ENUMERATION OF ALL PRIMITIVE PYTHAGOREAN TRIPLES WITH HYPOTENUSE LESS THAN OR EQUAL TO N

EDUARDO CALOOY ROQUE

Abstract. All primitive Pythagorean triples with hypotenuse less than or equal to N can be counted with the general formulas for generating sequences of Pythagorean triples ordered by $c-b$. The algorithm calculates the interval $(1,m)$ such that $c=N$ then ν from each m is calculated to get the interval (n_1, n_ν) then $(m, n_\nu)=1$ is used for counting. It can be enumerated manually if N is small but for large N the algorithm must be implemented with any computer programming languages.

1. Introduction

It was shown in 1900 by D.H. Lehmer that the number of primitive Pythagorean triples with hypotenuse less than or equal to N is approximately equal to $N/2\pi$. Considering this we sought how to count and tabulate them. Thus we show here that all primitive Pythagorean triples with hypotenuse less than or equal to N can be counted with the general formulas for generating sequences of Pythagorean triples ordered by $c-b$. This can be done manually for small N or implemented with any computer programming languages for large N.

For example, if we let $N=100$ then we count manually and with a C/C++ script and tabulate with the $\LaTeX 2_\varepsilon$ spreadtab package which can construct tables similar to a spreadsheet. We also tabulated the approximate count against the exact count and their difference when $N=\{10^2,10^3,10^4,10^5,10^6,10^7\}$.

2. Gneral formulas for enumeration of Pythagorean triples

We have the following general formulas to aid in the enumeration of Pythagorean triples with $a<b<c$ and $c\leqslant N$. Let (a,b,c) be a Pythagorean triple then

$$(a,b,c)=\begin{cases}\left(m^2+2mn,2n^2+2mn,m^2+2mn+2n^2\right) \iff (\gamma,\beta)=\left(m^2,2n^2\right)\\ \left(2m^2+2mn,n^2+2mn,2m^2+2mn+n^2\right) \iff (\gamma,\beta)=\left(2m^2,n^2\right)\end{cases} \tag{1}$$

where $\alpha,\beta,\gamma\in\mathbb{Z}$ such that $\alpha=b-a$, $\beta=c-a$, $\gamma=c-b$, $\beta=\alpha+\gamma$ and $m,n\in\mathbb{N}$.

If $a<b<c$ then

$$n_\nu=\begin{cases}\left\lfloor\dfrac{m}{\sqrt{2}}\right\rfloor+\nu \iff (\gamma,\beta)=\left(m^2,2n^2\right)\\ \left\lfloor m\sqrt{2}\right\rfloor+\nu \iff (\gamma,\beta)=\left(2m^2,n^2\right)\end{cases} \tag{2}$$

where $\nu\in\mathbb{N}$.

Theorem 1. *Let (a,b,c) be a Pythagorean triple and m, N, $\nu\in\mathbb{N}$ then*

$$\nu=\begin{cases}\left\lfloor\dfrac{\sqrt{2N-m^2}-m}{2}\right\rfloor-\left\lfloor\dfrac{m}{\sqrt{2}}\right\rfloor \iff (\gamma,\beta)=(m^2,2n^2)\\ \left\lfloor\sqrt{N-m^2}-m\right\rfloor-\left\lfloor m\sqrt{2}\right\rfloor \iff (\gamma,\beta)=(2m^2,n^2)\end{cases} \tag{3}$$

Date: October 25, 2013.

$$m = \begin{cases} \left\lfloor \sqrt{\dfrac{2N}{1+\left(1+\sqrt{2}\right)^2}} \right\rfloor \iff (\gamma,\beta) = (m^2, 2n^2) \\ \left\lfloor \sqrt{\dfrac{N}{1+\left(1+\sqrt{2}\right)^2}} \right\rfloor \iff (\gamma,\beta) = (2m^2, n^2) \end{cases} \tag{4}$$

Proof. We have

$$c = \begin{cases} m^2 + 2mn + 2n^2 \iff (\gamma,\beta) = (m^2, 2n^2) \\ 2m^2 + 2mn + n^2 \iff (\gamma,\beta) = (2m^2, n^2) \end{cases} \tag{5}$$

Let $c = N$ then take m as a constant and complete the square to get

$$n_\nu = \begin{cases} \left\lfloor \dfrac{-m \pm \sqrt{2N - m^2}}{2} \right\rfloor \iff (\gamma,\beta) = (m^2, 2n^2) \\ \left\lfloor -m \pm \sqrt{N - m^2} \right\rfloor \iff (\gamma,\beta) = (2m^2, n^2) \end{cases} \tag{6}$$

Now evaluate the two formulas for n_ν and we have

$$\nu = \begin{cases} \left\lfloor \dfrac{-m \pm \sqrt{2N - m^2}}{2} \right\rfloor - \left\lfloor \dfrac{m}{\sqrt{2}} \right\rfloor \iff (\gamma,\beta) = (m^2, 2n^2) \\ \left\lfloor -m \pm \sqrt{N - m^2} \right\rfloor - \left\lfloor m\sqrt{2} \right\rfloor \iff (\gamma,\beta) = (2m^2, n^2) \end{cases} \tag{7}$$

We let $\nu = 0$ if $c > N$ thus

$$m = \begin{cases} \left\lfloor \pm\sqrt{\dfrac{2N}{1+\left(1+\sqrt{2}\right)^2}} \right\rfloor \iff (\gamma,\beta) = (m^2, 2n^2) \\ \left\lfloor \pm\sqrt{\dfrac{N}{1+\left(1+\sqrt{2}\right)^2}} \right\rfloor \iff (\gamma,\beta) = (2m^2, n^2) \end{cases} \tag{8}$$

Now take the positive value of the radicals and we have the desired formulas. □

3. All primitive Pythagorean triples with hypotenuse $\leqslant 100$

Theorem 2. *There are exactly 16 primitive Pythagorean triples with hypotenuse less than or equal to* 100.

Proof. Let $N = 100$ then we can count the number of primitive Pythagorean triples with hypotenuse less than or equal to 100. Since there are two cases, we determine for each case the interval $(1, m)$ then for each m calculate ν to get the interval (n_1, n_ν) and from n_1 to n_ν count considering $(m, n_\nu) = 1$ and whether m is odd in one case and n_ν is odd in the other.

Case $(\gamma,\beta) = (m^2, 2n^2), (m, n) = 1,$ *odd* m:

$$\text{Primitive Pythagorean triples lie in } \left(1, \left\lfloor \sqrt{\frac{2(100)}{1+(1+\sqrt{2})^2}} \right\rfloor \right) = (1, 5).$$

We exclude even m. Now if $m = 1$ then $\nu = 6$ and $(n_1, n_{\nu=6}) = (1, 6)$. If $m = 3$ then $\nu = 3$ and $(n_1, n_{\nu=3}) = (3, 5)$. If $m = 5$ then $\nu = 1$ and $(n_1, n_{\nu=1}) = (4, 4)$. Considering $(m, n_\nu) = 1$ then the sum of primitive Pythagorean triples for these intervals is $6 + 2 + 1 = 9$.

Case $(\gamma,\beta) = (2m^2, n^2), (m.n) = 1,$ *odd* n:

$$\text{Primitive Pythagorean triples lie in } \left(1, \left\lfloor \sqrt{\frac{100}{1+(1+\sqrt{2})^2}} \right\rfloor \right) = (1, 3).$$

If $m = 1$ then $\nu = 7$ and $(n_1, n_{\nu=8}) = (2, 8)$. If $m = 2$ then $\nu = 5$ and $(n_1, n_{\nu=5}) = (3, 7)$. If $m = 3$ then $\nu = 2$ and $(n_1, n_{\nu=2}) = (5, 6)$. We exclude even n_ν and considering $(m, n_\nu) = 1$ then the sum of primitive Pythagorean triples for these intervals is $3 + 3 + 1 = 7$.

Thus there are exactly $9 + 7 = 16$ primitive Pythagorean triples with $c \leqslant 100$. □

4. Pseudocode

```
Algorithm: Counting the number of primitive Pythagorean triples with c ⩽ N
Input: Input N
Output: Output ctr
begin
    ctr, n_1, n_ν;
    ctr = 0;
    // Case (γ, β) = (m^2, 2n^2), (m, n) = 1, odd m
    for i ← 1 to ⌊ sqrt( 2N / (1 + (1 + √2)^2) ) ⌋ do
        if i is odd then
            n_1 = ⌊ i / √2 ⌋ + 1;
            n_ν = ⌊ (√(2N − i^2) − i) / 2 ⌋;
            for j ← n_1 to n_ν do
                if (i, j) == 1 then
                    ctr = ctr + 1;
                end
            end
        end
    end
    // Case (γ, β) = (2m^2, n^2), (m, n) = 1, odd n
    for i ← 1 to ⌊ sqrt( N / (1 + (1 + √2)^2) ) ⌋ do
        n_1 = ⌊ i√2 ⌋ + 1;
        n_ν = ⌊ √(N − i^2) − i ⌋;
        for j ← n_1 to n_ν do
            if j is odd then
                if (i, j) == 1 then
                    ctr = ctr + 1;
                end
            end
        end
    end
    return ctr;
end
```

5. Table of D. H. Lehmer's approximation against the exact count

N	$N/2\pi$	Exact count	Difference
100	16	16	0
1000	159	158	-1
10000	1592	1593	1
100000	15915	15919	4
1000000	159155	159139	-16
10000000	1591546	1591579	33

6. Tables of primitive Pythagorean triples with $a < b < c$ and $c \leqslant 100$

Case $(\gamma, \beta) = (m^2, 2n^2), (m, n) = 1,$ *odd* m:

m	n	a	b	c
1	1	3	4	5
1	2	5	12	13
1	3	7	24	25
1	4	9	40	41
1	5	11	60	61
1	6	13	84	85
1	7	15	112	113

m	n	a	b	c
3	4	33	56	65
3	5	39	80	89
3	7	51	140	149
3	8	57	176	185
3	10	69	260	269
3	11	75	308	317
3	13	87	416	425

m	n	a	b	c
5	4	65	72	97
5	6	85	132	157
5	7	95	168	193
5	8	105	208	233
5	9	115	252	277
5	11	135	352	377
5	12	145	408	433

Case $(\gamma, \beta) = (2m^2, n^2), (m, n) = 1,$ *odd* n:

m	n	a	b	c
1	3	8	15	17
1	5	12	35	37
1	7	16	63	65
1	9	20	99	101
1	11	24	143	145
1	13	28	195	197
1	15	32	255	257

m	n	a	b	c
2	3	20	21	29
2	5	28	45	53
2	7	36	77	85
2	9	44	117	125
2	11	52	165	173
2	13	60	221	229
2	15	68	285	293

m	n	a	b	c
3	5	48	55	73
3	7	60	91	109
3	11	84	187	205
3	13	96	247	265
3	17	120	391	409
3	19	132	475	493
3	23	156	667	685

7. Conclusion

Enumeration of all primitive Pythagorean triples with hypotenuse $\leqslant N$ can be done by applying the general formulas for generating sequences of Pythagorean triples ordered by $c - b$ and theorem 1 .

The LaTeX 2_ε spreadtab package is used to obtain primitive Pythagorean triples and create the tables. Furthermore, the LaTeX 2_ε spreadtab and C/C++ scripts for enumeration are attached to allow the reader to check and verify the results.

Click to download the scripts here: . Extract by executing "tar xvzf codesko.tar.gz" then follow the instructions in the file README.txt.

We hope and pray that this humble work be of benefit to fellowmen and also give thanks to God our Father in heaven and our Lord Jesus Christ for this knowledge.

Proverbs 3:13

Happy is the man that finds wisdom, and the man that gets understanding.

Proverbs 9:10

The fear of the Lord is the beginning of knowledge: and the knowledge of the Holy One is understanding.

References

[1] James J. Tattersall: *Elementary Number Theory in Nine Chapters*, (1999)

[2] Christian Tellechea: *Spreadtab*, v0.4b, (2012)

[3] Eduardo Calooy Roque: *On General Formulas for generating sequences of Pythagorean Triples ordered by* $c - b$, (2013). Available at http://vixra.org