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Positional codes of complex numbers and vectors

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Annotation

In book the theory of positional coding of complex numbers and vectors is considered. The method of search of the radix of coding is described and the various radix of coding are offered. Some variants of construction of binary codes of complex numbers and vectors are allocated.

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Introduction

Let's briefly stop on history of a question. Computer arithmetics of complex mathematical objects originates in Shannon article about positional codes of real numbers on the negative radix [1]. This idea, very likely, for the first time has been realized in Poland [2] and has induced (apparently) several authors to development of methods of coding of complex numbers. Practically simultaneously the Knuth [3] has offered system of coding on the basis $j\sqrt{2}$. Khmelnik [4] has offered some systems, including on the bases $j\sqrt{2}$ and (-1+j). The basis (-1+j) has considered Penney [5] later. Khmelnik in the dissertation [6] has considered a complex of questions of designing of the arithmetic device for operations with complex numbers. These results developed then in works [7, 8, 9, 11, 12, 13, 14, 33, 34, 35, 44, 45, 46, 47].

In several works [16, 17, 18] methods of construction of multipliers of complex numbers are considered. So the basic attention is given ways of realization of these devices on the chip. For this purpose redundant systems of coding which, in opinion of authors, allow to construct big regular schemes are offered. However at that other operations with offered to codes (for example, division) are not considered.

For codes of real numbers the method "digit-by-digit" [19, 20] for calculation of elementary functions by a hardware is known. It can be generalized on positional codes of complex numbers, that for the first time has been made Khmelnik in [6, 11]. So frequently it is enough to have hardware realization only for potentiation and taking the logarithm as since through these functions in complex area it is possible to express all elementary functions. Besides this method is applicable for construction of algorithms of the hardware decision of the transcendental equations and systems of such equations. At use of codes complex (instead of real) numbers the class of such equations extends, and algorithms of their decision essentially become simpler. In [6, 11] one of such algorithms is described.

The further development of idea of positional coding has gone on a way of construction of positional codes of vectors [21, 22], matrixes [36, 37], functions [23, 24, 25, 33], geometrical figures [22, 26, 27, 28, 32, 38, 39]. It is necessary to note, that codes of geometrical figures can be considered as codes of numerical arrays and for them effective search algorithms [29, 30, 31] can be constructed. Many are generalized from these results in the book [32].

The preference given to positional codes, follows, mainly, from that with them arithmetic operations are very simply carried out. So, without dependence from object of coding, addition of positional codes is connected to distribution of carries from younger categories to grown-ups, and multiplication will consist of shifts (that is renumbering categories) and additions. The mentioned above method "digit-by-digit" in general is applicable only in a combination to positional system of coding.

It is important to note, that in programming for offered computers the existing mathematical methods, not taking into account, naturally, specific opportunities of these computers are used. It is possible to hope, that at diffusion of such computers will be found not only other methods of the decision of problems, but also other unexpected scopes as it continuously occurs to existing computers. For example, there is a theory of functions spatial complex variable [40]. The algebra of four-dimensional vectors [21, 32], offered for their coding, coincides with the algebra of spatial complex numbers used in [40]. In this connection there is an opportunity of development of computer arithmetics of spatial complex numbers with calculation hardware of function of this variable, as the further generalization of a method "digit-by-digit" (just as it has been made for complex numbers [6, 11]). In it there is a practical sense because the theory of functions spatial complex variable is used in very difficult problems of theoretical physics [40].

In the offered project it is possible to find out many analogies to traditional computer arithmetics. It is possible to specify a number of books, where this arithmetics is in detail considered [41, 42, 43].

1. About a method of positional coding.

In this section will be considered positional codes many-dimensional vectors Z, based on their representation in kind of decomposition,

$$Z = \sum_m r_m f(\rho,\ m), \qquad (1)$$

where m - number of the category,

ρ - basis of coding, number or vector,

$f(\rho,\ m)$ - base function from number and the bases,

r - category of decomposition, number or vector, accepting significance from limited set

$$A_R = \{a_0, a_1, a_2, \dots, a_j, \dots, a_{R-1}\},$$

containing R of various sizes a_j. The positional code of a vector Z, appropriate to this decomposition, has a kind

$$K(Z) = \dots \sigma_m \dots,$$

where σ_m - digit, designating size r_m.

The formula (1) includes operation of addition and multiplication. For existence of algorithms of operations with such decomposition (or, that one and too, with positional codes) addition and multiplication should be associative and commutative, as well as to be subject to the distributive law. Hence, for a opportunity of positional coding of some set of objects this set should make ring. To such requirement set of real numbers and set of manu-dimensional vectors satisfies, in which operations of addition and multiplication on number are determined. For real numbers the positional systems are known. For indicated set of vectors a positional system with the real basis will be below constructed.

Set of complex numbers makes ring and for it positional systems on the real and complex basis will be also constructed.

For construction of a positional system of manu-dimensional vectors on vector basis operation of multiplication of vectors, subordinated set forth above laws should be determined. In other words, algebra in manu-dimensional vector space should be determined. It is made below.

In the beginning we consider two ways of coding of vectors, and then we pass to the more general and strict description of a method of positional coding.

2. Two ways of synthesis of codes of complex numbers

The positional codes of manu-dimensional vectors can be received some composition of codes of real numbers on the negative basis. In the beginning we consider this method in application to coding of complex numbers. Here and further *j* - imaginary unit.

Let X_α and X_β - real numbers, given by binary decomposition on to the basis ρ =-2, that is

$$X_\alpha = \sum_{(m)} \alpha_m \rho^m, \qquad X_\beta = \sum_{(m)} \beta_m \rho^m.$$

To these decomposition there correspond codes

$$K(X_\alpha) = \ldots \alpha_m \ldots, \qquad K(X_\beta) = \ldots \beta_m \ldots$$

There are two ways of association of these two codes in a united code of complex number. The **first** of them a pair of categories consists that α_m and β_m is designated one digit σ_m. Thus a code

$$K(Z) = \ldots \sigma_m \ldots$$

of complex number Z = $X_\alpha + jX_\beta$ on the basis ρ=-2 with categories, accepting one of four significances will be formed:

$$\sigma_m \in \{ 0, 1, j, 1+j \}.$$

Let's consider complex function of the real whole argument

$$\rho_2(m) = \begin{cases} (-2)^{m/2} \text{ if } m - \text{even} \\ j(-2)^{\frac{m-1}{2}} \text{ if } m - \text{odd} \end{cases} \tag{2}$$

Thus the considered code of complex number on the radix (-2) with complex values of categories can be considered as a code of complex number on the radix (ρ_2) with bits. To this code there corresponds decomposition of complex number as $Z = \sum_m (\sigma_m \rho_2^m)$, where bits $\sigma_m = \begin{cases} \alpha_m \text{ if } m - \text{even} \\ j \cdot \beta_m \text{ if } m - \text{odd} \end{cases}$. For an illustration we shall write down codes of some characteristic numbers in this system:

```
K(2) = 10100, K(-2) = 100, K(-1) = 101,
K(j) = 10, K(-j) = 1010, K(2j) = 101000.
```

The **second** way consists in construction of a sequence of interleaved categories α_m and β_m

$$\dots\beta_{m+1}\alpha_{m+1}\beta_m\alpha_m\beta_{m-1}\alpha_{m-1}\dots$$

We designate $\alpha_m = \sigma_{2m}$, $\beta_m = \sigma_{2m+1}$ and we copy a indicated sequence in other kind:

$$\dots\sigma_{k+3}\sigma_{k+2}\sigma_{k+1}\sigma_k\sigma_{k-1}\sigma_{k-2},$$

where $k=2m$. This sequence is binary code

$$K(Z) = \dots\sigma_m\dots$$

of some complex number Z. It is possible to show (and it will be made below), that the code, received in such a way, is a binary code on the radix

$$\rho = \pm j\sqrt{2},$$

and coded number

$$Z = X_\alpha + \rho X_\beta.$$

Thus, some composition of binary codes of real numbers on the basis $\rho = -2$ will form codes of complex numbers. At fulfillment algebraic the addition of complex numbers such codes can be considered as simple set of codes of real numbers and to execute the same operation with each pair of real numbers independently. At the same time with such codes operations of multiplication, and with the codes of the second type - and division are feasible. Thus the operations of multiplication and division consist, as usually, of cycles "shift - addition".

3. Method of coding of points of the many-dimensional space

A method of coding of points of the manu-dimensional Euclidian of the space should establish someconformity between these points and codes from some set. This conformity, generally speaking, can be not mutual - unequivocal. But for a opportunity of unequivocal decoding to each code there should correspond only one point of the coded space. At the same time even the limited area of the space contains nondenumerable set of points. Hence, set of appropriate codes also nondenumerably and among them there should be codes with infinite number of categories (***infinite codes***). However in practice of calculations ***final codes***, and the set of final codes boundedly can be used only.

That in these conditions to preserve conformity between codes and points of the space, naturally the limited coded area *G* decompose on limited set of the areas δ of determined size and configurations so that each point of the area *G* was in one of areas δ. Then between set of final codes and set of the areas δ it is possible to establish mutual - unequivocal conformity.

Such way of coding of points of the manu-dimensional space is approximate. Really, all points $Z_j \in \delta_i$ there corresponds a unique code K_i. However at decoding of a code K_i a unique point Z_i will be formed. We designate A radius - vector of a point *Z* by a symbol $\overline{Z}$. Difference $\Delta Z_j = |\overline{Z}_j - \overline{Z}_i|$ defines a absolute error of coding of a point Z_j.

By way of illustration we consider fig. 1, where the area Z_j of the two-dimensional space, broken on area δ is represented.

On this drawing the area δ_i = ABCD is allocated, and area δ_i belongs also its bottom (AD) and right (CD) border. In the area δ_i a basic point Z_i and some point $Z_j \in \delta_i$ is allocated. Length of a section ΔZ_j characterizes a absolute error of coding of a point Z_j.

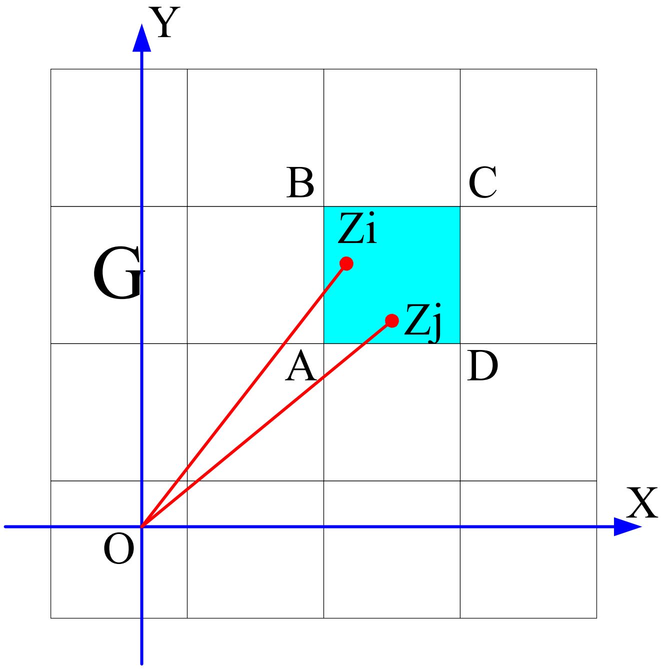


Fig. 1.

So, the stated principle of coding of points of the many-dimensional space consists in following:

- the limited area *G* of the coded space is divided on limited set of the equal areas δ_i (*i*=1, 2,..., *N*), and

$$G = \bigcup \delta_i \text{ and } \delta_i \cap \delta_j = \varnothing \text{ at } i \neq j;$$

- set of final codes K_i (*i*=1, 2,..., *N*) is defined;
- between the areas and the codes establish mutual - unequivocal conformity.

At observance of these conditions we shall speak, that the ***system of coding*** of the area *G* of the many-dimensional space satisfies to a ***principle of coding*** and the area *G* is coded with step-type behaviour δ. The following two lemmas are obvious.

Lemma 1. System of coding of the area *G* satisfies principle coding, if *V*=*NU*, and back, where

U - volume of the area,

V - volume areas *G*,

N - capacity of set of final codes.

Lemma 2. System of coding of the area G, the coding satisfying principle, is ***total*** (that is, any point there corresponds a final code), ***nonredundant*** (that is, each point there corresponds a unique final code) and ***approximate*** (that is, subset of points - vectors, the module of a difference of which does not exceed some size, there corresponds one final code).

Consider set of n-digit codes of a kind

$$K = \alpha_{n-1} \ldots \alpha_k \ldots \alpha_1 \alpha_0 , \tag{3}$$

where α_k - digit, accepting one of R_k significances, $R_k > 1$ and integer.

Lemma 3. If the system of coding satisfies to a principle of coding, at a increase of word length of final codes and preservation of step-type behaviour of coding volume of the coded area is increased also, as capacity of set of final codes, and back.

Proof. The capacity of set of final codes

$$N_n = \prod_{k=1}^{n} R_k . \tag{4}$$

Let this set of codes satisfies to a principle of coding and codes the area G_n with step-type behaviour δ. Pursuant to of lemma 1 quantity of the areas, contained in areas δ, also equally, and area G_n has we increase volume

$$V_n = N_n U . \tag{5}$$

Now word length of codes at a unit, that is, we add the category α_n, accepting one of R_n significances. Obviously

$$N_{n+1} = R_n N_n . \tag{6}$$

Let the new set of codes also satisfies to a principle of coding and codes the area G_{n+1} with the same step-type behaviour δ. Quantity of the areas δ, contained in areas G_{n+1}, equally N_{n+1}, that is, the area G_{n+1} has volume

$$V_{n+1} = N_{n+1} U . \tag{7}$$

Combining three last formulas, we find, that

$$V_{n+1} = R_{n+1} V_n , \tag{8}$$

that is, the direct part lemma is proven.

On a condition of a return part lemma formulas (5), (6), (8) are fair. From them follows (7), whence pursuant to lemma1 we receive the proof of a return part the given lemma.

We consider now a positional system of coding. In this system to each positional code

$$K(Z) = \alpha_n \ldots \alpha_k \ldots \alpha_m$$

there corresponds a point Z coded many-dimensional space, possessing decomposition of a following kind:

$$Z = \sum_{k=m}^{n} \alpha_k \rho^{\ k}, \tag{9}$$

where ρ - basis of coding,

k - number of the category,

α_k - *k*-category of a code (digit or quantitative equivalent, corresponding to it in decomposition), accepting one of R_k significances.

We notice, as ρ and α_k are also points coded of the many-dimensional space. The positional code refers to as ***infinite***, if $m = -\infty$, and - ***final***, if m is limited. Number *n* refers to as by length of a positional code. If R_k =*R*, decomposition and code refer to as refers *R*-decomposition and *R*-codes. So, we shall consider size *a*, accepting significance from sets

$$A_R = \{a_0, a_1, a_2, \ldots, a_j, \ldots, a_{R-1}\}, \tag{10}$$

containing *R* of various sizes a_j. In practice of positional coding essentially that *R* is limited and does not exceed several units.

Positional code of a point *Z* on the basis ρ we shall designate and to record also as follows

$$< Z >_\rho = \alpha_n \ldots \alpha_k \ldots \alpha_1 \alpha_0, \alpha_{-1} \alpha_{-2} \ldots \alpha_m, \tag{11}$$

placing a point between zero and (-1)-category (index - basis will not be indicated, if significance of the basis clearly from a context). A vector (point) *Z*, in a code of which, shall name ρ - ***whole***. Accordingly are defined ρ - ***fractional*** (***correct*** and ***wrong***) vectors *Z*. In particular,

$$< \rho >_\rho = 10. \tag{12}$$

Set $< \rho \ , \ A_R >$ of the basis of coding ρ we and shall A_R name sets as a ***system of positional coding***. We shall speak, that the point of the many-dimensional Euclidian space represent in the given system of positional coding, if to it there corresponds decomposition of a kind (9) and positional code of a kind (11), in which the categories accept significances from set (10).

The task consists of construction of such positional systems of coding, in which represent the any point of the given space and are thus executed conditions of completeness, nonredundant and approximate, certain in lemma 2.

The sense of construction of positional systems consists of simplification of fulfillment of arithmetic operations with points (vectors) of the many-dimensional space. On the other hand, existence of positional codes, based on decomposition (9), probably only in the event that in the given space operations of summation of vectors are determined and multiplication of a vector on the basis ρ (which can also be a vector).

In one- and two-dimensional spaces multiplication on the basis ρ (the multiplication on real or complex number) corresponds to a increase of a module vector-multiplicand in $|\rho|$ since, that is,

$$\text{if } Z_2 = Z_1\rho, \text{ then } |Z_2|=|Z_1||\rho|. \qquad (13)$$

It should once again note, that to multiplication $Z_1\rho$ there corresponds shift of a code $< Z_1 >$ on one category to the left *in any space*. We require, that the condition (13) was executed also *for any coded space* and we prove some condition of existence of a positional system, using these two fact.

Theorem 1. Necessary and sufficient condition that any point of the *h*-dimensional Euclidian space, in which is satisfied condition (13), represent in the given system of positional coding, is a condition the

$$|\rho|^h = R. \qquad (14)$$

Proof. Each code $< Z_2 >_\rho$ of length *(n + h)* at $m = -\infty$ can be received by shift on *h* of categories to the left some code $< Z_1 >_\rho$ of length *n*. But pursuant to (12) such shift equivalent to multiplication on the basis, that is, $Z_2 = Z_1\rho^h$. Thus from a ratio (13) follows, that $|Z_2| = |Z_1||\rho|^h$. Hence, the linear sizes of the coded area are increased in $|\rho|^h$ since (besides the coded area, generally speaking, is turned concerning the previous situation). Thus, the volumes of the areas G_n and G_{n+h} are connected by a ratio

$$V_{n+h} = |\rho|^h V_n. \qquad (15)$$

Obviously, the restriction *m* does not change volume of the coded area. There is only discreteness behaviour of coding $\delta = G_{m-1}$. Taking into account (14), from (15) we receive

$$V_{n+h} = RV_n. \qquad (16)$$

Comparing (16) and (8), from lemma 3 we find, that the system of positional coding at $m < -\infty$ satisfies to a principle of coding, that is, owing to lemma 2, is total, nonredundant and approximate. The theorem is proven.

4. Arithmetic systems of coding

Among positional systems of coding the heaviest interest present such, to which simple algorithms of addition and multiplication are aplicable. Just such systems we and consider hereinafter, but previously define them more strictly.

Definition 1. System $<\rho \quad , \quad A_R>$ of positional coding refers to as ***arithmetic***, if following conditions are executed

- number (-1) is ρ - whole,
- the sum and product of any pairs of vectors, belonging to to set A_R, are ρ - whole.

We notice, that the condition (13) can be executed and for a non-arithmetic system.

Lemma 4. If in arithmetic to positional system represent vectors Z_1 and Z_2, in this system represent and vectors $-Z_1$, $-Z_2$, Z_1+Z_2, $Z_1 Z_2$.

Validity lemma follows from that, as will be shown below, for arithmetic positional systems there are algorithms of arithmetic operations.

Definition 2. Positional a system $<\rho \quad , \quad A_R>$ refers to as ***normal***, if $A_R = B_R$, where

$$B_R = \{ \quad 0, \quad 1, \quad 2, \quad \ldots, \quad R-1 \quad \}.$$

Lemma 5. Normal a system, in which

$$R = \sum_{k=1}^{n} \alpha_k \rho^k, \qquad (17)$$

$$-R = \sum_{k=1}^{w} \beta_k \rho^k, \qquad (18)$$

that is, the codes of numbers R and -R are ρ-whole and have zero significance of the zero category, is arithmetic.

Proof. Any number from set B_R $0 \le a_j \le (R-1)$. Hence, for numbers from this set ratio $-a_j = a_k - R$ and $a_j + a_k = a_m + R$, if $a_j + a_k \ge R$. are executed. Taking into account Of a condition lemma, we conclude, that the numbers $(-a_j)$ and $(a_j + a_k)$ are ρ-whole. Obviously, the product $a_j a_k$ can be presented by a sum of numbers from set B_R. On induction by virtue of existence of algorithm of addition we conclude, that such sum is also ρ-whole. Thus, the conditions of definition 1 are executed. Hence, the considered system is arithmetic.

Lemma 6. Normal a system, in which number R has decomposition of a kind (17) and

$$R = \sum_{k=1}^{m} \alpha_k , \qquad (19)$$

is arithmetic.

Proof. As follows from (17) and (19), in lemma systems, in which

$$R = \sum_{k=1}^{n} \alpha_k \rho^k = \sum_{k=1}^{n} \alpha_k .$$

are considered.

We consider following algorithm:

$$\begin{array}{rrrrrrrrl}
\alpha_3 & \alpha_2 & \alpha_1 & 0 & & & & & \text{carries} \\
 & \alpha_3 & \alpha_2 & \alpha_1 & 0 & & & & \text{carries} \\
 & & \alpha_3 & \alpha_2 & \alpha_1 & 0 & & & \text{carries} \\
 & & & \alpha_3 & \alpha_2 & \alpha_1 & 0 & = \langle R \rangle_\rho & \text{addend 1} \\
 & & & & \beta_2 & \beta_1 & 0 & = \langle X \rangle_\rho & \text{addend 2} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & & \text{sum}
\end{array}$$

Here the code of number *R* develops with code of some number *X*, the categories of which will be formed so that

$$\alpha_1 + \beta_1 = R \text{ and } \alpha_1 + \alpha_2 + \beta_2 = R .$$

Thus and owing to (19) the addition of digits each column will form number *R*, which forms carry and zero category of a sum. As a result of Infinite carries and zero sum will be formed. Hence, *X*=-*R*. Obviously, such algorithm of formation of a code of number -*R* execute at any *R*, appropriate to decomposition (17) or, that one and too, at any code of number *R* of a kind

$$\langle R \rangle_\rho = \alpha_m \ldots \alpha_2 \alpha_1 0 .$$

Result of this algorithm is a code of number -*R* of a kind

$$\langle -R \rangle_\rho = \beta_w \ldots \beta_2 \beta_1 0 .$$

This code corresponds to decomposition (18). By this lemma is proven.

We notice, that the decomposition (17) and (18) can be considered as a system of two power equations concerning unknown ρ. Deciding it, it is possible, generally speaking, to define some system of coding. However such reception not always causes to positive result because the given system or not solvable analytically, or not composes, or gives as the decision of result, the of theorem not satisfying to condition 1, or gives as the decision real number.

Lemmas 4, 5, 6 will be used further at search of normal positional systems of coding.

5. Codes of real numbers

For real numbers dimension of the coded space h=1. Hence, for positional codes of real numbers it is necessary to observe a condition

$$|\rho| = R.$$

Positional codes of real numbers are widely known and widespread, in which $\rho = R$ and the categories accept significances from set B_R. Here concern usual decimal (R =10) and binary (R =2) codes. However such codes it is impossible to represent negative real numbers, to representation of which has to apply artificial receptions, in particular, to use return and additional codes, that causes a number of inconveniences.

In a too time there are two ways of construction of positional codes, suitable for the image real - positive and negative numbers. First of them consists that to categories give positive and negative significances from set $D_R = \{ -r_1, -r_1+1, \ldots, -1, 0, 1, \ldots, r_2-1, r_2 \}$ and $R = r_1 + r_2 + 1, \quad r_1 \neq 0, \quad r_2 \neq 0,$ and basis, still, leave equal R (at $r_1 = 0$ the set D_R is transformed into set B_R). Other way is based on application of the negative basis ρ = -R. Thus the sizes of categories can accept significances or from set B_R, or from set D_R. So, the known results, relating to positional coding of real numbers, are formulated as follows.

Theorem 2. Any real positive number represent in systems

$< R, B_R >$, $< R, B_R >$, $< R, D_R >$, $< R\ D_R >$.

So, exists four systems of coding of real numbers:

1. system $< R, B_R >$, for example < 5, { 0, 1, 2, 3, 4 } >;
2. system $< R\ B_R >$, for example < 5, { -2, -1, 0, 1, 2 } >;
3. system $< R\ D_R >$, for example < -5, { 0, 1, 2, 3, 4 } >;
4. system $< R\ D_R >$, for example < -5, { -2, -1, 0, 1, 2 } >.

We put examples penta-codes of codes of some numbers in considered systems, designating of size -1 and -2 digit $\bar{1}$ and $\bar{2}$:

1. K(16)= +31, K(-13)= -23*,*
2. K(16)= 1$\bar{2}$1, K(-13)= $\bar{1}$22,
3. K(16)= 121, K(-13)= 32,
4. K(16)= 121, K(-13)= $\bar{1}\bar{2}$2.

Here it should pay attention on that codes in the first of these systems are accompanied by marks " + " and "-", which are away in all other systems, as far as in them a mark of number together with a module are defined by significances of categories of a code.

It is important to note, that among indicated systems there are only two systems of binary coding, namely system with figures { 0, 1 } and basis "2" and "-2".

6. Codes of complex numbers

We prove in the beginning some theorems of existence of normal arithmetic systems of coding with the complex basis, designating through *j* a seeming unit.

Theorem 3. Any complex number represent in a normal system of coding on the complex basis ρ and this system is arithmetic, if

$$|\rho \quad| = \sqrt{R} \qquad (20)$$

and conditions (17), (19) are executed.

Proof. For complex numbers the dimension of the coded space *h*=2 and at any ρ is satisfied condition (13). From here and from (20) follows, that conditions of the theorem 1. hence, any complex number represent in the given system of coding are executed. Further, the conditions (17) and (19) are conditions lemma 6. Hence, the given system is arithmetic.

The theorem 3 enables to reduce the proof of theorems about representability of any complex number in a normal system of coding and arithmeticality of this system to the proof that is satisfied condition (19) and ρ is a complex root equation (17). Just by these method of the proof we and shall take advantage hereinafter.

Theorem 4. Any complex number represent in a normal system of coding on the complex basis

$$<\rho \quad = \sqrt{2}e^{\pm j\pi/2}; \quad B_2 > \text{ or } <\rho \quad = -1 \pm j; \quad \{0, \quad 1\} >$$

and this system is arithmetic.

Proof. We assume, that $<2>_\rho = 1100$. This condition it is equivalent to a equation $\rho^{\ 3} + \rho^{\ 2} = 2$. Its decision coincides the condition of the given theorem. Hence, is satisfied condition (17). Obviously, that the condition (19) is also executed, as far as *R*=2. By virtue of the theorem 3 the given the theorem is proven. We notice, that in this system $<R>_\rho = 1100, \quad <-R>_\rho = 11100$.

Theorem 5. Any complex number represent in a normal system of coding on the complex basis ρ and this system is arithmetic, if

$$\rho \quad = \sqrt{R}e^{j\varphi}, \quad \varphi \quad = \pm\ arcCos\ (-\beta \ /2\sqrt{R}), \quad \beta \ < (R,\ 2\sqrt{R})_{\min}$$

and β - the whole positive number.

Proof. We assume, that $<R>_\rho = 1\alpha_2\alpha_1 0$, where $1 + \alpha_2 + \alpha_1 = R, \quad \alpha_2 = \beta - 1$. This condition equivalent to a equation

$\rho^3+(\beta-1)\rho^2+(R-\beta)\rho=R$. Its decision gives significance, adduced in conditions of theorems. By virtue of the theorem 3 the given the theorem is proven.

For a illustration we record codes of some characteristic numbers in this system, having designated through $\overline{\rho}$ number, integrated to number ρ:

$K(R)=1(\beta-1)(R-\beta)0$,	$K(-R)=1\beta 0$,
$K(-1)=1\beta(R-1)$,	$K(\overline{\rho})=1(\beta-1)(R-\beta)$,
$K(-\overline{\rho})=1\beta$,	$K(-\rho)=1\beta(R-1)0$,
$K(\rho-\overline{\rho})=2\beta$,	$K(\rho+\overline{\rho})=1\beta(R-\beta)$.

In connection that β can accept some significances at constant R, there are some types of positional codes in systems of a considered kind. As a example in table 1 possible codes of number R are adduced at various R and β.

For an illustration we shall write down codes of some characteristic numbers in system with the basis $\rho=\frac{1}{2}\left(-1+j\sqrt{7}\right)$. Having designated through $\overline{\rho}$ number, conjugate to number ρ: $K(2)=1010$, $K(-2)=110$, $K(-1)=111$, $K(\overline{\rho})=101$, $K(-\rho)=1110$, $K(-\overline{\rho})=11$, $K\left(j\sqrt{7}\right)=10101$, $K\left(-j\sqrt{7}\right)=1110011$.

Table 1. Codes of number R.

$R\backslash\beta$	**1**	**2**	**3**	**4**	**5**
2	1010				
3	1020	1110			
4	1030	1120	1210		
5	1040	1130	1220	1310	
6	1050	1140	1230	1320	
7	1060	1150	1240	1330	1420
8	1070	1160	1250	1340	1430
9	1080	1170	1260	1350	1440

From systems of the theorem 5 it is possible to allocate groups with fixed significance of argument of the basis, for example

$\varphi = \pm 2\pi / 3$, if $\beta = \sqrt{R}$, that is, at R=4, 9, 16, 25, ...;

$\varphi = \pm 3\pi / 4$, if $\beta = \sqrt{2R}$, that is, at R=8, 18, 32, 50,...;

$\varphi = \pm 5\pi / 6$, if $\beta = \sqrt{3R}$, that is, at R=12, 27, 48, 75,..;

We consider now a positional system of a more general kind.

Theorem 6. Any complex number represent in a system of coding

$< \rho = 2e^{j\pi/3}, \quad A_4 >, \quad A_4 = \{ \quad 0, \quad 1, \quad , e^{2j\pi/3}, \quad e^{-2j\pi/3} \}$ and this system is arithmetic.

Proof. We notice, that $(-2)^k = l_k \rho^k$, where

$$l_k = (\quad 1, \quad e^{2j\pi/3}, \quad e^{-2j\pi/3} \quad)$$

accordingly at k = (3*m,* 3*m*+1, 3*m*+2), where *m*-whole. Obviously, $l_k \in A_4$. Hence, any degree of number "-2" אלטגאעסהונן in a indicated system of coding by one category. By virtue of the theorem 2 any real number X represent in a kind of decomposition on the basis "-2". But each category of such decomposition, presenting degree of number "-2" or 0, can be replaced by the category of decomposition in a indicated system of coding, that is, any real number represent in this system of coding.

Any complex number *Z* can be submitted as

$$Z = u_1 + u_2 e^{2j\pi/3} + u_3 e^{-2j\pi/3},$$

where $u_1, \ u_2, \ u_3$ - some real numbers. In this sum all making represent in a indicated system as far as cofactors of real numbers $u_1, \ u_2, \ u_3$ belong to set A_4 . If this system is arithmetic, in it represent and given sum, that is, any complex number. Remains to show, that the indicated system is arithmetic. For it we make tables of pairs multiplication, summation and table of inverting (multiplication on "-1") figures from set A_4 - see tables 1.2, 3, 4. For convenience of record these figures are designated by symbols 0, 1, *c, d.* As it is visible from these tables, in a considered system of coding all conditions of definition 1. Hence are executed, this system is arithmetic, as was required to show.

Table 2. One-digit multiplication

*	**0**	**1**	**c**	**d**
0	0	0	0	0
1	0	1	c	d
c	0	c	d	1
d	0	d	1	c

Table 3. One-digit summation

+	**0**	**1**	**c**	**d**
0	0	1	c	d
1	1	dc0	1d	dc
c	c	1d	d10	c1
d	d	dc	c1	c10

Table 4. Inverting category

x	**0**	**1**	**c**	**d**
-x	**0**	**c1**	**dc**	**1d**

Table 4. Inverting category.

X	0	1	c	d
-X	0	c1	dc	1d

We notice, that in this system very idle simple a kind there are codes of complex numbers of a kind $e^{jk60^{\circ}}$, where *k* - integer - see table 4a. Besides for this system in table 4b codes of numbers 2^k and $(-2)^k$, where *k* - integer.

Table 4a. Codes of numbers $e^{jk60^{\circ}}$.

φ	0	60	120	180	240	300
code	**00**	**1d**	**0c**	**c1**	**0d**	**dc**

Table 4b. Codes of numbers 2^k *and* $(-2)^k$.

k	$(-2)^k$	2^k
-4	0.000d	0.000d
-3	0.001	0.0c1
-2	0.0c	0.0c
-1	0.d	1.d
0	1	1
1	c0	dc0
2	d00	d00
3	1000	c1000
4	c000	c000

Further we only shall more strictly state the results received in section 2.

Theorem 7. Any complex number Z represent in a positional system $< \rho = -R, \quad A_{R^2} >$, where set A_{R^h} consists of complex numbers $r_m = \alpha_m^1 + j\alpha_m^2$ and numbers $\alpha_m \in B_R$.

In particular, there is a system <-2, {0,1,j,1+j}>.

Theorem 8. Any complex number Z represent in a normal positional system, $< \pm j\sqrt{R}, \quad B_R >$.

For example, there is a system $< \pm j\sqrt{2}, \quad \{ 0, 1 \} >$.

For an illustration we shall write down codes of some characteristic numbers in system $\rho = j\sqrt{2}$: $K(2)$ = 10100, $K(-2)$ = 100, $K(-1)$ = 101, $K\left(j\sqrt{2}\right) = 10$, $K\left(-j\sqrt{2}\right) = 1010$.

Table 5. Binary systems of coding.

Preffered number systems	ρ	< 2 >	< -2>	<- 1>	Theorem	Fig.
System **1** $\rho_2(m)$	$\rho_2^m = \begin{cases} (-2)^{m/2} & \text{if } m-\text{even} \\ j(-2)^{m-1/2} & \text{if } m-\text{odd} \end{cases}$	10100	100	101	Formula (2)	1
System **2** $\rho = j\sqrt{2}$	$\pm j\sqrt{2}$	10100	100	101	Theorem 8	2
System **3** $\rho = j-1$	$-1 \pm j$	1100	11100	11101	Theorem 4	3
System **4** $\rho = \frac{1}{2}(j\sqrt{7}-1)$	$\frac{1}{2}\left(-1+j\sqrt{7}\right)$	1010	110	111	Theorem 5	4
	-2	110	10	11	Theorem 2	
	2	10			Theorem 2	

Obviously, for systems from theorems 7 and 8 the condition (14) is satisfied. The proof of these theorems is based on reasonings of section 2.

Let's result for an illustration and comparison binary codes of numbers in all specified systems of coding, including systems of coding on real (positive and negative) and complex to the radix - see table 5.

Further we shall stop on four binary systems of complex numbers - see a column «Preffered number systems» in table 5 in more detail. In figures the first 4 values of base function for the preffered number system are represented.

Fig. 1

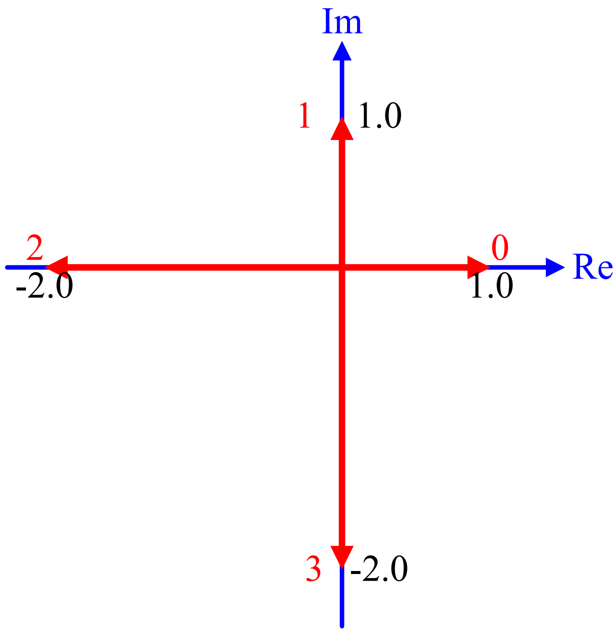


Fig. 2.

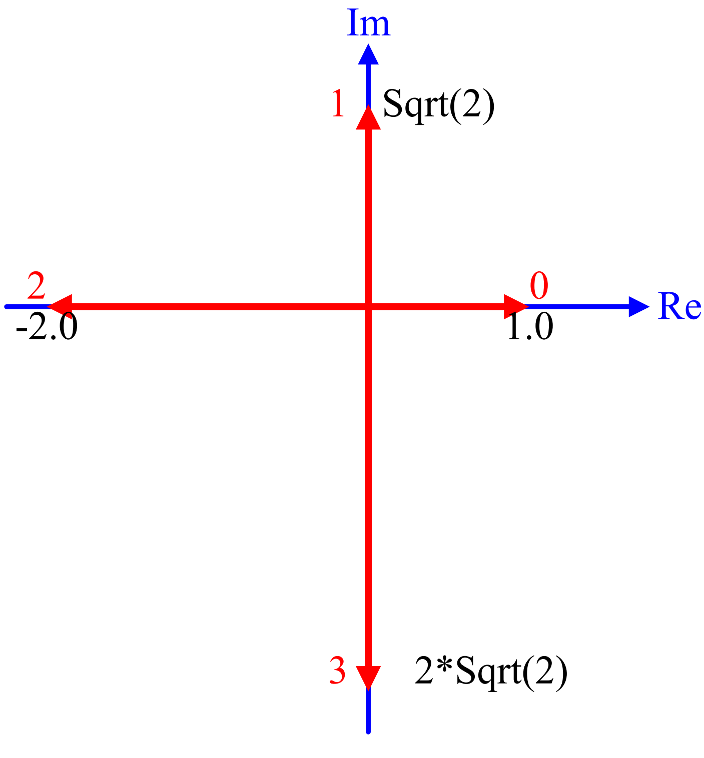


Fig. 3.

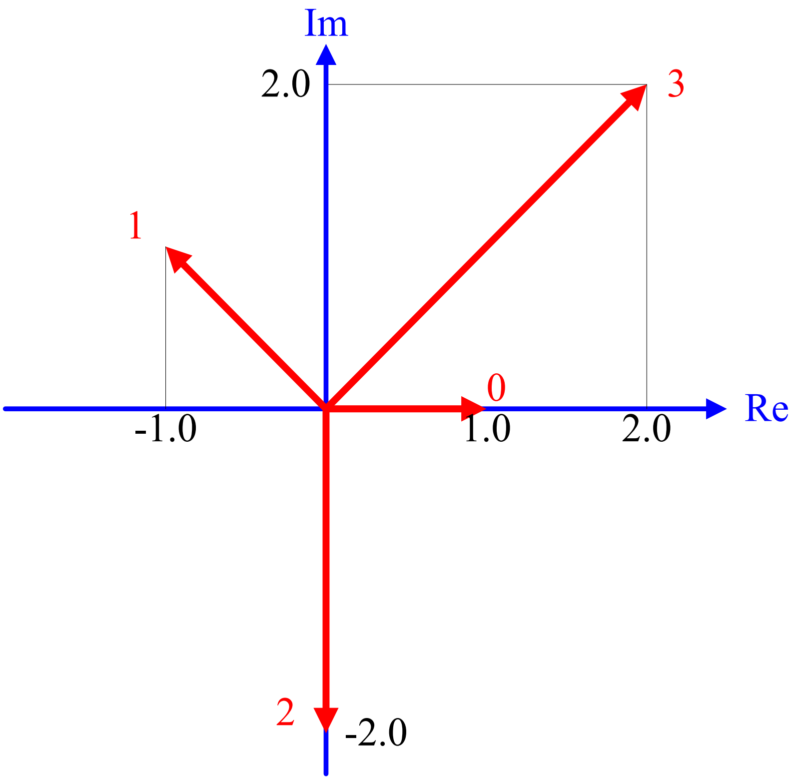
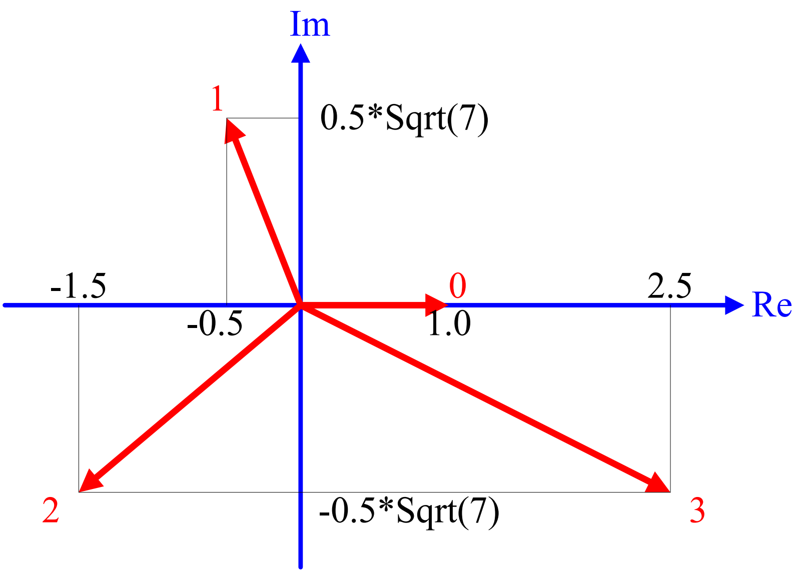


Fig. 4.



Let's result still table 6 four-valued codes of numbers ' 4 ' and '-4 ' in all systems of coding considered above (in this table '-1' it is designated by sign 'h').

Table 6. Four-valued coding system.

ρ	A_4	< 4 >	< - 4 >	**Theorem**
4	{0,1,2,3}	10		2
4	{-1,0,1,2}	10	h0	2
4	{-2,-1,0,1}	10	h0	2
-4	{0,1,2,3}	130	10	2
-4	{-1,0,1,2}	h0	10	2
-4	{-2,-1,0,1}	h0	10	2
$2e^{2j\pi/3}$	{0,1,2,3}	1120	120	5
$2e^{j\pi/3}$	{0,1,c,d}	d00	1d00	6
-2	{0,1,j,1+j}	100	1100	7
ρ_4	{0,1,2,3}	10300	100	8
$\pm 2j$	{0,1,2,3}	10300	100	8

7. Codes of many-dimensional vectors

The stated method of construction of codes of complex numbers can be integrated and is used for coding of multidimensional vectors. For it we consider set of real numbers $\{ X_i \}$, each of which is given by binary decomposition on the basis $\rho = -2$, that is

$$X_i = \sum_{(m)} \alpha_m^i \rho^m, \qquad (i=1, 2,..., n).$$

To each such decomposition there corresponds a code

$$K(X_i) = \ldots \alpha_m^i \ldots$$

We consider now *n*-dimensional vector

$$Z = E_1 X_1 + E_2 X_2 + \ldots + E_n X_n ,$$

where $\{ E_i \}$ - base of *n*-dimensional vector space. Set of codes $\{ K(X_i) \}$ it is possible thus interpret as a uniform code of a vector Z on the basis "-2". Each m-category of this code is represented by set $\{ \alpha_m^i \}$ binary categories. Having designated these sets by figures σ_m, we receive a code of a vector

$$K(Z) = \ldots \sigma_m \ldots,$$

appropriate to decomposition (1), where the vector

$$r_m = E_1 \alpha_m^1 + E_2 \alpha_m^2 + \ldots + E_i \alpha_m^i + \ldots + E_n \alpha_m^n$$

is represented by digit σ_m.

In particular, at n=2 codes of complex numbers on the basis "-2," considered higher will be formed. At n=3 codes of three-dimensional vectors will be formed, in which categories accept one of to eight significances:

$$r_m \in \{ 0, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{i+j}, \mathbf{i+k}, \mathbf{j+k}, \mathbf{i+j+k} \},$$

where **i, j, k** - unit vectors of rectangular coordinate axes.

For coding three-dimensional vectors, similarly previous function from real whole vector argument

$$\vartheta_2^m = \begin{Bmatrix} i(-2)^m \text{ if } m = 3k \\ j(-2)^{m-1} \text{ if } m = 3k+1 \\ k(-2)^{m-2} \text{ if } m = 3k+2 \end{Bmatrix},$$

can be entered. Thus the considered code of a three-dimensional vector on the basis (-2) with vector values of categories can be considered as a code of a three-dimensional vector on the basis (ϑ_2) with bits. To this code there corresponds decomposition of a vector as $Z = \sum_m \left(\alpha_m \vartheta_2^m\right)$.

We construct now, as earlier for complex numbers, sequence of interleaved(alternated) binary categories α_m^i:

$$\ldots\alpha_{m+1}^2\alpha_{m+1}^1\alpha_m^n\alpha_m^{n-1}\ldots\alpha_m^2\alpha_m^1\alpha_{m-1}^n\alpha_{m-1}^{n-1}\ldots$$

In other designations this sequence is a binary code

$$K(Z) = \ldots\alpha_k\ldots$$

some vector Z. Thus the basis of coding is also a vector $\rho = E_2\sqrt[n]{2}$, where E_2 - second unit vector of base $\{E_i\}$ n-dimensional vector space. The coded vector Z is defined(determined) in this case under the formula

$$Z = X_1 + \rho \quad X_2 + \ldots + \rho^{i-1} X_i + \ldots + \rho^{n-1} X_n.$$

Completely similarly positional codes of vectors (including complex numbers and multidimensional vectors) are under construction on the basis of association of positional codes of numbers - projections of vectors on the radix $\rho = -R$, where R - an integer. In this case, for example, instead of function ρ_2^m as the radix of coding of complex numbers function

$$\rho_R^m = \begin{Bmatrix} (-R)^{m/2} \text{ if } m - \text{even} \\ j(-R)^{m-1/2} \text{ if } m - \text{odd} \end{Bmatrix},$$

should be considered, instead of function ϑ_2^m as the radix of coding of three-dimensional vectors function

$$\vartheta_R^m = \begin{Bmatrix} i(-R)^{m/3} \text{ if } m = 3k \\ j(-R)^{m-1/3} \text{ if } m = 3k+1 \\ k(-R)^{m-2/3} \text{ if } m = 3k+2 \end{Bmatrix},$$

should be considered, etc.

Here we only more strictly state results, received in section 3. Thus special algebra in the *manu-dimensial Euclidian* space, described in chapter "Multiplication", section 2.

Theorem 7a. If in *h*-dimensial space with base $\{E_i\}$ is determined algebra, any point Z of this space represent in a positional system $<\rho = -R, \quad A_{R^h}>$, where set *A*=*r* consists of vectors

$$r_m = E_1\alpha_m^1 + E_2\alpha_m^2 + \ldots + E_i\alpha_m^i + \ldots + E_n\alpha_m^n,$$

and number $\alpha_m \in B_R$.

In particular, for complex numbers there are systems $<\rho = -R, \quad A_{R^2}>$, for example, < -2, { 0,1, j, 1 + j } >, and for three-dimensional vectors with orts **i, j, k** – oct-valued system, where each category accepts values $r_m \in$ { 0, **i, j, k, i+j, i+k, j+k, i+j+k** },

Theorem 8a. If in *h*-dimensial Euckidian space with base $\{E_i\}$ is determined algebra, any point *Z* represent in a normal positional system

$$<\rho = \pm E_2\sqrt[h]{R}, \quad B_R>.$$

In particular, for complex numbers there are systems $< \pm j\sqrt{R}, \quad B_R >$, for example, $< \pm j\sqrt{2}, \quad \{ 0, \quad 1 \} >$, and for three-dimensional vectors with orts **i, j, k** – binary system

$$< \pm j\sqrt[3]{2}, \quad \{ 0, \quad 1 \} >.$$

In last system we have:

<**i**>=1, <**-i**>=1001, <**2i**>=1001000, <**-2i**>=1000.

<**j**>=10, <**-j**>=10010, <**2j**>=10010000, <**-2j**>=10000.

<**k**>=100, <**-k**>=100100, <**2k**>=100100000, <**-2k**>=100000.

For three-dimensional vectors with orts **i, j, k** is also four-value a system $<\pm j\sqrt[3]{4}, \quad \{0,1,2,3\}>$, where <**4i**>=1003000 and <**-4i**>=1000.

Obviously, for systems from theorems 7a and 8a is satisfied condition (14). Proof of these theorems is based on reasons of section 3.

With positional codes of vectors operations algebraic addition, vector, scalar and special multiplication are feasible. Algorithms of these operations contain cycles algebraic addition of codes of numbers and shift of a code of a vector, that is, will be easily realized technically. It can be used at construction of processors, operating with vectors as a whole. Such processor requires more simple algorithm for the decision of tasks with vectors, and at the given algorithm works under the shorter program and has increased speed. For valuation of these sizes it is possible to indicate, for example, that the program to vector multiplication of vectors, given by three numbers, contains 6 operations of multiplication and 3 operations of subtraction.

References

1. Shannon C.E., **A Symmetrical Notation of Number**, Amer. Math. Month., 57, 1950, 90.
2. Shevchik U. **Universal digital machine**. Collection "Digital computers and programming", volume 2, publ. "Sowetskoe radio", 1967 (in Russian).
3. Knuth D.E., **An Imaginary Number System**, Communication of the ACM-3, 1960, **№** 4.
4. Khmelnik S.I. **Specialized computer for operations with complex numbers.** Questions of Radio Electronics, series XII, volume 2, 1964, http://lib.izdatelstwo.com/Papers2/s4.zip (in Russian).
5. Penney W., **A 'Binary' System for Complex Numbers**, Journal of ACM 12, No. 2, 1965, pp. 247-248.
6. Khmelnik S.I. **Methods of synthesis of the computers operating with complex numbers, and their use for the decision of some problems of antiaircraft defense.** Moscow, GSNII-5, inv. 7907, 1964 (in Russian).
7. Khmelnik S.I. **The adder of complex numbers codes.** Questions of Radio Electronics, series XII, volume 3, 1965 (in Russian).
8. Khmelnik S.I. **Positional coding of complex numbers**. Questions of Radio Electronics, series XII, volume 9, 1966, http://lib.izdatelstwo.com/Papers2/s5.zip (in Russian).
9. Khmelnik S.I. **Positional coding of complex numbers**, in book [10].
10. Pospelov D.A.. **Arithmetic bases of computers of discrete action**, publ. "Vysshaja shkola", 1970 (in Russian).
11. Khmelnik S.I. **The decision of navigating problems on computer, complex numbers operating on codes.** Questions of Specialized Radio Electronics, series "Telemehanics and control system", volume 6, 1971 (in Russian).
12. Khmelnik S.I. **The arithmetic unit for the digital computer.** USSR author's Certificate 266362, 1970 (in Russian).
13. Khmelnik S.I. **The arithmetic unit for operations with complex numbers.** USSR author's Certificate 377769, 1973 (in Russian).
14. Khmelnik S.I. **Devices of processors of digital units.** In book [15].
15. **Digital devices on microcircuits**, publ. "Energia", Moscow, 1975 (in Russian).
16. Aoki T., Amada H., Higuchi T. **Real/Complex Reconfigurable Arithmetic Using Redundant Complex Number Systems**. In Proc. 13th Symposium on Computer Arithmetic, 1997.

17. Y. Chang and K. Parhi. **High Performance Digit Serial Complex Number Multiplier-Accumulator**, Proc. Int. Conf. on Computer Design,1998.
18. Nielsen A. M. and Kornerup P. **Redundant Radix Representation of Ring**, IEEE Transactions on Computers, Vol. 48 (11), November 1999
19. Baikov V.D., Smolov V.B. **The specialized processors: iterative algorithms and structures**, publ. "Radio I swiaz", Moscow, 1985 (in Russian).
20. Miller J. M., **Elementary Functions. Algorithms and Implementation**. Birkhauser, 1997, Boston.
21. Khmelnik S.I. **The Vectors Coding**, journ. "Cibernetics", Ac. of Sc., Republic of Ukraine, 1969, №5 (in Russian)
22. Khmelnik S.I. **Algebra of Many-dimensional Vectors and Spatial Figures Coding**, journ. "Avtomatika i Vychislitelnaya Tekhnika", Ac. of Sc. of Latvia Republic, 1971, №1 (in Russian)
23. Khmelnik S.I. **Coding of functions,** journ. "Cibernetics", Ac. of Sc., Republic of Ukraine, 1966, **№**4 (in Russian).
24. Khmelnik S.I. **Some types of positional codes of functions,** journ. "Cibernetics", Ac. of Sc., Republic of Ukraine, 1970, **№**5 (in Russian).
25. Khmelnik S.I. **Algorithms of coding and decoding of functional series.** Collection "Digital computers and programming", volume 8, 1974 (in Russian).
26. Khmelnik S.I. **The Planar Figures Coding**, journ. "Avtomatika i Vychislitelnaya Tekhnika", Ac. of Sc. of Republic of Latvia, 1970, №6 (in Russian)
27. Khmelnik S.I. **Digital Device for Image Geometrical Transformations**. USSR author's Certificate. 333573, 1972 (in Russian)
28. Khmelnik S.I. **Device for Image Geometrical Transformations**. USSR author's Certificate 1030816, 1983 (in Russian)
29. Khmelnik S.I. **High-performance Search Procedures**. The Third International Symposium in Information Theory, part II, Tallinn, 1973 (in Russian)
30. Khmelnik S.I, **Multi-criteria Assignment Problem** , journ. "Isvestiya of USSR Ac. of Sc., Tekhnicheskaya Kibernetika", №4, 1977 (in Russian)
31. Khmelnik S.I. **Search procedures with geometrical codes,** journ. "Cibernetics", Ac. of Sc., Republic of Ukraine, 1990, **№**6 (in Russian).

32. S. Khmelnik. **Computer Arithmetic of Vectors, Figures and Functions,** Publishing «Mathematics in Computers Comp.», Moscow– Tel-Aviv, 1995 (in Russian)
33. Khmelnik S. **Method and system for implementing a coprocessor.** Canadian Intellectual Property Office. Application No 2,293,953. Priority 05.01.00.
34. S. Khmelnik**. Method and System for Processing Complex Numbers.** International patent application under PCT. PCT/CA01/00007, WO 01 50332 A2. Priority 05.01.00.
35. Khmelnik S. **Method and System for Processing Complex Numbers.** United States Patent US 2003154226 (A1), Application No. US 20020189195, Priority 05.06.02, http://worldwide.espacenet.com/publicationDetails/biblio?
36. Khmelnik S. **Method and System for Processing Matrices of Complex Numbers.** Canadian Intellectual Property Office. Application No 2,339,919. Priority 07.03.01.
37. Khmelnik S. **Method and System for Processing Matrices of Complex Numbers and Complex Fast Fourier Transformations.** International patent application under PCT. PCT/CA02/00295, WO 02 071254 A2. Priority 07.03.01.
38. Khmelnik S. **Method and System for Processing Geometrical Figures.** Canadian Intellectual Property Office. Application No 2,349,924. Priority 07.07.01.
39. S. Khmelnik. **Method and System for Processing Geometrical Figures.** International patent application under PCT. PCT/CA02/00835, WO 02 099625 A2. Priority 07.06.01.
40. Eliseev V.I. **Introduction in methods of the theory of functions spatial complex variable**. Moscow, NIAT, 1990, http://www.maths.ru/ (in Russian)
41. Wanhammar L. **DSP Integrated Circuits**, Academic Press, 1999.
42. Parhami B.. **Computer Arithmetic. Algorithms and Hardware Design**, Oxford University Press, 2000.
43. Flynn M.J., Oberman S.F. **Advanced Computer Arithmetic Design**, A wiley interscience Publication John Wiley & Sons, Inc., 2001.
44. Khmelnik S. **Method and System for Processing Complex Numbers.** European Patent Office, EP 1248993. Priority 12.07.01.
45. Khmelnik S.I. **Coding of Complex Numbers and Vectors**, publ. «Mathematics in Computers», Israel, 2004, ISBN 978-0-557-74692-7, http://mic34.com/Magazine/94846.pdf (in Russan)

46. Solomon I. Khmelnik, Inna S. Doubson, Semion M. Khmelnik, Alexander E. Viduetsky. **Complex Numbers Arithmetic Unit**. publ. «Mathematics in Computers», Israel, 2004, ISBN 978-0-557-74692-7, http://mic34.com/Magazine/94846.pdf (in Russan)
47. Khmelnik S.I. **Computer Arithmetic of Complex Numbers and Vectors. Theory, Hardware, Modeling**, publ. «Mathematics in Computers», Israel, Printed in USA, Lulu Inc., ID 560836, 2006., (in Russan)