking flux. Let us now change, based on the Stokes theorem, in the force equation  $X_e = \int \mathbf{E} \cdot d\mathbf{l}_e$  from the curvilinear integral taken over the closed electric circuit with a length of  $\ell_e$  to the integral  $\int \text{rot} \mathbf{E} \cdot d\mathbf{f}_M$  over the magnetic circuit cross section  $f_M$ . In the similar way one can change in the force equation  $X_M = \int \mathbf{H} \cdot d\mathbf{l}_M$  from the curvilinear integral over the closed magnetic circuit with a length of  $\ell_M$  to the integral  $\int \text{rot} \mathbf{H} \cdot d\mathbf{f}_e$  over the surface  $f_e$  covering the electric circuit. Then (11) and (12) become:

$$\int \text{rot} \mathbf{E} \cdot d\mathbf{f}_M = -\int (d\mathbf{B}/dt) d\mathbf{f}_M. \quad (13)$$

$$\int \text{rot} \mathbf{H} \cdot d\mathbf{f}_e = \int (d\mathbf{D}/dt) d\mathbf{f}_e; \quad (14)$$

Or in the differential form

$$\text{rot} \mathbf{E} = -d\mathbf{B}/dt, \quad (15)$$

$$\text{rot} \mathbf{H} = d\mathbf{D}/dt. \quad (16)$$

<sup>1)</sup> Availability of such relationships once again confirms that electricity and magnetism are two independent phenomena with interrelation between them seen only in dynamics.

These equations differ from the corresponding Maxwell's equations in that they contain the total time derivatives of electric and magnetic induction vectors. This is not a surprise since the primary equations of energodynamics (1) as well contain the exact differentials of polarization and magnetization vectors. It is significant that Maxwell himself initially defined emf as well thru the total derivative  $d\Phi/dt$  of magnetic flux  $\Phi$  (K. Polivanov, 1982).

To form equations (15) and (16) into a more common shape, let us expand the derivatives  $d\mathbf{B}/dt$  and  $d\mathbf{D}/dt$  on the assumption that polarization and magnetization processes are absent in a system under consideration ( $\mathbf{P}, \mathbf{M} = 0$ ). Since  $(\partial\mathbf{E}/\partial\mathbf{r}) = \rho_e$ , whereas magnetic analogs to the free charge  $\rho_e$  do not exist, we have:

$$d\mathbf{B}/dt = (\partial\mathbf{B}/\partial t); \quad d\mathbf{D}/dt = (\partial\mathbf{D}/\partial t) + \mathbf{j}_e. \quad (17)$$

Here  $\mathbf{j}_e = (\mathbf{v}_e \cdot \nabla)\mathbf{D} = \rho_e \mathbf{v}_e$  – conduction current caused by the free charge transfer.

Thus finally:

$$\text{rot } \mathbf{E} = -(\partial\mathbf{B}/\partial t), \quad (18)$$

$$\text{rot } \mathbf{H} = \mathbf{j}_e + (\partial\mathbf{D}/\partial t). \quad (19)$$

As for another couple of Maxwell's equations:

$$\text{div } \mathbf{D} = \rho_e, \quad \text{div } \mathbf{B} = 0, \quad (20)$$

the first of them is a direct consequence from expression (1) written in the differential form since the electric displacement vector  $\mathbf{D}$  is identical by implication to the vector  $\mathbf{Z}_{eV}$ . Second term of the relationship (20) just states the fact of the absence of magnetic “monopolies” analogous to the electric charge  $\rho_e$ .

The thermodynamic derivation of electromagnetic field equations set herein forth rebuts the popular opinion that the Maxwell's equations are non-derivable from whatever primary laws. At the same time this derivation discloses a number of assumptions laid into their foundation. First of all, electromotive and magnetomotive forces have been defined in (11) and (12) for closed electric and magnetic circuits. Hence the Maxwell's equations are inapplicable to non-closed electric currents and their elements. This is what Maxwell himself stressed. Secondly, such processes as the re-polarization in semiconductors and the magnetic circuit re-magnetization were excluded from consideration. As a result, the losses from irreversibility of the electric-to-magnetic (and vice versa) field conversion were allowed for only the electric form of energy (due to conduction currents). The magnetic loss (re-magnetization, eddy currents, etc.) which lead to a magnetic analog of the conduction current was not taken into consideration in that case. It was assumed further that the electric and magnetic fluxes in equations (11) and (12) were homogeneous so that the power balance  $N_e = N_M$  was observed for each of the local zones of a system under investigation (otherwise changing to the differential form (15)-(16) of the above equations becomes incorrect). These circumstances cause some incompleteness of the Maxwell's equations and disable interpreting on their basis a number of effects hardly prone to analysis [3].

## References

1. *Etkin V.A.* To the thermodynamic theory of non-linear irreversible processes.// Russian Journal of Physical Chemistry, 1985, **59**(3), pp. 2246...2249 (translated from *Zhurnal Fizicheskoi Khimii*, 1985, **59**(3), 560...567).
2. *De Groot S.R., Mazur R.* Non-Equilibrium Thermodynamics:– Amsterdam, 1962.
3. *Etkin V.A.* Energodynamics (Thermodynamic Fundamentals of Synergetics) – New York, 2011. – 480 p.
4. *Landau L., Pitaevskii L., Lifshitz E.* Electrodynamics of Continuous Media. 2nd edition, Butterworth-Heinemann, 1984.