Theory of Motion

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October 22, 2013

By examining the theory of relativity, we postulate that every massive particle specifies a space-time body frame in a universal entity, which may be referred to as ether. As a result, the four-dimensional theory of general accelerating motion is developed. It is seen that the relative motion of particles is actually the result of relative four-dimensional rotation of their corresponding space-time body frames. Consequently, the governing geometry of relative motion is non-Euclidean.

1. Introduction

Based on the Lorentz ether theory, Poincaré in 1905 proposed the relativity principle as a general law of nature, including electrodynamics and gravitation. Minkowski realized that Poincaré's theory could be best understood in a four dimensional space, known as "Minkowski spacetime", in which time and space are not separated entities but intermingled in a four dimensional space-time. Consequently, one realizes that the Lorentz transformations form a group, and the Maxwell's equations are covariant under this group. Although the Lorentz transformation is fundamental in this development, Poincaré's theory of relativity does not clearly explain its physical meaning and cannot clarify the fundamental relativistic meaning of Minkowski space-time as a single entity. Despite the fact that Poincaré's theory shows a relationship between pure Lorentz transformation and hyperbolic rotation, it does not specify the fundamental mechanism of this rotation and what is rotating. Thus, Poincaré's theory does not clearly resolve the fundamental aspects of Minkowski space-time, including its geometry, and does not give further insight into the accelerating motion. This is the origin of most of the troubles within the theory of relativity and electrodynamics, including the geometrization of the theory of relative accelerating motion.

Early investigators of relativity, such as Robb, Varičak, Lewis, Wilson, and Borel have noticed and extensively investigated the non-Euclidean geometric character of uniform relative motion, where hyperbolic geometry governs the velocity addition law [1-5]. Interestingly, it has been shown that non-Euclidean geometry is the origin of the famous Thomas-Wigner rotation [5]. However, the importance of this non-Euclidean geometry and its affinity with the Minkowski space-time in theory of relativity has not been appreciated and its fundamental meaning has remained a mystery. In addition, there has not been any consistent development for relativistic accelerating relative motion.

These difficulties suggest that we reconsider the theory of relativity and look for a more fundamental interpretation of the Lorentz transformation to understand the meaning of the governing non-Euclidean geometry. Therefore, we develop the fundamental vortical theory of motion, which is based on establishing the relation between space-time and matter. This shows that a massive particle specifies its own space-time body frame and the motion of the particle is the four-dimensional rotation of its space-time body frame. As a result of this development, we realize why the laws of relativity are covariant and follow the language of non-Euclidean geometry. It is also seen that the orthogonal transformations similar to Lorentz transformations are not restricted to relative uniform motion. The relative motion of accelerating particles is also represented by varying orthogonal transformations. Interestingly, this development revives the idea of the ether as a fundamental entity in the universe. The space-time body frames of particles are different representations of the ether.

We organize the current paper in the following manner. In Section 2, we provide an overview of elementary relativistic kinematics of a particle. Subsequently, in Section 3, we present the fundamental four-dimensional character of the particle and its vortical motion. Afterwards, in Section 4, we develop the general accelerating relative motion and velocity addition law. Next we discuss the concept of ether in the context of these developments in Section 5. A summary and general conclusion is presented in Section 6.

2. Preliminaries and Relativistic Kinematics of a Particle

As an inertial reference frame, a four-dimensional coordinate system $x_1 x_2 x_3 x_4$ is considered such that $x_1 x_2 x_3$ is the usual space and x_4 is the axis measuring time with imaginary values, such that $x_4 = ict$. This Minkowki space-time is shown symbolically in Figure 1 by considering a two dimensional space and one time direction. Throughout this paper, we refer to this as our specified inertial reference frame. The unit four-vector bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ are defined by

$$\begin{aligned} \mathbf{e}_1 &= (1,0,0,0), \\ \mathbf{e}_2 &= (0,1,0,0), \\ \mathbf{e}_3 &= (0,0,1,0), \\ \mathbf{e}_4 &= (0,0,0,1). \end{aligned} \tag{1}$$

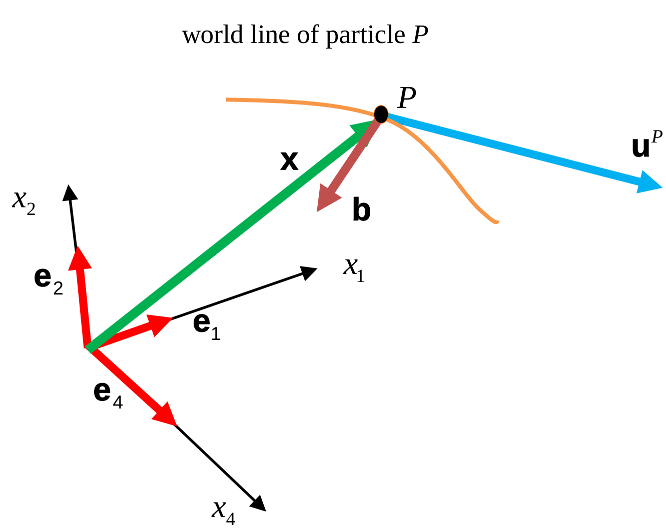

Figure 1. World line of the particle in inertial reference frame.

The space-time position four-vector of the particle P can be represented by

$$\mathbf{X} = x_\mu \mathbf{e}_\mu . \tag{2}$$

However, for simplicity, we sometimes write

$$\mathbf{X} = (\mathbf{x}, x_4) = (\mathbf{x}, ict) = (x, y, z, ict) , \tag{3}$$

or even

$$x_\mu = (\mathbf{x}, x_4) , \tag{4}$$

and also often use x in place of $\mathbf{X}$. The position of the massive particle in the inertial reference frame describes a path known as the world line.

By considering two neighboring positions $\mathbf{X}$ and $\mathbf{X} + d\mathbf{X}$ on the world line, we have

$$d\mathbf{X} = dx_\mu \mathbf{e} = (d\mathbf{x}, icdt) = (\mathbf{v}, ic)dt . \tag{5}$$

The three-vector $\mathbf{v} = v\mathbf{e}_t$ is the velocity of the particle, where $\mathbf{e}_t$ is the tangential unit three-vector in the direction of $\mathbf{v}$. The square length of this infinitesimal four-vector

$$ds^2 = d\mathbf{X} \bullet d\mathbf{X} = dx_\mu dx_\mu = d\mathbf{x}^2 - c^2 dt^2 = -c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right), \tag{6}$$

is the scalar invariant under all Lorentz transformation. The proper time between the events $d\tau$ is defined by

$$d\tau = dt\sqrt{1 - v^2/c^2} . \tag{7}$$

Therefore,

$$ds = icd\tau . \tag{8}$$

By using the concept of rapidity ξ, where

$$\tanh \xi = \frac{v}{c} , \tag{9}$$

we obtain

$$dt = d\tau \cosh \xi . \tag{10}$$

The four-vector velocity $\mathbf{u}^P = u_\mu^P \mathbf{e}_\mu$ is the rate of change of the position vector of the particle $\mathbf{x}$ with respect to its proper time

$$\mathbf{u}^P = \frac{d\mathbf{x}}{d\tau}. \tag{11}$$

In terms of components, this relation becomes

$$u_\mu^P = \frac{dx_\mu}{d\tau}. \tag{12}$$

For simplicity, we may drop the superscript ^{P} and write $\mathbf{u} = u_\mu \mathbf{e}_\mu$. The space and time components of $\mathbf{u} = (\mathbf{u}, u_4)$ are

$$\mathbf{u} = \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = c \sinh \xi \mathbf{e}_t, \tag{13}$$

and

$$u_4 = \frac{ic}{\sqrt{1 - v^2/c^2}} = ic \cosh \xi. \tag{14}$$

Therefore, the four-vector velocity can be represented as

$$\mathbf{u} = c(\sinh \xi \mathbf{e}_t,\ \mathrm{i} \cosh \xi). \tag{15}$$

The length of the four-vector velocity is a constant, since

$$u_\mu u_\mu = \mathbf{u}^2 + u_4{}^2 = -c^2, \tag{16}$$

which shows the four-velocity is time-like.

The four-acceleration $\mathbf{b} = b_\mu \mathbf{e}_\mu$ is defined as

$$\mathbf{b} = \frac{d\mathbf{u}}{d\tau} = \frac{d^2\mathbf{x}}{d\tau^2}. \tag{17}$$

In terms of components, this relation becomes

$$b_\mu = \frac{du_\mu}{d\tau} = \frac{d^2 x_\mu}{d\tau^2} . \tag{18}$$

The four-acceleration is always perpendicular to the four-vector velocity, as shown in Figure 1, where

$$\mathbf{u} \bullet \mathbf{b} = u_\mu b_\mu = 0 . \tag{19}$$

It can be easily shown that

$$\mathbf{b} = \left[\cosh^2 \xi \mathbf{a} + \cosh^4 \xi \left(\frac{\mathbf{v} \bullet \mathbf{a}}{c^2} \right) \mathbf{v}, i \cosh^4 \xi \left(\frac{\mathbf{v} \bullet \mathbf{a}}{c} \right) \right], \tag{20}$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} . \tag{21}$$

is the three-vector acceleration of the particle. The length of the four-vector acceleration can be found to be

$$|\mathbf{b}|^2 = b_\mu b_\mu = \cosh^4 \xi a^2 + \cosh^6 \xi \left(\frac{\mathbf{v} \bullet \mathbf{a}}{c} \right)^2 . \tag{22}$$

Since $b_\mu b_\mu$ is positive, the four-acceleration is space-like.

It should be noticed that all four-tensors in this article are either of first or second order. For simplicity, we use the same symbols for their vector and matrix representations. Therefore, the relation

$$\mathbf{u} \bullet \mathbf{b} = 0 , \tag{23}$$

can be written in matrix form

$$\mathbf{u}^T \mathbf{b} = 0 . \tag{24}$$

What we have presented is the well-known relativistic kinematics of a particle. However, it is seen that the accelerating motion of the particle can be considered as the result of a four-dimensional rotation with important consequences, as will be shown in the following.

3. Vortical Theory of Motion

As we mentioned, Poincaré's theory of relativity can be extended by establishing the relation between matter and space-time. It is postulated that the massive particle specifies its own space-time body frame $x'_1x'_2x'_3x'_4$, in which the particle has an attached four-velocity $\mathbf{u}^P$ with magnitude c in the time direction x'_4. Thus, the four-vector velocity of the particle in this system is represented by

$$u'^P_\mu = (0,0,0,ic) \ . \tag{25}$$

while its representation in our inertial reference frame system $x_1x_2x_3x_4$ is

$$u^P_\mu = c(\sinh\xi\mathbf{e}_t,\ \mathrm{i}\cosh\xi). \tag{26}$$

This is shown in Figure 2.

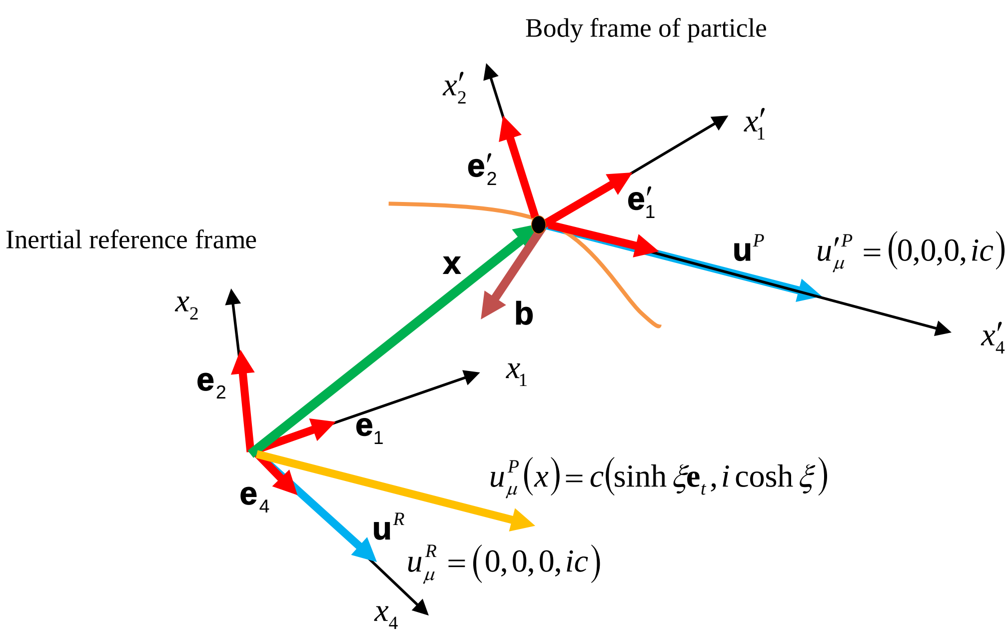

Figure 2. Inertial reference frame and body frame.

Within the present theory every massive particle specifies a space-time coordinate system, and conversely every space-time frame has a massive particle associated with it. Therefore, the reference frame $x_1x_2x_3x_4$ is specified with a massive particle, having the attached four-velocity $\mathbf{u}^R$ and magnitude *c* in the time direction x_4, where

$$u_\mu^R = (0,0,0,ic). \tag{27}$$

The orientation of the body frame of the particle *P* relative to the inertial reference frame system is specified by the orthogonal transformation four-tensor $\boldsymbol{\Lambda}(\mathbf{x})$ with orthogonality condition

$$\boldsymbol{\Lambda}(\mathbf{x})\boldsymbol{\Lambda}^T(\mathbf{x}) = \boldsymbol{\Lambda}^T(\mathbf{x})\boldsymbol{\Lambda}(\mathbf{x}) = \mathbf{1}, \tag{28}$$

or

$$\Lambda_{\mu\alpha}\Lambda_{\nu\alpha} = \Lambda_{\alpha\mu}\Lambda_{\alpha\nu} = \delta_{\mu\nu}. \tag{29}$$

where $\delta_{\mu\nu}$ is the Kronecker delta in four dimensions. This variable orthogonal transformation $\boldsymbol{\Lambda}(\mathbf{x})$ is a general Lorentz transformation, which can be written as

$$\begin{aligned}\boldsymbol{\Lambda}(\mathbf{x}) &= \begin{bmatrix}\mathbf{Q} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{1}+(\cosh\xi-1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t\\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{Q}+(\cosh\xi-1)\mathbf{Q}\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{Q}\mathbf{e}_t\\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi\end{bmatrix}.\end{aligned} \tag{30}$$

Here $\mathbf{Q}$ is a three-dimensional orthogonal tensor representing spatial rotation of the body frame. The pure Lorentz transformation or boost part of this transformation depends on the rapidity vector $\boldsymbol{\xi}$ of the particle, even for accelerating motion. It is seen that the relations among the base unit four-vectors of the body frame and inertial systems are

$$\mathbf{e}'_\mu = \Lambda_{\mu\nu}\mathbf{e}_\nu, \tag{31}$$

or

$$\mathbf{e}_\mu = \Lambda_{\nu\mu}\mathbf{e}'_\nu. \tag{32}$$

Therefore, the angles among these directions are such that

$$\cos(\mathbf{e}'_\mu, \mathbf{e}_\nu) = \Lambda_{\mu\nu}, \tag{33}$$

$$\cos(\mathbf{e}_\mu, \mathbf{e}'_\nu) = \Lambda_{\nu\mu}. \tag{34}$$

The inertial reference frame and the body frame of the particle both have attached four-vector velocities $\mathbf{u}^R$ and $\mathbf{u}^P$ in their own space-time frames, respectively. The attached four-vector velocity $\mathbf{u}^P$ is rotating with the body frame of the particle, such that between its components in this frame and the inertial frame, we have

$$\mathbf{u}'^P = \mathbf{\Lambda}(\mathbf{x})\mathbf{u}^P(\mathbf{x}). \tag{35}$$

We may drop the superscript ^{P} and write

$$\mathbf{u}' = \mathbf{\Lambda}(\mathbf{x})\mathbf{u}(\mathbf{x}), \tag{36}$$

or in terms of components as

$$u'_\mu = \Lambda_{\mu\nu} u_\nu. \tag{37}$$

The Lorentz transformation (37) relates the components of four-vector velocity $\mathbf{u}'$ of particle P in its frame and its components of four-vector velocity $\mathbf{u}$ in the inertial reference frame of particle R.

This relation can also be written as

$$\mathbf{u}(\mathbf{x}) = \mathbf{\Lambda}^T(\mathbf{x})\mathbf{u}'. \tag{38}$$

By taking the derivative of (28) with respect to the invariant proper time of the particle, we obtain

$$\frac{d\mathbf{\Lambda}^T}{d\tau}\mathbf{\Lambda} + \mathbf{\Lambda}^T\frac{d\mathbf{\Lambda}}{d\tau} = \mathbf{0}. \tag{39}$$

Now by defining the four-tensor $\mathbf{\Omega}$

$$\mathbf{\Omega} = \frac{d\mathbf{\Lambda}^T}{d\tau}\mathbf{\Lambda}, \tag{40}$$

we can see that the relation (39) becomes

$$\mathbf{\Omega} + \mathbf{\Omega}^T = \mathbf{0}\,. \tag{41}$$

In terms of components, this relation can be written as

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0\,. \tag{42}$$

which shows that $\mathbf{\Omega}$ is an anti-symmetric four-tensor. From our knowledge in non-relativistic rigid body dynamics, we realize that the four-tensor $\mathbf{\Omega}$ represents the four-tensor angular velocity of the space-time body frame of the particle measured in the inertial reference system. Interestingly, by using the relation (39), we rewrite (40) as

$$\frac{d\mathbf{\Lambda}}{d\tau} = -\mathbf{\Lambda}\mathbf{\Omega}\,, \tag{43}$$

which is the equation of motion of the body frame of the particle in terms of its four-tensor angular velocity $\mathbf{\Omega}(\mathbf{x})$.

The anti-symmetric tensor $\mathbf{\Omega}(\mathbf{x})$ in terms of elements in the inertial reference frame is written

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & -i\eta_1/c \\ \omega_3 & 0 & -\omega_1 & -i\eta_2/c \\ -\omega_2 & \omega_1 & 0 & -i\eta_3/c \\ i\eta_1/c & i\eta_2/c & i\eta_3/c & 0 \end{bmatrix}. \tag{44}$$

This is the general form of the four-tensor angular velocity $\mathbf{\Omega}$. By introducing three-vectors $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$, this four-tensor can be symbolically represented by

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{R}_{\boldsymbol{\omega}} & -i\frac{1}{c}\boldsymbol{\eta} \\ i\frac{1}{c}\boldsymbol{\eta}^T & 0 \end{bmatrix}, \tag{45}$$

where the anti-symmetric matrix $\mathbf{R}_{\boldsymbol{\omega}}$, corresponding to the three-vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, is defined by

$$\mathbf{R}_{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{46}$$

It should be noticed that, from (45), the elements $\Omega_{4i} = -\Omega_{i4} = i\eta_i / c$ are imaginary. The components of $\Omega_{\mu\nu}$ can be formally interpreted as follows:

(i) The angular velocities ω_1, ω_2 and ω_3 generate the space rotation of the body frame $x'_1 x'_2 x'_3 x'_4$ in the planes $x_2 x_3$, $x_3 x_1$ and $x_1 x_2$, respectively.

(ii) The imaginary angular velocities $\frac{i}{c}\eta_1$, $\frac{i}{c}\eta_2$ and $\frac{i}{c}\eta_3$ generate hyperbolic rotation of the body frame $x'_1 x'_2 x'_3 x'_4$ in the planes $x_1 x_4$, $x_2 x_4$ and $x_3 x_4$, respectively. The quantities η_1, η_2 and η_3 are the rate of change of boost of the body frame, along the x_1, x_2 and x_3 axes, respectively.

Therefore, the space-time body frame of the particle rotates relative to the inertial system with angular velocity tensor $\mathbf{\Omega}$, which is a combination of circular and hyperbolic angular velocities $\boldsymbol{\omega}$ and $\frac{1}{c}\boldsymbol{\eta}$. Note that these interpretations of three-vector velocities are not completely compatible with our notion of rotation in non-relativistic kinematics. Although we still use the notations $\boldsymbol{\omega}$ and $\frac{i}{c}\boldsymbol{\eta}$, and call these angular velocities, these vectors cannot be taken as proper angular velocity vectors. Basically these interpretations are consistent only for an inertial co-moving observer, who measures the components of

$$\Omega'_{\mu\nu} = \Lambda_{\mu\alpha}\Lambda_{\nu\beta}\Omega_{\alpha\beta}, \tag{47}$$

where

$$\mathbf{\Omega}' = \begin{bmatrix} \mathbf{R}_{\omega'} & -i\frac{1}{c}\boldsymbol{\eta}' \\ i\frac{1}{c}\boldsymbol{\eta}'^{T} & 0 \end{bmatrix}. \tag{48}$$

This co-moving observer can consider the vectors $\boldsymbol{\omega}'$ and $\frac{i}{c}\boldsymbol{\eta}'$ as proper angular velocity vectors.

The inverse tensor transformation of (47),

$$\Omega_{\mu\nu} = \Lambda_{\alpha\mu}\Lambda_{\beta\nu}\Omega'_{\alpha\beta}, \tag{49}$$

shows that a combination of the circular and hyperbolic angular velocities $\boldsymbol{\omega}'$ and $\frac{1}{c}\boldsymbol{\eta}'$ gives the vectors $\boldsymbol{\omega}$ and $\frac{1}{c}\boldsymbol{\eta}$ under a non-Euclidean geometry transformation. One can see the appearance of paradoxical effects similar to Thomas-Wigner precession [6] in this transformation. However, we now know there is no paradox at all and these effects are simply the result of the governing non-Euclidean geometry.

By noticing that the four-vector velocity of the particle is attached to its own body frame in its time direction, we realize that the four-acceleration of the particle is the result of the four-dimensional rotation of its body frame. Let us derive an expression for four-acceleration of the particle in terms of the four-tensor angular velocity $\mathbf{\Omega}(\mathbf{x})$. By taking the proper time derivative of (38), we obtain

$$\frac{d\mathbf{u}(\mathbf{x})}{d\tau} = \frac{d\mathbf{\Lambda}^{T}(\mathbf{x})}{d\tau}\mathbf{u}'. \tag{50}$$

Then by substituting from (36), we have

$$\frac{d\mathbf{u}(\mathbf{x})}{d\tau} = \frac{d\mathbf{\Lambda}^{T}(\mathbf{x})}{d\tau}\mathbf{\Lambda}(\mathbf{x})\mathbf{u}(\mathbf{x}). \tag{51}$$

Using the definition of four-tensor angular velocity $\mathbf{\Omega}$ in (40), we finally obtain

$$\frac{d\mathbf{u}(\mathbf{x})}{d\tau}=\mathbf{\Omega}(\mathbf{x})\mathbf{u}(\mathbf{x}), \tag{52}$$

or

$$\frac{du_{\mu}}{d\tau}=\Omega_{\mu\nu}u_{\nu}\,. \tag{53}$$

It is seen that the four-acceleration $\frac{d\mathbf{u}(\mathbf{x})}{d\tau}$ is the result of the instantaneous four-dimensional rotation of the four-velocity $\mathbf{u}$ attached to the body frame and rotating with four-tensor angular velocity $\mathbf{\Omega}(\mathbf{x})$. The space and time components of this equation are

$$\frac{d\mathbf{u}}{d\tau}=\boldsymbol{\omega}\times\mathbf{u}-\frac{i}{c}\boldsymbol{\eta}u_{4}, \tag{54}$$

and

$$\frac{du_{4}}{d\tau}=i\frac{1}{c}\boldsymbol{\eta}\bullet\mathbf{u}\,. \tag{55}$$

However, we notice that the more complete equation of motion for the particle is the equation of instantaneous four-dimensional rotation of its space-time body frame (43), that is

$$\frac{d\mathbf{\Lambda}}{d\tau}+\mathbf{\Lambda}\mathbf{\Omega}=\mathbf{0}\,, \tag{56}$$

which can also be written as

$$\frac{d\Lambda_{\mu\nu}}{d\tau}+\Lambda_{\mu\alpha}\Omega_{\alpha\nu}=0\,. \tag{57}$$

Now, for any four-vector $\mathbf{S}$ attached to the body frame of the particle, we have

$$S'_{\mu}=\Lambda_{\mu\nu}S_{\nu}\,. \tag{58}$$

This shows that

$$S'_{\mu}S'_{\mu}=\Lambda_{\mu\nu}\Lambda_{\mu\rho}S_{\nu}S_{\rho}=\delta_{\mu\nu}S_{\nu}S_{\rho}=S_{\nu}S_{\nu}=\text{const}\,, \tag{59}$$

which means the length of the four-vector velocity is $\mathbf{S}$ invariant. As a result of this, the rate of change of this four-vector relative to the inertial frame is

$$\frac{dS_\mu}{d\tau} = \Omega_{\mu\nu} S_\nu \,, \tag{60}$$

or

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{\Omega S} \,. \tag{61}$$

What we have demonstrated is that the Lorentz transformations are not restricted to relative uniform motion. The relative motion of accelerating particles as a four-dimensional vortical motion is also represented by varying orthogonal Lorentz transformations. The general Lorentz transformation must be written for attached four-vectors such as velocities, not positions, that is

$$u'_\mu = \Lambda_{\mu\nu} u_\nu \,. \tag{62}$$

If the particle has uniform motion, which means $\Lambda_{\mu\nu}$ is constant, we can integrate (62) and obtain

$$x'_\mu = \Lambda_{\mu\nu} x_\nu \,. \tag{63}$$

The relation (63) is the traditional form of Lorentz transformation for four-vector positions among inertial systems. However, this relation is not valid when the particle is accelerating. What we have developed here is the extension of Poincaré's relativity for accelerating systems, which may be called the rotational theory of relativity.

It should be noticed that the equation of motion in the form of (53) has been mentioned in the context of Fermi-Walker transport for an accelerating observer [7], where a natural moving frame is instantly attributed to the particle. However, the fundamental meaning of the relation (53) regarding to the four-dimensional character of the particle and vortical motion of its space-time body frame has not been realized.

4. General Relative Motion and Velocity Addition Law

The non-Euclidean character of uniform relative motion was noticed by early investigators of the theory of relativity [1-5]. Varićak demonstrated that the Einstein formula for the combination of constant velocities shows that rapidity combines by the triangle rule in hyperbolic space. This is a fundamental result for the hyperbolic geometry, which was demonstrated later by other approaches [1]. Now, we realize that this property holds for four-vectors and four-tensors. Inertial observers relate the components of four-vectors and four-tensors by Lorentz transformations. This is the origin of non-Euclidean geometry governing the three-vector and three-tensors. The hyperbolic geometric character of addition of uniform three-vector velocities is the manifest of this fact. We demonstrate that this property holds even for accelerating particles in the following.

Consider two particles *A* and *B* moving with velocities $\mathbf{v}_A = \mathbf{v}_A(t)$ and $\mathbf{v}_B = \mathbf{v}_B(t)$ relative to the inertial reference system $x_1x_2x_3x_4$. The four-vector velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ are attached four-vectors to the space-time body frames $x_1'x_2'x_3'x_4'$ and $x_1''x_2''x_3''x_4''$ of *A* and *B*, such that

$$(\mathbf{u}_A)_A = \mathbf{\Lambda}_A\mathbf{u}_A, \tag{64}$$

$$(\mathbf{u}_B)_B = \mathbf{\Lambda}_B\mathbf{u}_B. \tag{65}$$

This is depicted in Figure 3. It should be noticed that $(\mathbf{u}_A)_A$ and $(\mathbf{u}_B)_B$ are representations of the two four-vector velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ measured by observers attached to their corresponding body frames *A* and *B*, where

$$(\mathbf{u}_A)_A = \mathbf{u}'_A = (0,0,0,ic), \tag{66}$$

$$(\mathbf{u}_B)_B = \mathbf{u}''_B = (0,0,0,ic). \tag{67}$$

Therefore, we have the interesting relation

$$(\mathbf{u}_A)_A = (\mathbf{u}_B)_B = (0,0,0,ic). \tag{68}$$

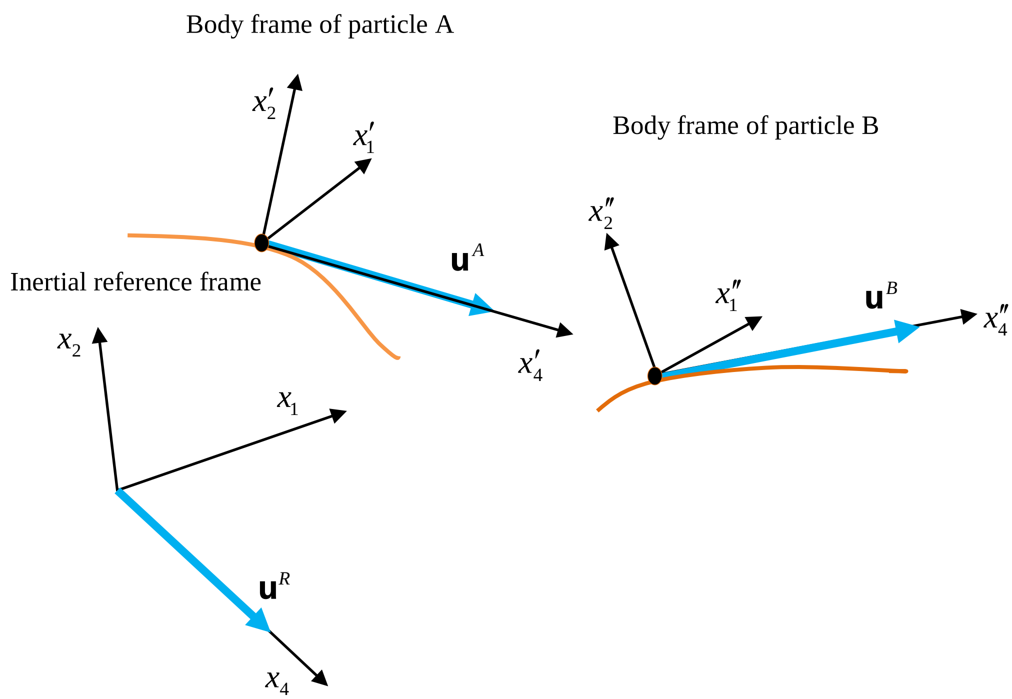

Figure 3. Body frames in relative motion.

However, we notice that an inertial observer in the inertial reference frame measures different components for four-vectors $\mathbf{u}_A$ and $\mathbf{u}_B$. The equality (68) for components of $(\mathbf{u}_A)_A$ and $(\mathbf{u}_B)_B$ is the origin of the non-Euclidean character of relative motion, which we next explore.

The transformations $\mathbf{\Lambda}_A = \mathbf{\Lambda}_A(t)$ and $\mathbf{\Lambda}_B = \mathbf{\Lambda}_B(t)$ represent the orientation of the body frames of particles A and B relative to the inertial reference frame. For these transformations, we explicitly have

$$\mathbf{\Lambda}_A(t) = \begin{bmatrix} \mathbf{Q}_A + (\cosh\xi_A - 1)\mathbf{Q}_A\mathbf{e}_{tA}\mathbf{e}_{tA}^T & i\sinh\xi_A\mathbf{Q}_A\mathbf{e}_{tA} \\ -i\sinh\xi_A\mathbf{e}_{tA}^T & \cosh\xi_A \end{bmatrix}, \tag{69}$$

and

$$\mathbf{\Lambda}_B(t) = \begin{bmatrix} \mathbf{Q}_B + (\cosh\xi_B - 1)\mathbf{Q}_B\mathbf{e}_{tB}\mathbf{e}_{tB}^T & i\sinh\xi_B\mathbf{Q}_B\mathbf{e}_{tB} \\ -i\sinh\xi_B\mathbf{e}_{tB}^T & \cosh\xi_B \end{bmatrix}. \tag{70}$$

By using (68) to combine (64) and (65), we obtain

$$\mathbf{u}_A = \boldsymbol{\Lambda}_A^T \boldsymbol{\Lambda}_B \mathbf{u}_B . \tag{71}$$

The orientation of the body frame *B* relative to *A* at time *t* is denoted by $\boldsymbol{\Lambda}_{B/A}$ where

$$\boldsymbol{\Lambda}_B = \boldsymbol{\Lambda}_A \boldsymbol{\Lambda}_{B/A} . \tag{72}$$

From this relation, we have

$$\boldsymbol{\Lambda}_{B/A} = \boldsymbol{\Lambda}_A^T \boldsymbol{\Lambda}_B . \tag{73}$$

Therefore, (71) becomes

$$\mathbf{u}_A = \boldsymbol{\Lambda}_{B/A} \mathbf{u}_B , \tag{74}$$

which can also be written as

$$\mathbf{u}_B = \boldsymbol{\Lambda}_{B/A}^T \mathbf{u}_A . \tag{75}$$

We notice that $\boldsymbol{\Lambda}_{B/A}$ is the relative Lorentz transformation from body frame *A* to body frame *B* measured by the inertial reference frame at time *t*. Therefore, all the relations are relative to this observer at time *t*.

Now we derive the relations relative to the observer attached to the body frame *A*. For this, we notice that the velocity of particle *B* relative to particle *A* measured by the observer in the body frame of *A* is

$$\left(\mathbf{u}_B\right)_A = \left(\mathbf{u}_{B/A}\right)_A = \boldsymbol{\Lambda}_A \mathbf{u}_B . \tag{76}$$

By substituting for $\mathbf{u}_B$ from (65), we obtain

$$\left(\mathbf{u}_B\right)_A = \left(\mathbf{u}_{B/A}\right)_A = \boldsymbol{\Lambda}_A \boldsymbol{\Lambda}_B^T \left(\mathbf{u}_B\right)_B . \tag{77}$$

We also have the obvious relation

$$\left(\mathbf{u}_A\right)_A = \left(\boldsymbol{\Lambda}_{B/A}\right)_A \left(\mathbf{u}_{B/A}\right)_A , \tag{78}$$

which can be written as

$$\left(\mathbf{u}_{B/A}\right)_A = \left(\mathbf{\Lambda}_{B/A}\right)_A^T \left(\mathbf{u}_A\right)_A. \tag{79}$$

By comparing (77) and (79) and using (68), we obtain the relation

$$\left(\mathbf{\Lambda}_{B/A}\right)_A^T = \mathbf{\Lambda}_A \mathbf{\Lambda}_B^T, \tag{80}$$

which can be written as

$$\left(\mathbf{\Lambda}_{B/A}\right)_A = \mathbf{\Lambda}_B \mathbf{\Lambda}_A^T. \tag{81}$$

By using the relation (73), we obtain

$$\left(\mathbf{\Lambda}_{B/A}\right)_A = \mathbf{\Lambda}_A \mathbf{\Lambda}_{B/A} \mathbf{\Lambda}_A^T. \tag{82}$$

Interestingly, this is the transformation for tensor $\mathbf{\Lambda}_{B/A}$ from the inertial reference frame to the body frame A. What we have is the development of the general theory of relative motion. This transformation clearly shows that the geometry governing the attached three-vector and three-tensors is generally non-Euclidean.

Explicitly from (77), we have

$$\left(\mathbf{u}_{B/A}\right)_A = \begin{bmatrix} \mathbf{Q}_A + (\cosh\xi_A - 1)\mathbf{Q}_A \mathbf{e}_{tA}\mathbf{e}_{tA}^T & i\sinh\xi_A \mathbf{Q}_A \mathbf{e}_{tA} \\ -i\sinh\xi_A \mathbf{e}_{tA}^T & \cosh\xi_A \end{bmatrix} \begin{bmatrix} c\sinh\xi_B \mathbf{e}_{tB} \\ ic\cosh\xi_B \end{bmatrix}. \tag{83}$$

From this, we obtain the relations

$$\left(\sinh\xi_{B/A}\mathbf{e}_{tB/A}\right)_A = -\sinh\xi_A\cosh\xi_B \mathbf{Q}_A\mathbf{e}_{tA} + \mathbf{Q}_A[\mathbf{1} + (\cosh\xi_A - 1)\mathbf{e}_{tA}\mathbf{e}_{tA}^T]\sinh\xi_B\mathbf{e}_{tB}, \tag{84}$$

$$\left(\cosh\xi_{B/A}\right)_A = \cosh\xi_A\cosh\xi_B - \sinh\xi_A\sinh\xi_B\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}. \tag{85}$$

These relations are the manifest of hyperbolic geometry governing the velocity addition law, which applies even for accelerating particles. This property holds for all attached four-vectors and four-tensors. Inertial observers relate components of attached four-vectors and four-tensors by Lorentz transformations. This is the origin of non-Euclidean geometry governing the three-vector and three-tensors. As we saw, the addition of three-vector velocities follows hyperbolic geometry. Thus,

$$\begin{aligned}&\left(\sinh\xi_{B/A}\mathbf{e}_{tB/A}\right)_A\\&\quad=\mathbf{Q}_A\left\{\left[(\cosh\xi_A-1)\sinh\xi_B\left(\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}\right)-\sinh\xi_A\cosh\xi_B\right]\mathbf{e}_{tA}+\sinh\xi_B\mathbf{e}_{tB}\right\},\end{aligned}\tag{86}$$

$$\left(\cosh\xi_{B/A}\right)_A=\cosh\xi_A\cosh\xi_B-\sinh\xi_A\sinh\xi_B\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}\,.\tag{87}$$

If the relative rapidity $\left(\xi_{B/A}\right)_A$ is nonzero, we can divide (86) by (87) to obtain

$$\begin{aligned}&\left(\tanh\xi_{B/A}\mathbf{e}_{tB/A}\right)_A\\&\quad=\mathbf{Q}_A\frac{\left[(\cosh\xi_A-1)\tanh\xi_B\left(\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}\right)-\sinh\xi_A\right]\mathbf{e}_{tA}+\tanh\xi_B\mathbf{e}_{tB}}{\cosh\xi_A-\sinh\xi_A\tanh\xi_B\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}}.\end{aligned}\tag{88}$$

We should notice that these relations hold despite the fact that the transformation

$$x'_\mu=\Lambda_{\mu\nu}x_\nu\,.\tag{89}$$

is only valid among non-accelerating systems.

5. Ether as the Fundamental Entity in Universe

The rotational theory of relativity shows that there is a relationship between space-time and massive particles. Every particle specifies its space-time body frame relative to the inertial reference frame. Now the important question concerns the very existence of these space-time systems. It is seen that we are compelled to postulate the existence of a universal entity, which is independent of any special space-time frame. It is in this universal entity, in which particles and their corresponding space-time body frame exist. However, we should notice the ambiguity in introducing the concept of ether. We used the term ether for the universal entity in which a massive observer specifies a space-time. How can we visualize ether, when the concepts of where and when cannot be applied? It should be noticed that this ether is different from the ether conceived by earlier proponents of the ether theory. They considered the ether some sort of matter filling the space. Ether was the term used to describe a medium for the propagation of electromagnetic waves. Ether was considered to be separate from matter and that particles, such as electrons, serve as source of vortices in this ether [8].

Therefore, it is noticed that our ether is something in which a particle specifies a four-dimensional orthogonal system with three real and one imaginary axis called space-time. In other words, the space-time body frames of particles are different representations of ether. However, it is well justified to call our fundamental universal entity the historical ether out of respect, which now is represented by four-dimensional space-time systems. The acceleration of a particle is the result of the instantaneous rotation of its space-time body frame in the ether.

We notice that clarifying the concepts of ether and space-time is an important step toward understanding the theory of relativity.

6. Conclusions

The rotational theory of relativity shows that every particle specifies a four-dimensional space-time body frame in a universal entity, here referred to as ether, and moves in the time direction with speed *c*. As a result, the relative motion of particles is actually the result of relative four-dimensional rotation of their corresponding Minkowki space-time body frames. This aspect of space-time shows that the pure Lorentz transformations represent the relative four-dimensional orientation among the space-time body frames of uniformly translating particles. Inertial observers in these frames relate the components of four-vectors and four-tensors by Lorentz transformation. This is the origin of non-Euclidean geometry governing the three-vector and three-tensor components. The hyperbolic geometry of the velocity addition law for uniform motions is the manifest for this fact.

We also realize that the orthogonal transformations are not restricted to relative uniform motion. The relative motion of accelerating particles is also represented by varying orthogonal transformations. This establishes the general vortical theory of motion. The acceleration of a particle is the result of the instantaneous rotation of its space-time body frame in the ether. This instantaneous rotation is specified by a four-dimensional angular velocity tensor in the inertial reference frame. The hyperbolic part of this rotation is in fact the accelerating motion. However, there is also a circular space rotation, which is observed in some phenomena, such as the spin precession of a stationary charged particle in a magnetic field.

Although Lorentz, Poincare, Einstein, Minkowski, Robb, Varičak, Lewis, Wilson, Borel, and others have developed important aspects of the theory of relativity, the fundamental meaning of space-time and its relation with the ether has not been appreciated. Through the developed rotational theory of relativity, one appreciates the work of those who questioned the fifth postulate in Euclidean geometry about parallel lines and considered the possibility of non-Euclidean geometry. It is stunning to see that the rules of motion are governed by non-Euclidean geometry, because all motion is a four-dimensional rotation of space-time of the body frame. Furthermore, we realize that the theory of motion is a model for hyperbolic geometry.

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