Solutions of the problems "Goldbach-Euler" and "infinitely many twin primes"

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In math there are some classic problems of number theory which have not been solved yet.Two of these problems are as below:

1. «Pair of twin primes » (where difference is equal to 2 such as pairs of twin prime numbers (3;5); (5;7); (11;13); …) are infinite.
2. «It is possible to show any even number, starting from 4, as a sum of two prime numbers »

2nd problem is known as «Goldbach-Euler problem».

In order to solve these problems we have compiled a table determining if the numbers like 6m-1 and 6m+1 $(m \in N)$ are prime or composite. (Table 3)

We have solved these problems as below by using some facts and conclusions besides Table 3.

First, let's start with 2nd problem. First problem will be solved along the proof process, too. Since 4=2+2; 6=3+3; 8=3+5, then we can solve the problem for the numbers greater than 8. It is clear that, we should look at the natural numbers in the form of 12n-2; 12n; 12n+2; 12n+4; 12n+6 and 12n+8 where $n \in N$.

According to the divisibility rule for 6, natural numbers are divided into 6 groups as following:

1) 6m; 2) 6m+1; 3) 6m+2; 4) 6m+3; 5) 6m+4; 6) 6m+5 $m \in N$

Since, these groups, which get values greater than 3 are composite numbers, except second and sixth, so any prime number which is greater than 3 should be in the form of 6m+1 or 6m+5 (6m-1).

Conclusion 1. Any prime number greater than 3 can be shown as 6m+1 or 6m-1.

Let's write all groups of even natural numbers greater than 8 as following, which we showed above as 12n-2; 12n; 12n+2;12n+4; 12n+6;12n+8 :

$$\left.\begin{array}{ll} 12n-2=(6k_1-1)+(6k_2-1); & k_1 \in N; k_2 \in N \;\; k_1+k_2=2n \\ 12n=(6k_1-1)+(6k_2+1) & \text{«\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_»} \\ 12n+2=(6k_1+1)+(6k_2+1) & \text{«\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_»} \\ 12n+4=(6k_1-1)+(6k_2-1) & k_1 \in N; k_2 \in N \;\; k_1+k_2=2n+1 \\ 12n+6=(6k_1-1)+(6k_2+1) & \text{«\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_»} \\ 12n+8=(6k_1+1)+(6k_2+1) & \text{«\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_»} \end{array}\right\} \quad (1)$$

Now let's figure out the conditions of being prime of the numbers which are in the form of 6m-1; 6m+1, where $m \in N$. In order to do so, let's first look for the conditions of being composite of the numbers 6m-1; 6m+1 ($m \in N$) .

It is obvious that, there cannot be prime multiplier of 2 and 3 in the composite numbers of 6m-1 and 6m+1. So, if 6m-1 and 6m+1 are composite numbers, then each prime multiplier of these numbers are not less than 5.

Therefore,

$k_1; k_2 \in N$

$$\left.\begin{array}{l} 6m+1=(6k_1-1)(6k_2-1) \cup 6m+1=(6k_1+1)(6k_2+1) \\ 6m-1=(6k_1-1)(6k_2+1) \cup 6m-1=(6k_1+1)(6k_2-1) \end{array}\right\} \quad (2)$$

It should be noted that if there are more than 2 multipliers in the $6m \pm 1$, then number of each of $6k_1$-1 $6k_1$+1; $6k_2$-1 $6k_2$+1 should be odd. We consider number of multipliers as two, while it doesn't affect the whole calculation (it is as (6p-1)(6q-1)(6t-1)=6k-1).

Let's write 4 possible cases of (2) more detailed:

$6m_1+1=(6k_1-1)(6k_2-1)=6(6k_1k_2-(k_1+k_2))+1$; $m_1=6k_1k_2-(k_1+k_2)$

$6m_2-1=(6k_1+1)(6k_2-1)=6(6k_1k_2-(k_1-k_2))-1$; $m_2=6k_1k_2-(k_1-k_2)$

$6m_3-1=(6k_1-1)(6k_2+1)=6(6k_1k_2-(k_1-k_2))-1$; $m_3=6k_1k_2-(k_2-k_1)$

$6m_4+1=(6k_1+1)(6k_2+1)=6(6k_1k_2+(k_1+k_2))+1$; $m_4=6k_1k_2+(k_1+k_2)$

So, we got following result :

in case of $k_1;k_2 \in N$

$$\left.\begin{array}{l} \text{a) if } m_1=6k_1k_2-(k_1+k_2)\text{, then } 6m_1+1 \text{ is a composite number} \\ \text{b) if } m_2=6k_1k_2-(k_1-k_2)\text{, then } 6m_2-1 \text{ is a composite number} \\ \text{c) if } m_3=6k_1k_2-(k_2-k_1)\text{, then } 6m_3-1 \text{ is a composite number} \\ \text{d) if } m_4=6k_1k_2+(k_1+k_2)\text{, then } 6m_4+1 \text{ is a composite number} \end{array}\right\} \quad (3)$$

It is seen from the expressions of m_1, m_2, m_3 and m_4 (3), that m_1 and m_4 is symmetric with respect to k_1 and k_2. m_2 and m_3 is also symmetric with respect to k_1 and k_2.

If
$$\left.\begin{array}{l} m_1=\varphi(k_1;k_2)=6k_1k_2-(k_1+k_2) \\ m_2=f(k_1;k_2)=6k_1k_2-(k_1-k_2) \end{array}\right\}$$

then, $m_3=f(-k_1;-k_2); m_4=\varphi(-k_1;-k_2)$

If we take these into consideration, we should look for the solution within the condition of $m_1<m_2\leq m_3<m_4$ assuming that $k_1 \geq k_2$.

Then we will get following conclusions from the equalities of (3) which we showed above.

Conclusion 2 : Where $k_1,k_2 \in N$

$$\left.\begin{array}{l} \text{If } m=\begin{cases} m_1=6k_1k_2-(k_1+k_2) \\ m_4=6k_1k_2+(k_1+k_2) \end{cases} \text{ then } 6m+1 \\ \text{If } m=\begin{cases} m_2=6k_1k_2-(k_1-k_2) \\ m_3=6k_1k_2-(k_2-k_1) \end{cases} \text{ then } 6m-1 \end{array}\right\} \quad (4)$$

is a composite number.

Conclusion 3 :

$$\left.\begin{array}{l} \text{If } m\in(m_1\cup m_4)\setminus((m_1\cup m_4)\cap(m_2\cup m_3)) \text{ then } 6m-1 \\ \text{If } m\in(m_2\cup m_3)\setminus((m_1\cup m_4)\cap(m_2\cup m_3)) \text{ then } 6m+1 \end{array}\right\} \quad (5)$$

is a prime number.

Conclusion 4 : If $m\notin(m_1\cup m_2\cup m_3\cup m_4)$, then 6m-1 and 6m+1 are twin primes.

Last conclusion is also seen from the following table:

$m \in$	$m_1 \cup m_4 / \varphi$	$m_2 \cup m_3 / \varphi$	$\varphi = (m_1 \cup m_4) \cap (m_2 \cup m_3)$	$m \notin (m_1 \cup m_2 \cup m_3 \cup m_4)$
6m-1	P	C	C	P
6m+1	C	P	C	P

In the table, P is for prime numbers; C is for composite numbers.

And now let's figure out ways to find m_1, m_2, m_3 and m_4. It is clear that,

$$(k_1;k_2) = \begin{cases} (1;1),\ (2;1),\ (3;1)\ ,...,\ (a;1) a \in N \\ (2;2),\ (3;2),\ (4;2),\ ...,\ (a+1;2) a \in N \\ (3;3),\ (4;3),\ (5;3),\ ...,\ (a+2;3) a \in N \\ ---------------------- \\ ---------------------- \\ ---------------------- \\ (a;a),\ (a+1;a),\ (a+2;a),\ ...,\ (2a-1;a) a \in N \end{cases} \quad (6)$$

In (6), there are a rows and a columns. So, we can write (6) more generally as following:

$$(k_1;k_2)=\{(a;a); (a+1;a); \ldots, (2a-1;a)\} \quad (7) \qquad a \in N$$

Now let's consider the values of $(k_1;k_2)$ in (7) in the expressions of m_1, m_2, m_3 and m_4 in (3).

$k_1=k_2=a \Rightarrow m_1=6a^2-2a;\ m_2=6a^2;\ m_3=6a^2;\ m_4=6a^2+2a$

$(k_1;k_2)=(a+1;a) \Rightarrow m_1=6a^2+4a-1;\ m_2=6a^2+6a-1;\ m_3=6a^2+6a+1;\ m_4=6a^2+8a+1$

$(k_1;k_2)=(a+2;a) \Rightarrow m_1=6a^2+10a-2;\ m_2=6a^2+12a-2;\ m_3=6a^2+12a+2;\ m_4=6a^2+14a+2$

Let's write the expressions of m_1, m_2, m_3 and m_4 in the following table:

m / $(k_1;k_2)$	m_1	m_2	m_3	m_4
(a;a)	$6a^2-2a$	$6a^2$	$6a^2$	$6a^2+2a$
(a+1;a)	$6a^2+4a-1$	$6a^2+6a-1$	$6a^2+6a+1$	$6a^2+8a+1$
(a+2;a)	$6a^2+10a-2$	$6a^2+12a-2$	$6a^2+12a+2$	$6a^2+14a+2$
————	————	————	————	————
————	————	————	————	————
————	————	————	————	————

Table 1

We can easily show that in the Table 1, m_1 and m_2 are arithmetic series where the difference of columns is 6a-1; m_3 and m_4 are arithmetic series where the difference of columns is 6a+1.

So, where a, $n \in N$

$$\left.\begin{aligned} m_1&=(6a-1)(n+a-1)-a \\ m_2&=(6a-1)(n+a-1)+a \\ m_3&=(6a+1)(n+a-1)-a \\ m_4&=(6a+1)(n+a-1)+a \end{aligned}\right\} \quad (8)$$

If we consider (8) in Table 1, then we will get Table 2.

	m_1 (6a-1)(n+a-1)-a	m_2 (6a-1)(n+a-1)+a	m_3 (6a+1)(n+a-1)-a	m_4 (6a+1)(n+a-1)+a
a=1	5n-1	5n+1	7n-1	7n+1
a=2	11(n+1)-2	11(n+1)+2	13(n+1)-2	13(n+1)+2
a=3	17(n+2)-3	17(n+2)+3	19(n+2)-3	19(n+2)+3
a=4	23(n+3)-4	23(n+3)+4	25(n+3)-4	25(n+3)+4
———	———————	———————	———————	———————
———	———————	———————	———————	———————
———	———————	———————	———————	———————

Table 2

In order to show values of all rows of Table 2 as a table, let's convert Table 2 to Table 3:

	$n\ge 1$ $n\in N$				$n\ge 2$ $n\in N$				$n\ge 3$ $n\in N$				$n\ge 4$ $n\in N$		$n\in N$ $n\ge 5$				....
	$\overset{m_1}{5n-1}$	$\overset{m_2}{5n+1}$	$\overset{m_3}{7n-1}$	$\overset{m_4}{7n+1}$	$\overset{m_1}{11n-2}$	$\overset{m_2}{11n+2}$	$\overset{m_3}{13n-2}$	$\overset{m_4}{13n+2}$	$\overset{m_1}{17n-3}$	$\overset{m_2}{17n+3}$	$\overset{m_3}{19n-3}$	$\overset{m_4}{19n+3}$	$\overset{m_1}{23n-4}$	$\overset{m_2}{23n+4}$	$\overset{m_1}{29n-5}$	$\overset{m_2}{29n+5}$	$\overset{m_3}{31n-5}$	$\overset{m_4}{31n+5}$	
1	4	6	6	8															....
2	9	11	13	15	(20)	(24)	(24)	28											....
3	14	16	(20)	22	(31)	35	37	(41)	(48)	(54)	(54)	60							....
4	19	21	27	29	42	46	(50)	(54)	65	(71)	73	(79)	(88)	96					....
5	(24)	26	(34)	(36)	53	(57)	63	67	82	(88)	(92)	98	(111)	(119)	140	(150)	(150)	(160)	....
6	29	(31)	(41)	43	64	68	76	80	99	105	(111)	117							
7	(34)	(36)	(48)	(50)	75	(79)	(89)	93	(116)										
8	39	(41)	55	(57)	(86)	90	102	(106)											
9	44	46	62	64	(97)	101													
10	49	51	(69)	(71)	108														
11	(54)	56	76	78															
12	59	61	83	85															
13	64	66	90	(92)															
14	(69)	(71)	(97)	99															
15	74	76	(104)	(106)															
16	(79)	81																	
17	84	(86)																	
18	(89)	91																	
19	94	96																	
20	99	101																	
21	(104)																		

Table 3

It is obvious from the table that,

if $t\in N$, then we can write:

$$\left.\begin{aligned} m_1&=(6t-1)n-t\\ m_2&=(6t-1)n+t\\ m_3&=(6t+1)n-t\\ m_4&=(6t+1)n+t \end{aligned}\right\} \quad (9)$$

It should be noted that in the table, if we multiply the numbers without parentheses in $m_1 \cup m_4$ with 6 and subtract 1, then we will get prime numbers; and similarly if we multiply the numbers without parentheses in $m_2 \cup m_3$ with 6 and add 1, then we will get prime numbers. In the table, if we multiply the numbers with 6 which are not in the interval [1;119] and subtract 1 or if we multiply with 6 and add 1, then we will get twin primes.

In the Table 3, the numbers in parentheses which are in the interval [1;119] are the numbers included in $(m_1 \cup m_4) \cap (m_2 \cup m_3)$ and if we multiply these numbers with 6 and add 1 or subtract 1 we get composite numbers.

Let us give some examples:

$29 \in m_1$ 6·29-1=173 is a prime number.

$102 \in m_3$ 6·102+1=613 is a prime number.

$30 \notin (m_1 \cup m_2 \cup m_3 \cup m_4)$ 6·30-1=179 is a prime number.

6·30+1=181 is a prime number.

179 and 181 are twin primes.

Now, let's show how to find the numbers in parentheses. These numbers should belong to the intersection of $m_1 \cap m_2; m_1 \cap m_3; m_2 \cap m_4; m_3 \cap m_4$. In order to find the same numbers in the columns of 5n-1 and 7n-1, we should find general solution of the indefinite equation of 5n-1=7k-1.

$$5n - 1 = 7k - 1 \Rightarrow 5n = 7k \Rightarrow \left.\begin{array}{l} n = 7t \\ k = 5t \end{array}\right\} t \in N$$

So, these numbers are in the form of 35t-1.

Therefore, $\begin{array}{ll} t = 1 \Rightarrow 34 & t = 3 \Rightarrow 104 \\ t = 2 \Rightarrow 69 & t = 4 \Rightarrow 139 \end{array}$

Or for the columns of 5n-1 and 11n+2 we can write 5n-1=11k+2⇒5n-11k=3

It is clear that, since GCF(5;11)=1**,** then there are numbers such as $\alpha; \beta \in z$ where $5\alpha - 11\beta = 1$.

Let's find α and β by using Euclid algorithm.

$$\begin{array}{r|l} 11 & 5 \\ -10 & 2 \\ \hline 5 & 1 \\ -5 & 5 \\ \hline 0 & \end{array}$$

1=11-5·2

Or 5·(-2)-11·(-1)=1

$\alpha = -2; \ \beta = -1$

$$\left.\begin{array}{l} 5n - 11k = 3 \\ 5 \cdot (-2) - 11 \cdot (-1) = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5n - 11k = 3 \\ 5 \cdot (-6) - 11 \cdot (-3) = 3 \end{array}\right\} \Rightarrow n = 11t - 6 \qquad k = 5t - 3$$

5n-1=5(11t-6)-1=55t-31

Using this method we can get following formulas.

m_1 5n-1	m_2 5n+1	m_3 7n-1	m_4 7n+1	m_1 11n-2	m_2 11n+2	m_3 13n-2	m_4 13n+2	……..
35t-1	35t+1							……..
55t+24	55t+31	77t+20	77t-20					……..
65t+24	65t+41	91t+41	91t-41	143t-2	143t+2	221t-51	221t+54	……..
85t+54	85t+31	119t+48	119t+71	187t+20	187t-20	247t-54	247t+54	……..
95t+54	95t+41	133t+41	133t+92	209t+130	209t+79	299t-119	299t+119	……..
115t+4	115t+111	161t+111	161t+50	253t+119				
145t+34	145t+111	203t+111						
155t+119	155t+136							
185t+179	185t+6							
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________	________	________	________	________	________	________	________	

Table 4

Where p and q are prime, if the expression of mt+γ in Table 4 is get from pn+α =qk+β, then m=pq and γ=pt_1+α=qt_2+β.

It is seen from the Table 3 that,

in the row of $\overset{m_1}{5n-1}$; $\overset{m_2}{5n+1}$; $\overset{m_3}{7n-1}$; $\overset{m_4}{7n+1}$; $\overset{m_1}{11n-2}$; $\overset{m_2}{11n+2}$; $\overset{m_3}{13n-2}$; $\overset{m_4}{13n+2}$; …;

m_1-s are arithmetic series as a_1=5n-1; d=6n-1;

m_2-s are arithmetic series as a_1=5n+1; d=6n+1;

m_3-s are arithmetic series as a_1=7n-1; d=6n-1; and

m_4-s are arithmetic series as a_1=7n+1; d=6n+1.

So, we know how to continue to fill the table downwards and rightwards.

Now, let's form sums as $k_1+k_2=2n$; $k_1+k_2=2n+1$ $(n \in N)$:

$k_1+k_2=2n$		$k_1+k_2=2n+1$	
1+(2n-1)=2n		1+2n=2n+1	
2+(2n-2)=2n		2+(2n-1)=2n+1	
3+(2n-3)=2n	(10)	3+(2n-2)=2n+1	(11)
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---------------		---------------	
---------------		---------------	
n+n=2n		n+(n+1)=2n+1	

In (10) and (11) 1st addends indicate $k_1$and 2nd addends indicate k_2.

Now let's show that in (10) and (11) there are at least 9 twins which those numbers do not belong to the numbers in parentheses in Table 3, or they do not exist in Table 3.

In other words, there are at least 18 twins of k_1, k_2 which belong to the set of

$$((m_1 \cup m_4) \setminus ((m_1 \cup m_4) \cap (m_2 \cup m_3)) \cup ((m_2 \cup m_3) \setminus ((m_1 \cup m_4) \cap (m_2 \cup m_3))$$

or none of them belongs to $(m_1 \cup m_2 \cup m_3 \cup m_4)$.

We can see from Table 3 that, there are 18 numbers in parentheses in the interval of [1;100] which are

20, 24, 31, 34, 36, 41, 48, 50, 54, 57, 69, 71, 79, 86, 88, 89, 92, 97 (12).

It is obvious that in the rest of (12) until 2n, number of them is not greater than n-9, there are at least 18 twins of k_1, k_2 which none of them is in parentheses, or in the table, or only one of them is not in the table.

Now, let's prove that, number of numbers in the series of (12) until 2n is not greater than n-9. We will use the following fact in order to prove our conclusion.

Fact: 1) If we write the numbers from 1 to 2n in an ascending order with a difference of 2 between each other, number of them will be n.

A={1, 3, 5, 7, …., 2n-1} n(A)=n

B={2, 4, 6, 8, …, 2n} n(B)=n

2) In the series of natural numbers in an ascending order (until 2n):

1 – if the number of steps is a_1- then it does not affect the whole amount;

2 – if the number of steps is a_2- then it decreases whole amount by a_2;

3 – if the number of steps is a_3- then it decreases whole amount by $2a_3$;

4 – if the number of steps is a_4- then it decreases whole amount by $3a_4$.

Therefore, similarly, steps of a_1, a_2, a_3, a_4, a_5, …, a_k decrease the whole amount by $a_2+2a_3+3a_4+4a_5+…+(k-1)a_k$.

So, number of them in the interval of [20;2n] will be $(2n-19)-(a_2+2a_3+3a_4+…+(k-1)a_k)$.

Now, let's evaluate the series of numbers(which are in parentheses) until 2n in the set of $(m_1 \cup m_4) \cap (m_2 \cup m_3)$, (number of terms until 2n in the series of 20, 24, 31, 34, 36, 41, 48, 50, 54, 57, 69, 71, 79, 86, 88, 89, 92, 97, … (13) which is in an ascending order). This problem will be solved by evaluating the number of terms of the numbers in an ascending order which are found as a result of putting natural numbers to t in Table 4.

Note: Since the numbers in (13) are taken from Table 4, value of each expression which belongs to the table should not exceed 2n.

Let's look at the difference of numbers of two expressions in Table 4, where coefficients of t are not mutually prime numbers . In those two expressions if the common factor of coefficients of t is 5, then difference between them is

$5n-1-(5k-1)=5m \in \{5, 10, \dots\}$

$5n-1-(5k+1)=5m-2 \in \{3, 8, 13, \dots\}$

$5n+1-(5k-1)=5m+2 \in \{2, 7, 12, \dots\}$

$5n+1-(5k+1)=5m \in \{5, 10, \dots\}$

For this cases the difference is at least 2, 3, 5.

In Table 4, if the common factor of the coefficients of t is 7 in two expressions, then difference between them is

$7n-1-(7k-1)=7m \in \{7, 14, \dots\}$

$7n-1-(7k+1)=7m-2 \in \{5, 12, \dots\}$

$7n+1-(7k-1)=7m+2 \in \{2, 9, \dots\}$

$7n+1-(7k+1)=7m \in \{7, 14, \dots\}$

In this case the difference is at least 2, 5, 7.

In Table 4, if the common factor of the coefficients of t is 11 in two expressions, then difference between them is

$11n-2-(11k-2)=11m \in \{11, 22, \dots\}$

$11n-2-(11k+2)=11m-4 \in \{7, 18, \dots\}$

$11n+2-(11k-2)=11m+4 \in \{4, 17, \dots\}$

$11n+2-(11k+2)=11m \in \{11, 22, \dots\}$.

In Table 4, it is clear that the difference is not less than 4, when the common factor of the coefficients of t is greater than11 in two expressions. So, in Table 4, if the common factor of the coefficients of t is 5 in two expressions, then difference of proper numbers is at least 2, 3, 5;

if the common factor is 7, then difference is at least 2, 5, 7;

if the common factor is 11, then difference is at least 4, 7, 11;

if the common factor is 13, then difference is at least 4, 9, 13;

if the common factor is 17, then difference is at least 6, 11, 17.

It is obvious that, if we continue the process, then difference will increase. So, if there is a common factor between two expressions in Table 4 and : if the common factor is 5 and 7, then the difference of proper numbers will be at least 2; if the common factor is greater than 7, then the difference will be at least 4. And now, let's evaluate the difference of proper numbers in Table 4, where coefficients of t are two mutually prime numbers in these expressions.

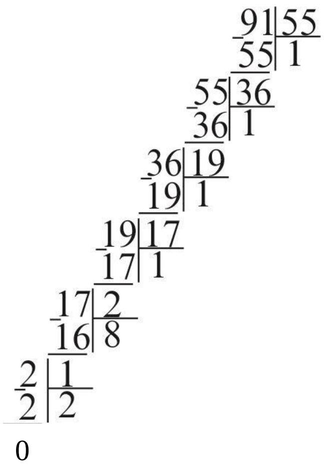
First, let's look at the problem on specific sample and then generalize the conclusion.

Let's look at 55t+24 and 91t+41 where coefficients of t are mutually prime. Let's indicate the difference of these expressions with m. $m \in z$

$$(55n+24)-(91k+41)=m \Rightarrow 55n-91k=m+17 \quad (14)$$

Since CGF(55;91) is 1, then there are numbers such as $\alpha;\beta \in N$, where $55\alpha-91\beta=1$

Now, let's find those α and β.



So,

1=17-8·2

2=19-17·1

17=36-19·1

19=55-36·1

36=91-55·1

Here we can write, 1=17-8·(19-17)=9·17-8·19=9·(36-19)-8·19=36·9-17·19=36·9-17·(55-36)=36·26-17·55=(91-55)·26-17·55=91·26-55·43

91·26-55·43=1$\Rightarrow$55·(-43)-91·(-26)=1.

We got general solution of

$$\left.\begin{aligned} 55\alpha - 91\beta &= 1 \\ 55\cdot(-43) - 91\cdot(-26) &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= 91t - 43 \\ \beta &= 55t - 26 \end{aligned}$$

If we take t=1 then, we will get $\alpha = 48;\ \beta = 29$.

So, $\left.\begin{aligned} 55n - 91k &= m + 17 \\ 55\cdot 48 - 91\cdot 29 &= 1 \end{aligned}\right\}$ (15)

Here we get $\left.\begin{aligned} 55n - 91k &= m + 17 \\ 55\cdot 48(m+17) - 91\cdot 29(m+17) &= m + 17 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} n &= 91t + 48(m+17) \\ k &= 55t + 29(m+17) \end{aligned}\right\}$ (16) $(t \in z)$

(55·(91t+48(m+17))+24)-(91·(55t+29(m+17))+41)=m

(5005t+2640(m+17)+24)-(5005t+2639(m+17)+41)=m (17).

In (17) by putting numbers of -4; -3; -2; -1; 0; 1; 2; 3 and 4 for m, we can write proper differences as following:

m=-4⇒(5005t+34344)-(5005t+34348)=-4

m=-3⇒(5005t+36984)-(5005t+36987)=-3

m=-2⇒(5005t+39624)-(5005t+39626)=-2

m=-1⇒(5005t+42264)-(5005t+42265)=-1

m=0⇒(5005t+44904)-(5005t+44904)=0

m=1⇒(5005t+47544)-(5005t+47543)=1

m=2⇒(5005t+50184)-(5005t+50182)=2

m=3⇒(5005t+52824)-(5005t+52821)=3

m=4⇒(5005t+55464)-(5005t+55460)=4

If we simplify the numbers by eliminating the multiples of 5005 , then we will get

(5005t+4314)-(5005t+4318)=-4

(5005t+1949)-(5005t+1952)=-3

(5005t+4589)-(5005t+4591)=-2

(5005t+2224)-(5005t+2225)=-1

(5005t+2499)-(5005t+2498)=1

(5005t+134)-(5005t+132)=2

(5005t+2774)-(5005t+2771)=3

(5005t+409)-(5005t+405)=4

4318+405=1952+2771=4591+132=2225+2498=4723

As a conclusion, unchanged sum which is equal to 4723 shows that there is the same density among the numbers in parentheses such as 55t+24 and 91t+41 when they are in an ascending order.

132; 134; 405; 409; 1949; 1952; 2224; 2225; 2498; 2499; 2771; 2774; 4314; 4318

difference 2 difference 4 difference 3 difference 1 difference 1 difference 3 difference 4

and 4589; 4591

difference 2

The order of the differences will be repeated periodically as below, if we add 5005t to each of these numbers, where $t \in N$:

2; 4; 3; 1; 1; 3; 4; 2

Now, let's look at general case, to the expressions of pqt_1+a and $p_1q_1t_2+b$ whose coefficients of t are mutually prime numbers in Table 4. Let's say $(pqt_1+a)-(p_1q_1t_2+b)=m$. Here, t_1 and t_2 are variables, $p; q; p_1; q_1 \in N$, $m; a; b \in z$. And $GCF(pq; p_1q_1)=1$.

Let's indicate the inequality as $pqt_1-p_1q_1t_2=m+b-a$, GCF $(pq; p_1q_1)=1 \Rightarrow \alpha, \beta \in N$, so there are numbers such as α and β, where $pq\alpha - p_1q_1\beta = 1$.

Therefore, $pq\alpha(m+b-a)-p_1q_1\beta(m+b-a)=m+b-a$.

From here,

$$pqt_1-p_1q_1t_2=m+b-a$$

$$pq\alpha(m+b-a)-p_1q_1\beta(m+b-a)=m+b-a$$

then, $t_1=p_1q_1t+\alpha(m+b-a)$; $t_2=pqt+\beta(m+b-a)$. Here $t \in z$.

If we consider t_1 and t_2 in $(pqt_1+a)-(p_1q_1+b)=m$, then $(pqp_1q_1t+pq(\alpha(m+b-a))+a)-(p_1q_1pqt+p_1q_1(\beta(m+b-a)+b)=m$.

By substituting m with (-m) we will get

$(pqp_1q_1t+pq(\alpha(-m+b-a))+a)-(p_1q_1pqt+p_1q_1(\beta(-m+b-a))+b)=-m$.

If x_{-m} and x_m are coordinates, where

$x_{-m}=pqp_1q_1t+pq(\alpha(-m+b-a))+a$

$x_m=pqp_1q_1t+pq(\alpha(m+b-a))+a$

then $x_{-m}+x_m=2(pqp_1q_1t+pq(\alpha(b-a)+a)$ will not depend on m.

So, our conclusion for specific case is also true for general case.

The sum is not dependent on m. So, the sum for p, q, a, b in any m is the same. Therefore, difference of the proper numbers will be 1 only if the coefficients of t are mutually prime numbers in any two expressions in Table 4. Also, the number of expressions is the same whose differences is 1, 2, 3, 4, … m, where the length of interval is pqp_1q_1. We have showed that the difference of two proper numbers is not 1, if the coefficients of t are not mutually prime numbers in any two expressions in Table 4. So, we came to the following conclusion :

Conclusion 5: In Table 3, in the series of (13) where the numbers in parentheses are in an ascending order, the number of consecutive numbers, whose difference is 4, are much more than the number of consecutive numbers whose difference is 1.

Conclusion 6: In each column, in Table 3, number of the numbers in parentheses (except repeated numbers in the columns which are in left) are not greater than number of remains in the column.

So, number of the numbers until 2n are not greater than (2n-19):2 ,or n-9 in the series of (13). Therefore, when they are in (10) and (11), there will not be any number such as (m) at least in 9 rows.

Now let's solve the second problem which we indicated in the first page. Addends are the numbers from 1 to 2n-1 in (10), and the numbers from 1 to 2n in (11). We showed that, number of those which belong to (13) are not greater than n-9. Then, in each of the series of (10) and (11), there are at least 9 rows which none of two addends in these rows exists in (13). Thus, appropriate (k_1)s and (k_2)s for those rows would be in following cases:

I $k_1;\ k_2 \in m_1 \cup m_4 \Rightarrow 6k_1-1,\ 6k_2-1$ are prime numbers

$k_1;\ k_2 \in m_2 \cup m_3 \Rightarrow 6k_1+1,\ 6k_2+1$ are prime numbers

II $\left.\begin{matrix} k_1 \in m_1 \cup m_4 \\ k_2 \in m_2 \cup m_3 \end{matrix}\right\} \Rightarrow 6k_1-1,\ 6k_2+1$ are prime numbers

III $\left.\begin{matrix} k_1 \in m_2 \cup m_3 \\ k_2 \in m_1 \cup m_4 \end{matrix}\right\} \Rightarrow 6k_1+1,\ 6k_2-1$ are prime numbers

IV $\left.\begin{array}{l} k_1 \in m_1 \cup m_4 \\ k_2 \notin Table3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 6k_1 - 1 \\ 6k_2 - 1 \\ 6k_2 + 1 \end{array}\right\}$ are prime numbers

V $\left.\begin{array}{l} k_1 \in m_2 \cup m_3 \\ k_2 \notin Table3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 6k_1 + 1 \\ 6k_2 - 1 \\ 6k_2 + 1 \end{array}\right\}$ are prime numbers

VI $\left.\begin{array}{l} k_1 \notin Table3 \\ k_2 \in m_1 \cup m_4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 6k_1 - 1 \\ 6k_1 + 1 \\ 6k_2 - 1 \end{array}\right\}$ are prime numbers

VII $\left.\begin{array}{l} k_1 \notin Table3 \\ k_2 \in m_2 \cup m_3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 6k_1 - 1 \\ 6k_1 + 1 \\ 6k_2 + 1 \end{array}\right\}$ are prime numbers

VIII $k_1; k_2 \notin Table3 \Rightarrow \left.\begin{array}{l} 6k_1 - 1 \\ 6k_1 + 1 \\ 6k_2 - 1 \\ 6k_2 + 1 \end{array}\right\}$ are prime numbers

If we consider I – VIII in (1), we will indicate that any even number greater than 8 is a sum of two prime numbers.

And now, let's show solution of the first problem. We indicated in the solution of previous problem that, any number which doesn't exist in Table 3, determines one pair of twin primes. That's why, indicating that the numbers which do not exist in Table 3 are infinite, is a solution of the problem.

In Table 3, let us assign the set of numbers which are not in the columns of

5n-1; 5n+1 as A_5;

7n-1; 7n+1 as A_7;

11n-2; 11n+2 as A_{11};

13n-2; 13n+2 as A_{13};

It is clear that,

A_5={5n; 5n+2; 5n+3}

A_7={7n; 7n+2; 7n+3; 7n+4; 7n+5}

A_{11}={11n; 11n+1; 11n+3; 11n+4; 11n+5; 11n+6; 11n+7; 11n+8; 11n+10}

-- .

Let's write sets of A_5; A_7; A_{11}; A_{13} … more generally :

$A_5=5n_1+p_1$; $p_1=\{0;2;3\}$; $n_1 \in N$

$A_7=7n_2+p_2$; $p_2=\{0;2;3;4;5\}$; $n_2 \in N$

$A_{11}=11n_3+p_3$; $p_3=\{0;1;3;4;5;6;7;8;10\}$; $n_3 \in N$ (18)

$A_{13}=13n_4+p_4$; $p_4=\{0;1;3;4;5;6;7;8;9;10;12\}$; $n_4 \in N$

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It is obvious that $A_5 \cap A_7$ is a set of natural numbers which do not exist in the first four columns of Table3, and $A_5 \cap A_7 \cap A_{11}$ is a set of natural numbers which do not exist in the first six columns of Table3.

Firstly, let us find a general formula for the numbers in $A_5 \cap A_7$.

$$5n_1+p_1=7n_2+p_2 \Rightarrow 5n_1-7n_2=p_2-p_1;$$

Since GCF(5;7)=1, for the variables n_1 and n_2, we can always find solutions with natural numbers for this equation. When solving the second problem, we have showed the solution of this kind of problems by using Euclid algorithm.

It is clear that, $\alpha = 3;\ \beta = 2$, in the equation of $5\alpha - 7\beta = 1$. From here, $5\cdot3-7\cdot2=1$

$$\text{So, } \left.\begin{array}{l} 5n_1 - 7n_2 = p_2 - p_1 \\ 5\cdot 3(p_2 - p_1) - 7\cdot 2(p_2 - p_1) = p_2 - p_1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} n_1 = 7t + 3(p_2 - p_1) \\ n_2 = 5t + 2(p_2 - p_1) \end{array}\right\} t \in z$$

$$\text{Then, } 5n_1 + p_1 = 7n_2 + p_2 = 5(7t + 3(p_2 - p_1) + p_1 \Rightarrow$$

$$\Rightarrow 5n_1 + p_1 = 7n_2 + p_2 = 35t + 15p_2 - 14p_1$$

As a result we get, $p_1=\{0;2;3\}$ $p_2=\{0;2;3;4;5\}$ and when $t \in z$ then the numbers which belong to $A_5 \cap A_7$ are found with the formula of $x(t)=35t+15p_2-14p_1$ (19). Here $t \in z$, $x(t) \in N$.

While $n(A_5)=3$; $n(A_7)=5$, then we can consider possible values of p_1 and p_2 in (19) and since $n(A_5)\cdot n(A_7)=3\cdot 5=15$, then we will get 15 different expressions of x(t) as following:

I $p_1=0$	$p_2=0 \Rightarrow 35t$	II $p_1=2$	$p_2=0 \Rightarrow 35t+7$	III $p_1=3$	$p_2=0 \Rightarrow 35t+28$
	$p_2=2 \Rightarrow 35t+30$		$p_2=2 \Rightarrow 35t+2$		$p_2=2 \Rightarrow 35t+23$
	$p_2=3 \Rightarrow 35t+10$		$p_2=3 \Rightarrow 35t+17$		$p_2=3 \Rightarrow 35t+3$
	$p_2=4 \Rightarrow 35t+25$		$p_2=4 \Rightarrow 35t+32$		$p_2=4 \Rightarrow 35t+18$
	$p_2=5 \Rightarrow 35t+5$		$p_2=5 \Rightarrow 35t+12$		$p_2=5 \Rightarrow 35t+33$

The multiples of 35 have been omitted in the expressions.

We can indicate all of fifteen expressions of x(t) in a short form with following formula :

$$x(t)=35t+a;\ a=\{0,2,3,5,7,10,12,17,18,23,25,28,30,32,33\} \quad (20)$$

It is clear from the formula that, there are infinite number of natural numbers in Table 3 which do not exist in the columns of $5n\mp1;\ 7n\mp1$.

And now let's show that there are infinite number of natural numbers in x(t), which do or do not exist in the columns of 11n-2; 11n+2 $n\geq 2$ in Table 3. We can write the numbers in 5th and 6th columns as 11k+q $k\geq 2\ k\in N$ and q={-2;2}.If we solve the expression of x(t)=35t+a which do not exist in the first four columns of Table 3, by writing as 35t+a=11k+q, then we will get

$$y(t)=385t+176a-175q \quad (21)$$

If we consider the values of a and q, we will see that there are infinite number of solutions of this equation. So, we get that there are infinite number of numbers in the 5th and 6th columns of table in x(t). And now let's show that there are infinite number of numbers in x(t) (in other words $A_5\cap A_7$) which do not exist in the 5th and 6th columns of Table 3. In order to do so, we must solve the equation of $35t+a=11n_3+p_3$ using the value of a in (20), and the value of p_3 in (18). The solution will be as

$$z(t)=385t+176a-175p_3 \quad (22)$$

It is obvious that there will be infinite number of numbers in $z(t)$.

So, we came to the following conclusion:

Conclusion 7: There are infinite number of natural numbers which do not exist in the first four columns of Table 3 and those which do or do not exist in next two columns are also infinite.

It is clear from the process that, there will always be solution for each created linear equation with two variables because the coefficients of the variables will be mutually prime numbers. If we continue the process rightward, then we will get the following conclusion:

Conclusion 8: There are infinite number of numbers which do not exist in Table 3. And this shows that, there are infinite number of twin primes.

Both of the problems have been solved.

References

E.S. Lyapin, A.E. Evseev, “Algebra and Number Theory”, vol 1&2, Moscow 1974 [In Russian]