A COROLLARY OF RIEMANN HYPOTHESIS

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Abstract. This paper use the results of the value distribution theory , got a significant conclusion by Riemann hypothesis

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First, we give some signs , definition and theorem in the value distribution theory , its contents see the references [1] and [2] .

Definition .

$$\log^+ x = \begin{cases} \log x & 1 \le x \\ 0 & 0 \le x < 1 \end{cases}$$

It is easy to see that $\log x \le \log^+ x$.

Set $f(z)$ is a meromorphic function in the region $|z| < R,\ 0 < R \le \infty$, and not identical to zero .

$n(r,f)$ represents the poles number of $f(z)$ on the circle $|z| \le r\,(0 < r < R)$, multiple poles being repeated . $n(0,f)$ represents the order of pole of $f(z)$ in the origin . For arbitrary complex number $a \ne \infty$, $n(r, \frac{1}{f-a})$ represents the zeros number of $f(z) - a$ in the circle $|z| \le r\ (0 < r < R)$, multiple zeros being repeated. $n(0, \frac{1}{f-a})$ represents the order of zero of $f(z) - a$ in the origin .

Definition .

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\varphi}) \right| d\varphi$$

$$N(r,f) \;=\; \int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt \;+\; n(0,f)\,\log r$$

Definition . $T(r,f) \,=\, m(r,f) \,+\, N(r,f)$.
$T(r,f)$ is called the characteristic function of $f(z)$.

LEMMA 1. If $f(z)$ is an analytical function in the region $|z| < R$ ($0 < R \leq \infty$) , then

$$T(r,f) \;\leq\; \log^{+}\, M(r,f) \;\leq\; \frac{\rho+r}{\rho-r}\, T(\rho\,,f)(\,0<r<\rho<R)$$

where $M(r,f) \,=\, \max_{|z|=r}\, |f(z)|$
The proof of the lemma see the page 57 of the references [1] .

LEMMA 2. Set $f(z)$ is a meromorphic function in the region $|z| < R$ ($0 < R \leq \infty$) , not identical to zero . Set $|z| < \rho$ ($0 < \rho < R$) is a circle , a_λ ($\lambda = 1,2,...,h$) and b_μ ($\mu = 1,2,...,k$) respectively is the zeros and the poles of $f(z)$ in the circle , appeared number of every zero or every pole and its order the same , and that $z = 0$ is not the zero or the pole of function $f(z)$, then in the circle $|z| < \rho$, We have the following formula

$$\log\,|\,f(0)\,| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log\,\left|f(\rho e^{i\varphi})\right| d\varphi \;-\; \sum_{\lambda=1}^{h} \log\,\frac{\rho}{|a_\lambda|} \;+\; \sum_{\mu=1}^{k} \log\,\frac{\rho}{|b_\mu|}$$

this formula is called Jensen formula .
The proof of the lemma see the page 48 of the references [1] .

LEMMA 3. Set function $f(z)$ is the meromorphic function in $|z| \leq R$, and

$$f(0) \;\neq\; 0,\; \infty,\; 1, \quad f'(0) \;\neq\; 0$$

then when $0 < r < R$, have

$$T(r,f) \;<\; 2\left\{N(R,\frac{1}{f}) \,+\, N(R,f) \,+\, N(R,\frac{1}{f-1})\right\}$$

$$+ \ 4 \ \log^+ |f(0)| \ + \ 2 \ \log^+ \frac{1}{R|f'(0)|} \ + \ 24 \ \log \frac{R}{R-r} \ + \ 2328$$

This is a form of Nevanlinna second basic theorems .

The proof of the lemma see the theorem 3.1 of the page 75 of the references [1] .

The need for behind, We will make some preparations.

LEMMA 4. If when $x \geq a$, $f(x)$ is a nonnegative degressive function , then below limits exist

$$\lim_{N \to \infty} \left(\sum_{n=a}^{N} f(n) \ - \ \int_a^N f(x)\, dx \right) \ = \ \alpha$$

where $0 \leq \alpha \leq f(a)$. in addition , if when $x \to \infty$, have $f(x) \to 0$, then

$$\left| \ \sum_{a \leq n \leq \xi} f(n) \ - \ \int_a^{\xi} f(\nu)\, d\nu \ - \ \alpha \ \right| \ \leq \ f(\xi - 1)\,, \qquad (\, \xi \geq a + 1\,)$$

The proof of the lemma see the theorem 2 of page 91 of the references [3] .

Set $s = \sigma + it$ is the complex number , when $\sigma > 1$, the definition of Riemann Zeta function is

$$\zeta(s) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^s}$$

When $\sigma > 1$, from the page 90 of the references [4], have

$$\log \zeta(s) \ = \ \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n}$$

where $\Lambda(n)$ is Mangoldt function .

LEMMA 5. For any real number t , have

(1)
$$0.0426 \le |\log\zeta(4+it)| \le 0.0824$$

(2)
$$|\zeta(4+it) - 1| \ge 0.0426$$

(3)
$$0.917 \le |\zeta(4+it)| \le 1.0824$$

(4)
$$|\zeta'(4+it)| \ge 0.012$$

PROOF.
(1)
$$|\log\zeta(4+it)| \le \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^4 \log n} \le \sum_{n=2}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} - 1 \le 0.0824$$

$$|\log\zeta(4+it)| \ge \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4} = 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \ge 0.0426$$

(2)
$$|\zeta(4+it) - 1| = \left| \sum_{n=2}^{\infty} \frac{1}{n^{4+it}} \right| \ge \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4}$$
$$= 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \ge 0.0426$$

(3)
$$|\zeta(4+it)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^{4+it}} \right| \le \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \le 1.0824$$

$$|\zeta(4+it)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^{4+it}} \right| \ge 1 - \sum_{n=2}^{\infty} \frac{1}{n^4} = 2 - \sum_{n=1}^{\infty} \frac{1}{n^4} = 2 - \frac{\pi^4}{90} \ge 0.917$$

(4)

$$| \zeta'(4+it) | = \left| \sum_{n=2}^{\infty} \frac{\log n}{n^{4+it}} \right| \geq \frac{\log 2}{2^4} - \sum_{n=3}^{\infty} \frac{\log n}{n^4}$$

from lemma 4 , have

$$\sum_{n=3}^{\infty} \frac{\log n}{n^4} = \int_3^{\infty} \frac{\log x}{x^4} dx + \alpha$$

where $0 \leq \alpha \leq \frac{\log 3}{3^4}$

$$\int_3^{\infty} \frac{\log x}{x^4} d\,x = -\frac{1}{3} \int_3^{\infty} \log x \; d\,x^{-3} = \frac{\log 3}{3^4} + \frac{1}{3} \int_3^{\infty} x^{-4} \, d\,x$$

$$= \frac{\log 3}{3^4} - \frac{1}{3^2} \int_3^{\infty} d\,x^{-3} = \frac{\log 3}{3^4} + \frac{1}{3^5}$$

therefore

$$\sum_{n=3}^{\infty} \frac{\log n}{n^4} \leq \frac{\log 3}{3^4} + \frac{1}{3^5} + \frac{\log 3}{3^4}$$

therefore

$$| \zeta'(4+it) | \geq \frac{\log 2}{2^4} - \frac{2\log 3}{3^4} - \frac{1}{3^5} \geq 0.012$$

The proof is complete .

Set $0 < \delta \leq \frac{1}{100}$, $c_1, c_2, \ldots,$ represents positive constant with only δ relevant in the article below .

LEMMA 6. When $\sigma \geq \frac{1}{2}, \; |t| \geq 2$, have

$$| \zeta(\sigma + it)| \leq c_1 \, |t|^{\frac{1}{2}}$$

The proof of the lemma see the theorem 2 of page 140 and the theorem 4 of page 142 , of the references [4] .

LEMMA 7. Set $f(z)$ is the analytic function in the circle $|z - z_0| \le R$, then for any $0 < r < R$, in the circle $|z - z_0| \le r$, have

$$| f(z) - f(z_0) | \le \frac{2r}{R - r} (A(R) - Ref(z_0))$$

where $A(R) = \max_{|z-z_0|\le R} Ref(z)$

The proof of the lemma see the theorem 2 of page 61 of the references [4] .

Now assume Riemann hypothesis is correct, abbreviation for RH . In other words , when $\sigma > \frac{1}{2}$, the function $\zeta(\sigma + it)$ has no zeros . Set the union set of the region $\sigma \ge \frac{1}{2} + \delta$, $|t| > 1$ and the region $\sigma > 2$, $|t| \le 1$ is the region D .

Therefore , the function $\zeta(\sigma + it)$ have neither zero nor poles in the region D , so , function $\log \zeta(\sigma + it)$ is a defined multi-valued analytic function in the region D . Every single value analytic branch differ $2\pi i$ integer times .

Assuming there are the points s_0 in the region D , satisfy $\zeta(s_0) = 1$ (If there is not such point s_0 , then the result of lemma 9 turns into $N(\rho, \frac{1}{\zeta - 1}) = 0$, the results of the theorem of this article can be obtained directly). For different single value analytic branch , the value of $\log \zeta(s_0) = \log 1$ are different , it can value $0, 2\pi ki, (k = \pm 1, \pm 2, ...)$. We select the single valued analytic branch of $\log \zeta(s_0) = \log 1 = 0$.

Because the region D is simple connected region , so the according to the single value theorem of analytic continuation (the theorem see the theorem 2 of page 276 of the references [5] and theorem 1 of page 155 of the references [6]), $\log \zeta(\sigma + it)$ is the single valued analytic function in the region D . in addition , when $\zeta(\sigma + it) = 1$, have $\log \zeta(\sigma + it) = 0$. In other words , 1 value point of $\zeta(\sigma + it)$ is the zero of $\log \zeta(\sigma + it)$.

Below, $\log \zeta(\sigma + it)$ always express a single valued analytic branch for we selected .

LEMMA 8. If RH is correct , then when $0 < \delta \le \frac{1}{100}$, $\sigma \ge \frac{1}{2} + 2\delta$, $|t| \ge 16$, we have

$$| \log \zeta(\sigma + it) | \le c_2 \log |t| + c_3$$

proof. In the lemma 7 , we choose $z_0 = 0,\ f(z) = \log \zeta(z + 4 + it),\ |t| \ge 16,\ R = \frac{7}{2} - \delta,\ r = \frac{7}{2} - 2\delta$. Because $\log \zeta(z + 4 + it)$ is the analytic function in the circle $|z-z_0| \le R$, so , from the lemma 7 , in the circle $|z-z_0| \le r$, we have

$$| \log\zeta(z+4+it) \ - \log\zeta(4+it) \ | \ \le \ \frac{7}{\delta} \ (\ A(R) - Re\log\zeta(4+it) \ )$$

hence

$$| \log\zeta(z+4+it) \ | \le \ \frac{7}{\delta} \ (\ A(R) + | \log\zeta(4+it) \ | \ ) + | \log\zeta(4+it) \ |$$

from the lemma 6 , have

$$A(R) = \max_{|z-z_0|\le R} \log | \ \zeta(z+4+it) \ | \le \ \frac{1}{2} \log|t| \ + \ \log c_1$$

from the lemma 5 , have

$$| \log\zeta(z+4+it) \ | \ \le \ c_2 \log|t| \ + \ c_3$$

because $|t| \ge 16$ is real number arbitrarily , so when $\sigma \ge \frac{1}{2}+2\delta$, we have

$$| \log\zeta(\sigma+it) \ | \ \le \ c_2 \log|t| \ + \ c_3$$

The proof is complete .

LEMMA 9. If RH is correct , then when $0 < \delta \le \frac{1}{100}$, $|t| \ge 16$, $\rho = \frac{7}{2} - 2\delta$, in the circle $|z| \le \rho$, we have

$$N\left(\ \rho \, , \ \frac{1}{\zeta(z+4+it) \ - 1} \ \right) \ \le \ \log\log|t| \ + \ c_4$$

proof. In the lemma 2 , we choose $f(z) = \log\zeta(z+4+it)$, $R = \frac{7}{2} - \delta$, $\rho = \frac{7}{2} - 2\delta$, a_λ $(\lambda = 1, 2, ..., h)$ is the zeros of function $\log\zeta(z+4+it)$ in the circle $|z| < \rho$, multiple zeros being repeated. The function $\log\zeta(z+4+it)$ has no poles in the the circle $|z| < \rho$, and $\log\zeta(4+it)$ not equal to zero , therefore we have

$$\log|\log \ \zeta(4+it)\,| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \log \ \zeta(4+it+\rho e^{i\varphi}) \right| \, d\varphi \ - \sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|}$$

from the lemma 5 and the lemma 8 , have

$$\sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|} \le \log\log|t| + c_4$$

because $z=0$ is neither the zero , nor pole of the function $\log\zeta(z+4+it)$, so if r_0 is a sufficiently small positive number , then

$$\sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|} = \int_{r_0}^{\rho} \left(\log\frac{\rho}{t}\right) \, dn(t,\frac{1}{f}) = \left[\left(\log\frac{\rho}{t}\right) n(t,\frac{1}{f})\right]\Big|_{r_0}^{\rho}$$

$$+ \int_{r_0}^{\rho} \frac{n(t,\frac{1}{f})}{t} \, dt = \int_{0}^{\rho} \frac{n(t,\frac{1}{f})}{t} \, dt = N\left(\rho, \frac{1}{f}\right)$$

$$= N\left(\rho, \frac{1}{\log\zeta(z+4+it)}\right) \ge N\left(\rho, \frac{1}{\zeta(z+4+it)-1}\right)$$

The proof is complete .

THEOREM . If RH is correct , then when $\sigma \ge \frac{1}{2} + 4\delta$, $0 < \delta \le \frac{1}{100}$, $|t| \ge 16$, we have

$$|\,\zeta(\sigma+it)\,| \le c_8 \,(\log|t|\,)^{c_6}$$

proof. In the lemma 3 , we choose $f(z) = \zeta(z+4+it)$, $|t| \ge 16$, from the lemma 5 , have $f(0) = \zeta(4+it) \ne 0, \infty, 1$, $f'(0) = \zeta'(4+it) \ne 0$, and $f'(0) = \zeta'(4+it) \ge 0.012$, $|\,f(0)\,| = |\,\zeta(4+it)\,| \le 1.0824$. We choose $R = \frac{7}{2} - 2\delta$, $r = \frac{7}{2} - 3\delta$. because $\zeta(z+4+it)$ is the analytic function , and have neither zero nor the poles in the circle $|z| \le R$, therefore

$$N\left(R, \frac{1}{f}\right) = 0 , \qquad N\,(R, f) = 0$$

from the lemma 9 , have

$$T\,(r, \zeta(z+4+it)\,) \le 2\log\log|t| + c_5$$

In the lemma 1 , we choose $R = \frac{7}{2} - 2\delta$, $\rho = \frac{7}{2} - 3\delta$, $r = \frac{7}{2} - 4\delta$, from the maximal principle , in the the circle $|z| \leq r$, we have

$$\log^{+} |\,\zeta(z + 4 + it)\,| \leq c_6 \log\log|t| + c_7$$

Since $|t| \geq 16$ is arbitrary real number, so when $\sigma \geq \frac{1}{2} + 4\delta$, have

$$\log^{+} |\,\zeta(\sigma + it)\,| \leq c_6 \log\log|t| + c_7$$

therefore

$$\log |\,\zeta(\sigma + it)\,| \leq c_6 \log\log|t| + c_7$$

therefore

$$|\,\zeta(\sigma + it)\,| \leq c_8 \, (\log|t|\,)^{c_6}$$

The proof is complete .

The result of this theorem is better than known results .

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