On The Frequency of Twin Primes

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Contents

Abstract:

The following document is an attempted (but failed) proof of the Twin Prime Conjecture by determining bounds for the number of twin prime pairs between a number and its square and then proving that the lower bound is always greater than 1 for sufficiently large numbers.

We will use proof by deduction to prove ∃ infinitely many twin primes. Define a twin prime pair H as a pair of integers $(k, k+2)$ s.t. $k, k+2 \in P$ $\{set\ of\ all\ primes\}$ & $\emptyset$ as the value $k+4, \forall H$.

$$\therefore\ k \not\equiv 0\ mod\ 2$$

$k \not\equiv 0\ mod\ 3$

$k \not\equiv 0\ mod\ 5$

$$\vdots$$

$$k \not\equiv 0\ mod\ t\ \in P < k$$

&

$$\therefore\ k+2 \not\equiv 0\ mod\ 2$$

$k+2 \not\equiv 0\ mod\ 3$

$k+2 \not\equiv 0\ mod\ 5$

$$\vdots$$

$$k+2 \not\equiv 0\ mod\ t\ \in P < k+2$$

∴ by definition of $\emptyset$ we can assert that:

$$\emptyset \not\equiv 2,4\ mod\ 2$$

$$\emptyset \not\equiv 2,4\ mod\ 3$$

$$\emptyset \not\equiv 2,4\ mod\ 5$$

$$\vdots$$

$$\emptyset \not\equiv 2,4\ mod\ t\ \in P < \emptyset$$

Define J' as a on the interval $= [a, b]$, $P_{m+1} \in P, m > 1$ & consider the interval:

$$Q_1 = [P_{m+1}, \quad {P_{m+1}}^2]$$

It follows from the sieve of Eratosthenes that if $R \in Z\ s.t.$

$$R \in Q_1\ \&$$

$$R \not\equiv 0 \; mod \; 2$$

$$R \not\equiv 0 \; mod \; 3$$

$$R \not\equiv 0 \; mod \; 5$$

$$\vdots$$

$$R \not\equiv 0 \; mod \; P_m$$

$\rightarrow R \in P$. Given this definition it follows that $\exists \; H \in Q_1$ iff $\exists \; (k, k+2)$ s.t. k, k+2 $\in P \rightarrow \exists \; H \in Q_1$ iff $\exists \; \emptyset \in Q_1$ since we can be sure that the largest possible k+2 $\in Q_1$ is ${P_{m+1}}^2 - 2$ which by definition of $\emptyset$ and primality of $(k, k+2)$ must satisfy:

$$\emptyset \not\equiv 2,4 \; mod \; 2$$

$$\emptyset \not\equiv 2,4 \; mod \; 3$$

$$\emptyset \not\equiv 2,4 \; mod \; 5$$

$$\vdots$$

$$\emptyset \not\equiv 2,4 \; mod \; P_m$$

Note $\exists \; {P_{m+1}}^2 - P_m + 1$ integers $\in Q_1$ and approximately:

$$\left(\frac{1}{2}\right)$$

$$\left(\frac{1}{3}\right)$$

$$\left(\frac{3}{5}\right)$$

$$\vdots$$

$$\left(\frac{P_m - 2}{P_m}\right)$$

Of all these integers, satisfy the corresponding incongruence that $\emptyset$ must satisfy.

$$\rightarrow \textit{The expected number of } \emptyset \in Q_1 = \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\ldots\left(\frac{P_m - 2}{P_m}\right)$$

$\rightarrow$ *The expected number of* $\emptyset \in Q_1 = E_\emptyset(P_{m+1})$

$$= \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{1}{2}\right)\prod_{i=2}^{m}\left[\frac{P_i - 2}{P_i}\right]$$

Lemma 1:

$$|Error|\ of\ E_\emptyset(P_{m+1}) \leq \pi(P_{m+1} - 1)$$

We begin by noting that given an interval:

$[a, b]$ *where* $a, b \in \mathbb{Z}$ *and* $a, b > 0$

That the total number of multiples of Q on the interval inclusive is contained in the interval:

$$\left[\frac{b-a+1}{Q} - \frac{Q-1}{Q}, \qquad \frac{b-a+1}{Q} + \frac{Q-1}{Q}\right]$$

We can write this statement more concisely as:

$$Error_{\left(\frac{1}{q}\right)} \in \left[-\frac{Q-1}{Q}, \frac{Q-1}{Q}\right]$$

The error generated for each term in the product $\left(\frac{1}{2}\right)\prod_{i=2}^{m}\left[\frac{P_i - 2}{P_i}\right]$ *is bounded correspondingly within*

the terms of $\pm\left(\frac{1}{2}\right) + \sum_{i=2}^{m} \pm\left[\frac{P_i - 1}{P_i}\right]$ *for example*:

$$Error_{\left(\frac{1}{2}\right)} \in \left[-\left(\frac{1}{2}\right), \qquad \left(\frac{1}{2}\right)\right]$$

$$Error_{\left(\frac{1}{3}\right)} \in \left[-\left(\frac{2}{3}\right), \qquad \left(\frac{2}{3}\right)\right]$$

$$\vdots$$

$$Error_{\left(\frac{1}{P_m}\right)} \in \left[-\left(\frac{P_m - 1}{P_m}\right), \qquad \left(\frac{P_m - 1}{P_m}\right)\right]$$

$$\therefore\ Error\ of\ E_\emptyset(P_{m+1}) \in \left[-\left(\frac{1}{2}\right) - \left(\frac{1}{3}\right) \ldots - \left(\frac{P_m - 1}{P_m}\right), \qquad \left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) \ldots + \left(\frac{P_m - 1}{P_m}\right)\right]$$

$$\rightarrow \; Error\; of\; E_{\emptyset}(P_{m+1}) \in \left[-\left(\frac{1}{2}\right) - \sum_{i=2}^{m}\left[\frac{P_i - 1}{P_i}\right], \quad \left(\frac{1}{2}\right) + \sum_{i=2}^{m}\left[\frac{P_i - 1}{P_i}\right]\right]$$

$$Note\; that{:}\; \left(\frac{P_m - 1}{P_m}\right) < 1 \;\forall P_m \in P$$

$$\therefore \; Error\; of\; E_{\emptyset}(P_{m+1}) \in \left[-1 - \sum_{i=2}^{m}[1], \quad 1 + \sum_{i=2}^{m}[1]\right] = [-m, \quad m]$$

$$= [-\pi(P_{m+1} - 1), \pi(P_{m+1} - 1)]$$

$$\rightarrow |Error|\; of\; E_{\emptyset}(P_{m+1}) \leq \pi(P_{m+1} - 1)$$

$$\&\; by\; corollary\; |Error|\; of\; E_{\emptyset}(P_{m+1}) \leq \pi(P_{m+1})$$

End Lemma:

$$\therefore \; The\; expected\; number\; of\; \emptyset \in Q_1 \in$$

$$R_1 = \left[\left(P_{m+1}{}^2 - P_{m+1} + 1\right)\left(\frac{1}{2}\right)\prod_{i=2}^{\pi(P_{m+1})-1}\left[\frac{P_i - 2}{P_i}\right] - \pi(P_{m+1}), \right.$$
$$\left. \left(P_{m+1}{}^2 - P_{m+1} + 1\right)\left(\frac{1}{2}\right)\prod_{i=2}^{\pi(P_{m+1})-1}\left[\frac{P_i - 2}{P_i}\right] + \pi(P_{m+1})\right]$$

Lemma 2:

$$\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\left(\frac{5}{7}\right)\ldots\left(\frac{P_n - 2}{P_n}\right) = \prod_{i=2}^{n}\left[\frac{P_i - 2}{P_i}\right] \geq \left(\frac{1}{3}\right)\prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right] = \left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\left(\frac{5}{7}\right)\ldots\left(\frac{P_{n-1}}{P_n}\right) = \left(\frac{1}{P_n}\right)$$

$$P_n > 2 \in \{odd\; numbers\} \rightarrow P_n - P_{n-1} \geq 2 \;\&\; P_n - P_{n-1} \in \{even\; numbers\} \;\forall n > 1$$

$$if\; P_n = 11 \rightarrow P_{n-1} = 7 \rightarrow \frac{P_{n-1}}{P_n} = \left(\frac{7}{11}\right) \leq \frac{9}{11} = \frac{P_n - 2}{P_n} \rightarrow \frac{P_{n-1}}{P_n} \leq \frac{P_n - 2}{P_n}.$$

$$\therefore if\; P_n - P_{n-1} = 2 \;\forall P_n \neq 11, \prod_{i=2}^{n}\left[\frac{P_i - 2}{P_i}\right] \geq \prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right] = \left(\frac{1}{P_n}\right)$$

$$if\; P_n - P_{n-1} \neq 2 \;\forall P_n, \prod_{i=2}^{n}\left[\frac{P_i - 2}{P_i}\right] \geq \prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right] = \left(\frac{1}{P_n}\right)$$

$$\therefore \prod_{i=2}^{n}\left[\frac{P_i - 2}{P_i}\right] \geq \prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right] = \left(\frac{1}{P_n}\right)$$

End Lemma:

$$\therefore\ R_1' = \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{2}\right) \prod_{i=2}^{\pi(P_{m+1})-1} \left[\frac{P_i - 2}{P_i}\right] - \pi(P_{m+1}) \geq R_2$$

$$= \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \prod_{i=3}^{\pi(P_{m+1})-1} \left[\frac{P_{i-1}}{P_i}\right] - \pi(P_{m+1})$$

$$= \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_m}\right) - \pi(P_{m+1})$$

Now we make an additional note that $P_{m+1} > P_m \because P$ is an ordered infinitely large set due to the work of Euclid[1].

$$\therefore R_2 = \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_m}\right) \geq R_3 = \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_{m+1}}\right) - \pi(P_{m+1})$$

Note that

$$\pi(x) < 1.25506 \frac{x}{\log(x)} \ \forall\, x \in \mathbb{R}, x \geq 17\ (Rosser, Schoenfield\ 2)$$

$$\therefore\ R_3 = \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_{m+1}}\right) - \pi(P_{m+1}) \geq R_4$$

$$= \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_{m+1}}\right) - 1.25506 \frac{P_{m+1}}{\log(P_{m+1})} \ \ \forall\, P_{m+1} \geq 17$$

Lemma 3:

$$\forall\ P_{m+1} \geq 17\ R_4 = \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_{m+1}}\right) - 1.25506 \frac{P_{m+1}}{\log(P_{m+1})} \geq 1, \forall\, P_{m+1} \geq 33912637.$$

$$\left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_{m+1}}\right) - 1.25506 \frac{P_{m+1}}{\log(P_{m+1})} \geq 1\ \forall\, P_{m+1} = 33912637$$

$$\frac{d}{dP_{m+1}}\left[\left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_{m+1}}\right) - 1.25506 \frac{P_{m+1}}{\log(P_{m+1})}\right] \geq 0\ \forall\, P_{m+1} \geq 33912637$$

$$\rightarrow \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{6}\right)\left(\frac{1}{P_{m+1}}\right) - 1.25506 \frac{P_{m+1}}{\log(P_{m+1})} \geq 1\ \forall\, P_{m+1} = 33912637$$

End Lemma:

Now note:

$$R_4 \geq 1\ \forall\, P_{m+1} \geq 33912637$$

$$R_3 \geq R_4 \geq 1 \; \forall \, P_{m+1} \geq 33912637$$

$$R_2 \geq R_3 \geq R_4 \geq 1 \; \forall \, P_{m+1} \geq 33912637$$

$$R_1' \geq R_2 \geq R_3 \geq R_4 \geq 1 \; \forall \, P_{m+1} \geq 33912637 \; \rightarrow \; R_1' \geq 1 \; \forall \, P_{m+1} \geq 33912637$$

$$\rightarrow \; The \; expected \; number \; of \; \emptyset \in Q_1 \geq 1 \; \forall \; P_{m+1} \geq 33912637$$

Conclusion:

$$\because \exists \; infinite \; Q \in P \; of \; arbitrarily \; large \; size \rightarrow \exists \; infinite \;\; P_{m+1} \geq 33912637$$

$$\therefore \; \exists \; infinitely \; many \; \emptyset$$

$$\therefore \; \exists \; infinitely \; many \; H$$

$$\therefore \; \exists \; infinitely \; many \; twin \; prime \; pairs$$

$$Q.E.D.$$

Further Details:

The method of error determination used in this proof was very loose and overall much greater than what is actually observed. Recall the following:

$$The \; error \; generated \; for \; each \; term \; in \; the \; product \; \left(\frac{1}{2}\right)\prod_{i=2}^{m}\left[\frac{P_i - 2}{P_i}\right] \; is \; bounded \; correspondingly \; within$$

$$the \; terms \; of \;\; \pm\left(\frac{1}{2}\right)\sum_{i=2}^{m}\left[\frac{P_i - 1}{P_i}\right] \; for \; example:$$

$$Error_{\left(\frac{1}{2}\right)} \in \left[-\left(\frac{1}{2}\right), \left(\frac{1}{2}\right)\right]$$

$$Error_{\left(\frac{1}{3}\right)} \in \left[-\left(\frac{2}{3}\right), \left(\frac{2}{3}\right)\right]$$

$$\vdots$$

$$Error_{\left(\frac{1}{P_m}\right)} \in \left[-\left(\frac{P_m - 1}{P_m}\right), \left(\frac{P_m - 1}{P_m}\right)\right]$$

Note that each successive term of Error is multiplied by the preceding term before its error is added and therefore:

$$|Net\ Error| \leq \left(\frac{1 * 2 * 4 \ldots p_m - 1}{2 * 3 * 5 \ldots p_m}\right) + \left(\frac{2 * 4 \ldots p_m - 1}{3 * 5 \ldots p_m}\right) \ldots + \left(\frac{p_m - 1}{p_m}\right)$$
$$= \left(\prod_{q=1}^{m} \frac{p_q - 1}{p_q}\right) + \left(\prod_{q=2}^{m} \frac{p_q - 1}{p_q}\right) \ldots + \left(\prod_{q=m}^{m} \frac{p_q - 1}{p_q}\right) = \sum_{i=1}^{m} \left[\prod_{q=i}^{m} \frac{p_q - 1}{p_q}\right]$$

Note that:

$$\left(\frac{p_m - 1}{p_m}\right) \geq \left(\prod_{q=r}^{m} \frac{p_q - 1}{p_q}\right) \forall r \in \mathbb{Z}, m \geq r > 0$$

If one excludes the final term then the same argument can be made for $\left(\frac{p_{m-1} - 1}{p_{m-1}}\right)$ and by induction for any arbitrary $\left(\frac{p_{m-i} - 1}{p_{m-i}}\right)$ granted enough terms from the end of the error sum are removed. This can be of value to bounding the prime counting function more tightly in exchange for more computation.

Works Cited:

1. "Euclid's Proof of the Infinitude of Primes (c. 300 BC)." *Euclid's Proof of the Infinitude of Primes (c. 300 BC)*. University of Tennessee at Martin, n.d. Web. 06 Feb. 2013. <http://primes.utm.edu/notes/proofs/infinite/euclids.html>.
2. Rosser, J. Barkley; Schoenfeld, Lowell (1962). "Approximate formulas for some functions of prime numbers.". *Illinois J. Math.* **6**: 64–94.. http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&handle=euclid.ijm/1255631807.