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Demystification of the Geometric Fourier Transforms

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Abstract. As it will turn out in this paper, the recent hype about most of the Clifford Fourier transforms is not worth the pain. Almost every one that has a real application is separable and these transforms can be decomposed into a sum of real valued transforms with constant multivecor factors. This fact makes their interpretation, their analysis and their implementation almost trivial.

Keywords: geometric algebra, Clifford algebra, Fourier transform, trigonometric transform, convolution theorem.

INTRODUCTION

In recent literature, three different approaches to hypercomplex Fourier transforms have been considered. As in [1], we can identify them as follows:

- A: Eigenfunction approach
- B: Generalized roots of -1 approach
- C: Characters of spin group approach

Approach A is studied in many papers like [2, 3, 4], approach B comprehensively in [5]. The third approach is followed in [6, 7]. In this paper, we will only work with transforms from Definition 1, which was introduced in [5]. Because even though, the concept of approach C differs very much from approach B, the resulting transforms can all be expressed as special cases of this definition.

Definition 1 (Geometric Fourier transform)**.** The general **geometric Fourier transform** (GFT) $\mathscr{F}_{F_1,F_2}(A)$ of a multivector field $A:\mathbb{R}^{p',q'}\to C\ell_{p,q}, p'+q'=m\in\mathbb{N}, p+q=n\in\mathbb{N}$ is defined by the calculation rule

$$\mathscr{F}_{F_1,F_2}(A)(u) := \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(x,u)} A(x) \prod_{f\in F_2} e^{-f(x,u)}\,\mathrm{d}^m x, \tag{1}$$

with two ordered finite sets $F_1=\{f_1(x,u),...,f_\mu(x,u)\}$, $F_2=\{f_{\mu+1}(x,u),...,f_\nu(x,u)\}$ of mappings $f_l(x,u):\mathbb{R}^m\times\mathbb{R}^m\to\mathscr{I}^{p,q}, \forall l=1,...,\nu$ and $x,u\in\mathbb{R}^m$.

Example 2. Definition 1 covers all known transforms from approach B and approach C, for example, the Clifford Fourier transform introduced by Jancewicz [8] and expanded by Ebling and Scheuermann [9] and Hitzer and Mawardi [10] or the one established by Sommen in [11] and re-established by Bülow [12]. Further we have the quaternionic Fourier transform by Ell [13] and later by Bülow [12], the spacetime Fourier transform and the two-sided transform by Hitzer [14, 15], the Clifford Fourier transform for color images by Batard et al. [6], the Cylindrical Fourier transform by Brackx et al. [16], the transforms by Felsberg [17] or Ell and Sangwine [18, 19].

In the following section, we will introduce a powerful tool for the analysis of the geometric Fourier transforms: the trigonometric transform. Then, we will first use it to reveal the true nature of the major subclass of separable GFTs, which covers almost every applied Clifford Fourier transform. In the last section of the paper, we will enjoy the advantages of this insight and show how it leads to a convolution theorem that is superior to the one in [20] because it requires less restrictions.

THE TRIGONOMETRIC TRANSFORM

In contrast to the hypercomplex definition in [20], we want to define real valued general trigonometric transforms, that only consist of scalar appearances of sines and cosines. Therefore we use the following notation.

Notation 3. For a multivector $A \in \mathscr{I}_{p,q} = \{B \in C\ell_{p,q}, B^2 \in \mathbb{R}^-\} \subset C\ell_{p,q}$ that squares to a negative real number and $j \in \{0,1\}$, we define

$$e_j^A := \begin{cases} \cos(||A||), & \text{if } j = 0, \\ \sin(||A||), & \text{if } j = 1 \end{cases} \tag{2}$$

with the norm $||A|| = \sqrt{A\overline{A}} \in \mathbb{R}$.

Lemma 4. *The exponential of a multivector $A \in \mathscr{I}_{p,q} \subset C\ell_{p,q}$ that squares to a negative real number satisfies*

$$e^A = \sum_{j \in \{0,1\}} (\frac{A}{||A||})^j e_j^A. \tag{3}$$

Definition 5 (Trigonometric transform)**.** Let $A : \mathbb{R}^m \to C\ell_{p,q}$ be a multivector field, $x, u \in \mathbb{R}^m$ vectors, F_1, F_2 two ordered finite sets of μ, respectively $\nu - \mu$, mappings $\mathbb{R}^m \times \mathbb{R}^m \to \mathscr{I}_{p,q} \subset C\ell_{p,q}$, and $j \in \{0,1\}^\mu, k \in \{0,1\}^{(\nu-\mu)}$ multi-indices. The **Trigonometric Transform** (TT) $\mathscr{F}_{F_1^j, F_2^k}$ is defined by

$$\mathscr{F}_{F_1^j, F_2^k}(A)(u) := \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j_l}^{-f_l(x,u)} A(x) \prod_{l=\mu+1}^{\nu} e_{k_l}^{-f_l(x,u)} \, \mathrm{d}^m x \tag{4}$$

with $e_j^{-f(x,u)}$ from Notation 3.

Remark 6. The $e_j^{-f(x,u)} \in \mathbb{R}$ are in the center of the geometric algebra, therefore there is no use with regards to content to distinguish the order of their appearances. It will be helpful though to stress their relation to the GFT.

Example 7. The standard cosine transform is a special case of this definition with $F_1 = \emptyset, F_2 = \{2\pi xu\}, k \in \{0,1\}^1 = 0$

$$\mathscr{F}_c(A)(u) = \int_{\mathbb{R}} A(x) \cos(xu) \, \mathrm{d}x = \mathscr{F}_{\emptyset, (2\pi xu)^0}(A)(u). \tag{5}$$

THE TRUE NATURE OF SEPARABLE GFT

The definition of separability has already been introduced in [5].

Definition 8. We call a mapping $f : \mathbb{R}^m \times \mathbb{R}^m \to C\ell_{p,q}$ x**-separable** or separable with respect to its first argument, if it suffices

$$f = ||f(x,u)|| i(u), \tag{6}$$

where $i : \mathbb{R}^m \to C\ell_{p,q}$ is a function that does not depend on x. Analogously we call it **separable** or separable with respect to both arguments, if it suffices

$$f = ||f(x,u)|| i, \tag{7}$$

whith constant $i \in C\ell_{p,q}$.

Analogously, a geometric Fourier transform that consists of separable mappings F_1, F_2 is called separable. Separability is a central quality for multiplication, shift and convolution properties of GFT. Almost every transform from approach B and C is separable. If there exist non separable transforms that are invertible at all, is an issue of current research. Therefore, the importance of this class of GFTs is obvious because the applications of a transform that puts a function into a space from which it may never return are rather sparse.

Example 9. From all the examples of special cases of Definition 1 in the introduction, only some cases of the two-sided transform [15] and the cylindrical transform [16] for dimensions higher than two are not separable.

In this section, we want to take a closer look at this vast class of GFTs. By expressing them by means of the trigonometric transforms, we will be able to reveal their true nature: they are combinations of simple real-valued transforms.

Theorem 10 (GFT decomposition into TT)**.** *Any geometric Fourier transform $\mathscr{F}_{F_1,F_2}$ with x-separable mappings $\forall l = 1,...,\nu : f_l(x,u) = ||f_l(x,u)||i_l(u)$ of a multivector field $A(x) = \sum_r a_r(x)e_r$ is the sum of real valued trigonometric transforms $\mathscr{F}_{F_1^j,F_2^k}(a_r)(u) \in \mathbb{R}$ with multivector factors*

$$\mathscr{F}_{F_1,F_2}(A)(u) = \sum_r \sum_{\substack{j\in\{0,1\}^{\mu},\\ k\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu}(-i_l(u))^{j_l}\mathscr{F}_{F_1^j,F_2^k}(a_r)(u)e_r \prod_{l=\mu+1}^{\nu}(-i_l(u))^{k_l}. \tag{8}$$

Corollary 11. *A geometric Fourier transform $\mathscr{F}_{F_1,F_2}$ with separable mappings $\forall l = 1,...,\nu : f_l(x,u) = i_l||f_l(x,u)||, i_l^2 \in \mathbb{R}^-$ of a multivector field $A(x) = \sum_r a_r(x)e_r$ is the sum of real valued trigonometric transforms $\mathscr{F}_{F_1^j,F_2^k}(a_r)(u) \in \mathbb{R}$ with constant multivector factors*

$$\mathscr{F}_{F_1,F_2}(A)(u) = \sum_r \sum_{\substack{j\in\{0,1\}^{\mu},\\ k\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu}(-i_l)^{j_l}\mathscr{F}_{F_1^j,F_2^k}(a_r)(u)e_r \prod_{l=\mu+1}^{\nu}(-i_l)^{k_l}. \tag{9}$$

That means, if we interpret a multivector valued signal as many signals, saved in 2^n channels, the separable geometric Fourier transforms can be interpreted as real valued transforms, that work one after another on each of the channels, get added in a certain way and written into certain channels depending on the multivector factor. As a result they can be interpreted, analyzed, and implemented with the same tools as the classical real-valued transforms.

CONVOLUTION THEOREM FOR NOT COORTHOGONAL EXPONENTS

So far in this paper, the description of the GFTs by means of the TT has mainly lead to the negative meaning of demystification. We showed, that most of GFTs do not differ very much from the real valued trigonometric transforms. That is why, one may regard them as not very interesting. Now, it is time to use the positive side of the demystification and exploit the simplicity to derive new properties, that could not be found without it.

In [20], we presented a convolution theorem for general geometric Fourier transforms with F_1, F_2 being coorthogonal, separable and linear with respect to the first argument. Coorthogonality can be interpreted as mutual commutation or anticommutation among the functions.

Example 12. Every example of special cases of Definition 1 from the introduction is coorthogonal.

Although coorthogonality is fulfilled by almost every popular geometric Fourier transform, we want to deduce a convolution theorem that holds for functions that are separable and linear with respect to the first argument but have arbitrary commutation properties. This formulation of the theorem is especially useful for the treatment of steerable Fourier transforms over the manifolds of square roots of minus one as in [21], which are generally not coorthoganal.

Definition 13. Let $A(x), B(x) : \mathbb{R}^m \to C\ell_{p,q}$ be two multivector fields. Their **convolution** $(A * B)(x)$ is defined as

$$(A * B)(x) := \int_{\mathbb{R}^m} A(y)B(x-y)\,\mathrm{d}^m y. \tag{10}$$

Theorem 14 (convolution)**.** *Let $A, B, C : \mathbb{R}^m \to C\ell_{p,q}$ be multivector fields with $A(x) = (C * B)(x)$ and F_1, F_2 be separable and linear with respect to the first argument, then the geometric Fourier transform of A satisfies the convolution property*

$$\mathscr{F}_{F_1,F_2}(A)(u) = \sum_{\substack{j,j'\in\{0,1\}^{\mu}\\ k,k'\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu}(-i_l(u))^{j_l+j'_l}\mathscr{F}_{F_1^j,F_2^k}(C)(u)\mathscr{F}_{F_1^{j'},F_2^{k'}}(B)(u) \prod_{l=\mu+1}^{\nu}(-i_l(u))^{k_l+k'_l}. \tag{11}$$

Proof. The transform satisfies

$$
\begin{aligned}
\mathscr{F}_{F_1,F_2}(A)(u) &= \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(x,u)} (C*B)(x) \prod_{f\in F_2} e^{-f(x,u)}\, \mathrm{d}^m x \\
&= \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(x,u)} \int_{\mathbb{R}^m} C(y)B(x-y)\, \mathrm{d}^m y \prod_{f\in F_2} e^{-f(x,u)}\, \mathrm{d}^m x \\
&\overset{x-y=z}{=} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(z+y,u)} C(y)B(z) \prod_{f\in F_2} e^{-f(z+y,u)}\, \mathrm{d}^m y\, \mathrm{d}^m z.
\end{aligned}
\tag{12}
$$

Since F_1 and F_2 are linear and separable with respect to the first argument, this equivalent to

$$
\mathscr{F}_{F_1,F_2}(A)(u) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{f\in F_1} (e^{-f(z,u)}e^{-f(y,u)}) C(y)B(z) \prod_{f\in F_2} (e^{-f(y,u)}e^{-f(z,u)})\, \mathrm{d}^m y\, \mathrm{d}^m z.
\tag{13}
$$

Now, we apply (**??**) with $\frac{f_l(x,u)}{|f_l(x,u)|} = i_l(u)$ and get

$$
\begin{aligned}
\mathscr{F}_{F_1,F_2}(A)(u) = \sum_{\substack{j,j\prime\in\{0,1\}^\mu \\ k,k'\in\{0,1\}^{(\nu-\mu)}}} & \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{l=1}^{\mu} (-i_l(u))^{j'_l} e_{j'_l}^{-f_l(z,u)} (-i_l(u))^{j_l} e_{j_l}^{-f_l(y,u)} C(y)B(z) \\
& \prod_{l=\mu+1}^{\nu} (-i_l(u))^{k_l} e_{k_l}^{-f_l(y,u)} (-i_l(u))^{k'_l} e_{k'_l}^{-f_l(z,u)}\, \mathrm{d}^m y\, \mathrm{d}^m z.
\end{aligned}
\tag{14}
$$

The separated exponentials are real valued and therefore in the center of the geometric algebra this is equivalent to

$$
\begin{aligned}
\mathscr{F}_{F_1,F_2}(A)(u) = \sum_{\substack{j,j\prime\in\{0,1\}^\mu \\ k,k'\in\{0,1\}^{(\nu-\mu)}}} & \prod_{l=1}^{\mu}(-i_l(u))^{j_l+j'_l} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j'_l}^{-f_l(z,u)} \prod_{l=1}^{\mu} e_{j_l}^{-f_l(y,u)} C(y) \\
& \prod_{l=\mu+1}^{\nu} e_{k_l}^{-f_l(y,u)} B(z) \prod_{l=\mu+1}^{\nu} e_{k'_l}^{-f_l(z,u)}\, \mathrm{d}^m y\, \mathrm{d}^m z \prod_{l=\mu+1}^{\nu} (-i_l(u))^{k_l+k'_l}.
\end{aligned}
\tag{15}
$$

Using Definition 5, leads to

$$
\begin{aligned}
\mathscr{F}_{F_1,F_2}(A)(u) = & \sum_{\substack{j,j\prime\in\{0,1\}^\mu \\ k,k'\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu}(-i_l(u))^{j_l+j'_l} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j'_l}^{-f_l(z,u)} \mathscr{F}_{F_1^j,F_2^k}(C)(u) \\
& B(z) \prod_{l=\mu+1}^{\nu} e_{k'_l}^{-f_l(z,u)}\, \mathrm{d}^m z \prod_{l=\mu+1}^{\nu} (-i_l(u))^{k_l+k'_l} \\
= & \sum_{\substack{j,j\prime\in\{0,1\}^\mu \\ k,k'\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu}(-i_l(u))^{j_l+j'_l} \mathscr{F}_{F_1^j,F_2^k}(C)(u) \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j'_l}^{-f_l(z,u)} \\
& B(z) \prod_{l=\mu+1}^{\nu} e_{k'_l}^{-f_l(z,u)}\, \mathrm{d}^m z \prod_{l=\mu+1}^{\nu} (-i_l(u))^{k_l+k'_l} \\
= & \sum_{\substack{j,j\prime\in\{0,1\}^\mu \\ k,k'\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu}(-i_l(u))^{j_l+j'_l} \mathscr{F}_{F_1^j,F_2^k}(C)(u) \mathscr{F}_{F_1^{j'},F_2^{k'}}(B)(u) \prod_{l=\mu+1}^{\nu} (-i_l(u))^{k_l+k'_l},
\end{aligned}
\tag{16}
$$

which completes the proof. $\square$

Corollary 15 (convolution)**.** *Let $A,B,C : \mathbb{R}^m \to C\ell_{p,q}$ be multivector fields with $A(x) = (C*B)(x)$ and F_1, F_2 each consist of functions in the center of $C\ell_{p,q}$, then the the GFT satisfy the simple product formula*

$$
\mathscr{F}_{F_1,F_2}(A)(u) = \mathscr{F}_{F_1,F_2}(C)(u)\mathscr{F}_{F_1,F_2}(B)(u).
\tag{17}
$$

CONCLUSIONS

In this paper, we show how any separable GFT can be decomposed into real-valued transforms with constant mulitivector factors. This has two big consequences. On one hand, it takes away most of their mystery and some of their fascination. But on the other hand, we provide a powerful tool for their comprehension and their analysis, which for example leads to a superior convolution theorem.

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