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The quest for conformal geometric algebra Fourier transformations

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Abstract. Conformal geometric algebra is preferred in many applications. Clifford Fourier transforms (CFT) allow holistic signal processing of (multi) vector fields, different from marginal (channel wise) processing: Flow fields, color fields, electro-magnetic fields, ... The Clifford algebra sets (manifolds) of $\sqrt{-1}$ lead to continuous manifolds of CFTs. A frequently asked question is: What does a Clifford Fourier transform of conformal geometric algebra look like? We try to give a first answer.

Keywords: Clifford geometric algebra, Clifford Fourier transform, conformal geometric algebra, horosphere.
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INTRODUCTION

Note: Please respect the Creative Peace License [13] when applying this work. Conformal geometric algebra is widely used in applications [8, 21]. Reasons are the elegant representation of geometric objects by products of points. The products of these objects form in turn new objects (via intersection, union, projection, ...). Conformal transformations are products of reflections at hyperplanes (versors). This even leads to the linearization of translations. Moreover, it is very interesting for the approximation abilities of conformal geometric algebra neural networks.

Clifford's geometric algebra $Cl(2,0)$ is a special case of Clifford algebras $Cl(p,q)$ of $\mathbb{R}^{p,q}$ over $\mathbb{R}$. $Cl(2,0) = Cl(\mathbb{R}^2)$ has a geometric product of vectors $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^2$

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}, \tag{1}$$

of scalar symmetric inner product and bivector antisymmetric outer product. It is the geometric algebra of the Euclidean plane and unifies 2D vector algebra, complex numbers and spinors in one algebra. Given an orthonormal vector basis $\{e_1,e_2\}$ of $\mathbb{R}^2$, the 4D ($2^2 = 4$) Clifford algebra $Cl(2,0)$ has a basis of 1 scalar, 2 vectors, and 1 bivector $\{1,e_1,e_2,e_{12} = \boldsymbol{i}\}$, where we define $\boldsymbol{i} = e_{12} = e_1e_2$. The basis bivector $\boldsymbol{i}$ squares to -1. Rotation by a versor (rotor) R is a product of two reflections

$$\boldsymbol{x}' = \widetilde{R}\boldsymbol{x}R, \qquad R = e^{\frac{1}{2}\varphi\boldsymbol{i}}, \qquad \widetilde{R} = e^{-\frac{1}{2}\varphi\boldsymbol{i}}. \tag{2}$$

CONFORMAL GEOMETRIC ALGEBRA OF THE EUCLIDEAN PLANE

The conformal model (see [8] and its references) of the Euclidean plane in $Cl(3,1)$ extends the basis of $\mathbb{R}^2$ by adding a plane $\{e_+,e_-\}$, $e_+^2 = 1$, $e_-^2 = -1$, $e_+\cdot e_- 1 = 0$ and thus generates $Cl(2+1,0+1) = Cl(3,1)$. We choose a null-basis, assigning origin and infinity vectors (like in projective geometry):

$$e_0 = \frac{1}{2}(e_- - e_+), \qquad e_\infty = e_- + e_+, \qquad e_0^2 = e_\infty^2 = 0, \qquad e_0\cdot e_\infty = -1. \tag{3}$$

The bivector E of the added origin and infinity plane is

$$E = e_+\wedge e_- = e_\infty\wedge e_0, \qquad E^2 = 1. \tag{4}$$

There are the multiplication properties

$$e_0E = -e_0, \qquad Ee_0 = e_0, \qquad e_\infty E = e_\infty, \qquad Ee_\infty = -e_\infty. \tag{5}$$

The full 16D basis of $Cl(3,1)$ is

$$\{1,\ e_1,e_2,e_0,e_\infty,\ e_{12} = \boldsymbol{i},e_1e_0,e_2e_0,e_1e_\infty,e_2e_\infty,E,\ \boldsymbol{i}e_0,\boldsymbol{i}e_\infty,e_1E,e_2E,\ \boldsymbol{i}E\}, \tag{6}$$

with scalars (1), vectors (4), bivectors (6), trivectors (4), pseudoscalars (1), and $Cl(2,0) \subset Cl(3,1)$. Note, that only bivectors and trivectors change sign under reversion (reversing the order of all vector factors).

We have a set of geometric objects $GO \subset Cl(3,1)$ described in the conformal model [8] as conformal points $P = \boldsymbol{p} + \frac{1}{2}\boldsymbol{p}^2 e_\infty + e_0$ (homogeneous, projective), point pairs $P_1 \wedge P_2$, flat point pairs $P \wedge e_\infty$, circles $C = P_1 \wedge P_2 \wedge P_3$ through three points, and lines $P_1 \wedge P_2 \wedge e_\infty$ (hyperplane), offset from the origin. A point P is on one of these objects (with blade Obj): $P \in Obj \Leftrightarrow P \wedge Obj = 0$. The products of these objects with their reverse give [8]

$$P\widetilde{P} = 0, \; Pp\widetilde{Pp} = -r^2 D^2, \; (P \wedge e_\infty)(\widetilde{P \wedge e_\infty}) = -1, \; Line\,\widetilde{Line} = -1, \; Circle\,\widetilde{Circle} = r^2 D^2 < 0, \tag{7}$$

where $D = \boldsymbol{p}_1 - \boldsymbol{p}_2 = 2r\boldsymbol{d}$, $\boldsymbol{d}^2 = 1$ for the point pair Pp, and D is a scalar multiple of e_{12} with negative square for the circle $Circle$. We therefore assume as norm for these objects $||Obj|| = \sqrt{-Obj\,\widetilde{Obj}}$. Note that for two conformal points P, Q the number $\sqrt{-P\widetilde{Q}} = |\boldsymbol{p} - \boldsymbol{q}|/\sqrt{2}$ is their distance.

Conformal transformations of $\mathbb{R}^2$ become versors (products of vectors) in $Cl(3,1)$: rotation around origin: $R = e^{\frac{1}{2}\varphi \boldsymbol{i}}$ (see (2)), translation by $\boldsymbol{t} \in \mathbb{R}^2$: $T = e^{\frac{1}{2}\boldsymbol{t}e_\infty}$, transversion by $\boldsymbol{t} \in \mathbb{R}^2$: $Tv = e^{\frac{1}{2}\boldsymbol{t}e_0}$ (a transversion composes inversion at the unit sphere around the origin, translation, and a second inversion at the unit sphere around the origin), and scaling by $e^\gamma, \gamma \in \mathbb{R}$: $S = e^{\frac{1}{2}\gamma E}$. These transformation versors V are applied to conformal objects A as

$$A \to A' = \widetilde{V} A V. \tag{8}$$

The inner product of $f, g : \mathbb{R}^{p,q} \to GO$, respectively its symmetric scalar part, are

$$(f,g) = -\int_{\mathbb{R}^{p,q}} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x}, \qquad \langle f,g \rangle = -\int_{\mathbb{R}^{p,q}} f(\boldsymbol{x}) * \widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x}. \tag{9}$$

The $L^2(\mathbb{R}^{p,q}; GO)$-quasi-norm (indicating distance in the case of conformal points) is

$$\|f\|^2 = \langle (f,f) \rangle, \qquad L^2(\mathbb{R}^{p,q}; GO) = \{f : \mathbb{R}^{p,q} \to GO \mid \|f\| < \infty\}. \tag{10}$$

Note, that for ensuring finite basis coefficient values of geometric objects in $GO \subset Cl(3,0)$ in the basis (6) of $Cl(3,1)$ the principal reverse operation of Clifford algebra can be used (reverse combined with changing the sign of every basis vector with negative square).

The Clifford algebra $Cl(3,1)$ is isomorphic to the (square) matrix algebras $\mathscr{M}(4,\mathbb{R})$. $Sc(f) = 0$ for every $f = \sqrt{-\mathbf{1}} \in Cl(3,1)$ [12, 19]. All $\sqrt{-\mathbf{1}} \in Cl(3,1)$ are computable with the Maple package CLIFFORD [2, 1, 18]. The square roots f of $-\mathbf{1}$ constitute a unique conjugacy class of dimension 8, with as many connected components as the group $\mathrm{G}(\mathscr{M}(4,\mathbb{R}))$ of invertible elements in $\mathscr{M}(4,\mathbb{R})$. For $\mathscr{M}(4,\mathbb{R})$, the centralizer (all elements in $Cl(3,1)$ commuting with f) and the conjugacy class of a square root f of $-\mathbf{1}$ both have $\mathbb{R}$-dimension 8 with two connected components.

CLIFFORD FOURIER TRANSFORMATIONS

We now consider a generalization of quaternion and Clifford Fourier transforms CFTs [3, 4, 10, 5, 6, 7, 9, 11, 14, 15, 16, 17, 20, 22] to conformal geometric algebra $Cl(3,1)$.

Definition 1 (CFT with respect to two square roots of -1)**.** *Let $f, g \in Cl(3,1)$, $f^2 = g^2 = -1$, be any two square roots of -1. The general Clifford Fourier transform (CFT) of $h \in L^1(\mathbb{R}^{p,q}; GO)$, with respect to f, g is*

$$\mathscr{F}^{f,g}\{h\}(\omega) = \int_{\mathbb{R}^{p,q}} e^{-fu(\boldsymbol{x},\omega)} h(\boldsymbol{x})\, e^{-gv(\boldsymbol{x},\omega)} d^n\boldsymbol{x}, \tag{11}$$

where $n = p+q$, $d^n\boldsymbol{x} = dx_1 \ldots dx_n$, $\boldsymbol{x}, \omega \in \mathbb{R}^{p,q}$, and $u, v : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$.

The square roots $f, g \in Cl(3,1)$ of -1 may be from any component of any conjugacy class. The above CFT is steerable in the continuous submanifolds of $\sqrt{-\mathbf{1}}$ in $Cl(3,1)$. We have the following properties of the general two-sided CFT: a Plancherel identity, respectively a Parseval identity, for functions $h_1, h_2, h \in L^2(\mathbb{R}^{p,q}; GO)$

$$\langle h_1, h_2 \rangle = \frac{1}{(2\pi)^n} \langle \mathscr{F}^{f,g}\{h_1\}, \mathscr{F}^{f,g}\{h_2\} \rangle, \qquad \|h\| = \frac{1}{(2\pi)^{n/2}} \left\| \mathscr{F}^{f,g}\{h\} \right\|. \tag{12}$$

For these identities to hold we need for $\sqrt{-1}$ that $\widetilde{f}=-f$, $\widetilde{g}=-g$.

We now seek to find the most suitable $\sqrt{-1}\in Cl(3,1)$. We can write every real $f=\sqrt{-1}$, $f\in Cl(3,1)$ as

$$f=\alpha+\boldsymbol{b}+\beta\boldsymbol{i}+(\alpha_\infty+\boldsymbol{b}_\infty+\beta_\infty\boldsymbol{i})e_\infty+(\alpha_0+\boldsymbol{b}_0+\beta_0\boldsymbol{i})e_0+(\alpha_E+\boldsymbol{b}_E+\beta_E\boldsymbol{i})E,$$

where $\alpha=0,\alpha_\infty,\alpha_0,\alpha_E,\beta,\beta_\infty,\beta_0,\beta_E\in\mathbb{R}$, $\boldsymbol{b},\boldsymbol{b}_\infty,\boldsymbol{b}_0,\boldsymbol{b}_E\in\mathbb{R}^2$.

From $f^2=-1$ we obtain the root equation (main condition for f)

$$f^2=\boldsymbol{b}^2-\beta^2-2\alpha_0\alpha_\infty+2\boldsymbol{b}_0\cdot\boldsymbol{b}_\infty+2\beta_0\beta_\infty+\alpha_E^2+\boldsymbol{b}_E^2-\beta_E^2=-1,$$

plus side conditions for zero non-scalar parts of f^2. By imposing that $\widetilde{f}=-f$ (necessary for Plancherel and Parseval identities), we abandon scalar, vector and pseudoscalar parts of f:

$$f=\beta\boldsymbol{i}+\boldsymbol{b}_\infty e_\infty+\beta_\infty\boldsymbol{i}e_\infty+\boldsymbol{b}_0e_0+\beta_0\boldsymbol{i}e_0+\alpha_EE+\boldsymbol{b}_EE. \tag{13}$$

and retain only bivector and trivector parts.

Regarding the trivector $\sqrt{-1}$ in $Cl(3,1)$ we can calculate the following. For $f=\boldsymbol{b}_EE$, $\boldsymbol{t},\boldsymbol{t}'\in\mathbb{R}^2$, $\boldsymbol{t}'=-\boldsymbol{b}_E\boldsymbol{t}\boldsymbol{b}_E$, we obtain

$$e^{-\boldsymbol{b}_EE}e_0e^{\boldsymbol{b}_EE}=e_0,\qquad e^{-\boldsymbol{b}_EE}\boldsymbol{t}e^{\boldsymbol{b}_EE}=\mathrm{ch}^2(|\boldsymbol{b}_E|)\boldsymbol{t}+\mathrm{sh}^2(|\boldsymbol{b}_E|)\boldsymbol{t}'+2\mathrm{ch}(|\boldsymbol{b}_E|)\mathrm{sh}(|\boldsymbol{b}_E|)(\boldsymbol{t}\wedge\frac{\boldsymbol{b}_E}{|\boldsymbol{b}_E|})E. \tag{14}$$

For $f=\beta_\infty\boldsymbol{i}e_\infty$ we obtain

$$e^{-\beta_\infty\boldsymbol{i}e_\infty}e_0e^{\beta_\infty\boldsymbol{i}e_\infty}=e_0-2\beta_\infty^2e_\infty-2\beta_\infty\boldsymbol{i}E. \tag{15}$$

Like for quaternion Fourier transformations this may help to separate signal symmetry components, but for now we set the trivector parts of $\sqrt{-1}$ in $Cl(3,1)$ aside. We keep only the four bivector parts.

If we only keep the bivector parts of $f=\sqrt{-1}$ in $Cl(3,1)$ we have

$$f=\beta\boldsymbol{i}+\boldsymbol{b}_\infty e_\infty+\boldsymbol{b}_0e_0+\alpha_EE. \tag{16}$$

We recognize that $e^{-\varphi\beta\boldsymbol{i}}h(\boldsymbol{x})e^{\varphi\beta\boldsymbol{i}}$ means a local rotation by $2\varphi\beta\in\mathbb{R}$ of the signal function $h:\mathbb{R}^{p,q}\to GO$. $e^{-b\boldsymbol{b}_\infty e_\infty}h(\boldsymbol{x})e^{b\boldsymbol{b}_\infty e_\infty}$, $b\in\mathbb{R}$, means a translation by $2b\boldsymbol{b}_\infty\in\mathbb{R}^2$ of the signal function $h:\mathbb{R}^{p,q}\to GO$. $e^{-c\boldsymbol{b}_0e_0}h(\boldsymbol{x})e^{c\boldsymbol{b}_0e_0}$, $c\in\mathbb{R}$, means a transversion over $2c\boldsymbol{b}_0\in\mathbb{R}^2$ of the signal function $h:\mathbb{R}^{p,q}\to GO$. $e^{-a\alpha_EE}h(\boldsymbol{x})e^{a\alpha_EE}$, $a\in\mathbb{R}$, means a scaling by the factor $e^{2a\alpha_E}$ of the signal function $h:\mathbb{R}^{p,q}\to GO$. Thus every term in a bivector $\sqrt{-1}$ in $Cl(3,1)$ has a clear geometric transformation interpretation. This could be of great advantage for the choice and application of conformal CFTs in image and signal processing.

Considering the side conditions for the zero non-scalar components in $f^2=-1$ let the angle $\Theta=\angle(\boldsymbol{b}_0,\boldsymbol{b}_E)$. For $\Theta\neq0,\pi$ we find that

$$\beta=\frac{1}{\sin\Theta}[\alpha_E\cos\Theta\pm\sqrt{\alpha_E^2+\sin^2\Theta}]. \tag{17}$$

Especially for $\Theta=\pi/2,3\pi/2$ we obtain

$$\beta=\pm\sqrt{1+\alpha_E^2}. \tag{18}$$

For the special case $\Theta=0$, $\alpha_E=0$, we have

$$\beta=\pm\sqrt{1+2|\boldsymbol{b}_0||\boldsymbol{b}_E|}. \tag{19}$$

For $\Theta=\pi$, $\alpha_E=0$, we have

$$\beta=\pm\sqrt{1-2|\boldsymbol{b}_0||\boldsymbol{b}_E|}, \tag{20}$$

whereas for $\Theta=\pi$, $\beta=0$, we have

$$\alpha_E=\pm\sqrt{2|\boldsymbol{b}_0||\boldsymbol{b}_E|-1}. \tag{21}$$

CONCLUSION

We have briefly introduced the conformal geometric algebra $Cl(3,1)$ model of the Euclidean plane. Then we defined Clifford Fourier transforms based on real $f=\sqrt{-1}$ in $Cl(3,1)$, and tried to select a Clifford Fourier transform by studying the manifold of $\sqrt{-1}$ in $Cl(3,1)$. A selection with clear geometric interpretation proved to be the bivector parts of $\sqrt{-1}$ in $Cl(3,1)$ for the construction of a CFT in conformal geometric algebra model of Euclidean plane. Then every term in a bivector $\sqrt{-1}$ in $Cl(3,1)$ has a geometric transformation interpretation. Setting $g=-f$ leads to conformal rotor transformation CFTs. We completed this by a detailed characterization of bivector $\sqrt{-1}$ in $Cl(3,1)$, taking all side conditions from non-scalar parts of ff, to be zero, into account. We therefore hope that this new conformal CFT with real bivector $\sqrt{-1}$ in $Cl(3,1)$ will be tried in applications. It still shares many properties of conventional FT (such as linearity, shift, modulation, partial differential, Plancherel, Parseval, convolution, ...).

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