The Uniqueness of Rational Structure and its Graph

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Abstract Galaxies are the basic components of the universe. A massive Hubble Space Telescope photos survey reveals that the diversity of galaxies in the early universe was as varied as the many galaxy types seen today. Therefore, understanding galaxies is the great challenge to humans. This paper deals with the disk-typed galaxies which is called spirals. In longer wavelength image, galaxy arms are mostly gone, and spiral galaxies fall to two types: ordinary and barred. The ordinary ones are basically an axi-symmetric disk whose stellar density decreases exponentially outwards. It is called the exponential disk. It is straightforward to show that any exponential disk has infinite nets of orthogonal curves such that the stellar density on one side of each curve is in constant ratio to the density on the other side of the curve. These curves are call proportion curves or Darwin curves. It happens that the Darwin curves of exponential disk are all golden spirals. Amazingly, astronomers found out that the arms of ordinary spiral galaxies are all golden spirals. Therefore, I had a proposition in 2004 that a two dimensional structure is called a rational one if there exists at least one orthogonal net of Darwin curves in the structure plane. Now in this paper, the mathematical solution to rational structure is completely obtained. We prove that rational structure is unique.

keywords: Rational Structure; Spiral Galaxies

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Contents

1 Rational Structure Equation

A structure is a two or three dimensional distribution of similar matters. This paper deals with two dimensional structure only. A curve in the plane of the distribution is called a proportion curve or a Darwin curve if the matter density on one side of the curve is in constant ratio to that on the other side of the curve. If there exists an orthogonal net of Darwin curves in the plane, the distribution of matters is called a rational structure. We found many evidences that galaxies are rational stellar distribution. We list a few examples. Firstly, galaxy components (disks and bars) can be fitted with rational structure. Secondly, spiral arms can be fitted with Darwin curves. Thirdly, rational structure dictates New Universal Gravity which explains constant rotation curves simply and elegantly.

Rational structure in two dimension

$$\rho(x, y) \tag{1}$$

means that not only there exists an orthogonal net of curves in the plane

$$x = x(\lambda, \mu), \ y = y(\lambda, \mu) \tag{2}$$

but also, along each curve, the matter density on one side of the curve is in constant ratio to the one on the other side. Such a curve is called a proportion curve or a Darwin curve. Such a distribution of matter is called a rational structure. Because the density ratio is equivalent to the derivative to the logarithm of the density

$$f(x, y) = \ln \rho(x, y) \tag{3}$$

we from now on, are only concerned with the logarithmic density $f(x, y)$. We know that, given the two partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \tag{4}$$

the structure $f(x, y)$ is determined provided that the Green's theorem is satisfied

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{5}$$

The above is the well–known identity of mixed derivatives. Now we are interested in rational structure. Instead of calculating the partial derivatives (4), we calculate the directional derivatives along the tangent direction to the above curves

$$\frac{\partial f}{\partial l_\lambda}, \quad \frac{\partial f}{\partial l_\mu} \tag{6}$$

where l_λ is the linear length on the *physical* plane (x, y) and is along the row curves whose parameter is λ while l_μ is the linear length along the column curves whose parameter is μ. Given the two partial derivatives (6), however, the structure $f(x, y)$ may not be determined. A similar Green's theorem, which is called rational structure equation, must be satisfied

$$\frac{\partial}{\partial \mu}\left(P\,\frac{\partial f}{\partial l_\lambda}\right) - \frac{\partial}{\partial \lambda}\left(Q\,\frac{\partial f}{\partial l_\mu}\right) = 0 \tag{7}$$

where

$$\begin{aligned} P(\lambda,\mu) &= \sqrt{x'^2_\lambda + y'^2_\lambda}, \\ Q(\lambda,\mu) &= \sqrt{x'^2_\mu + y'^2_\mu} \end{aligned} \tag{8}$$

are the lengths or the magnitudes of the vectors (x'_λ, y'_λ) and (x'_μ, y'_μ), respectively. Note that we have used the simple notation $x'_\lambda = \frac{\partial x}{\partial \lambda}$. From now on, we always use the similar simple notations.

It is straightforward to show that the partial derivatives of rational structure in the parameter space (λ, μ) are

$$f'_\lambda = P\,\frac{\partial f}{\partial l_\lambda}, \quad f'_\mu = Q\,\frac{\partial f}{\partial l_\mu} \tag{9}$$

Therefore, the rational structure equation (7) is nothing but the following simple equation

$$\frac{\partial^2 f}{\partial \lambda \partial \mu} = \frac{\partial^2 f}{\partial \mu \partial \lambda} \tag{10}$$

To simplify the expression of our equations, we introduce the important new notations for our directional derivatives along the tangent direction of row and column curves in the physical space (x, y)

$$u(\lambda,\mu) = \frac{\partial f}{\partial l_\lambda}, \quad v(\lambda,\mu) = \frac{\partial f}{\partial l_\mu} \tag{11}$$

The condition of rational structure is that u depends only on λ and v depends only on μ

$$u = u(\lambda), \; v = v(\mu) \tag{12}$$

The above two quantities are called the basic derivatives of rational structure. Now we prove the condition. Assume you walk along a row curve. The logarithmic ratio of the density on your left side to the immediate density on your right side is approximately the directional derivative of $f(x, y)$ along the column direction. That is, the logarithmic ratio is approximately the directional derivative $v(\lambda, \mu)$. Because $v(\lambda, \mu)$ is constant along the row curve (rational), $v(\lambda, \mu)$ is independent of λ, $v = v(\mu)$. Similarly, we can prove that $u(\lambda, \mu) = u(\lambda)$.

In the case of rational structure, the directional derivatives, $u = \frac{\partial f}{\partial l_\lambda}$ and $v = \frac{\partial f}{\partial l_\mu}$, are the functions of the single variables λ and μ, respectively (see the formula (12)). Therefore, the Green's theorem (7) turns out to be much simpler

$$u(\lambda) P'_\mu = v(\mu) Q'_\lambda \tag{13}$$

which is our rational structure equation [1,2]. The following equation is the necessary and sufficient condition for the curves to be orthogonal

$$x'_\lambda x'_\mu + y'_\lambda y'_\mu = 0 \tag{14}$$

In fact, rational structures depend only on the geometric curves, not the choice of coordinate parameters. Therefore, the general coordinate parameters $(\sigma,\, \tau)$ for the orthogonal net of curves are

$$\begin{cases} x = x(g(\sigma),\, h(\tau)), \\ y = y(g(\sigma),\, h(\tau)) \end{cases} \tag{15}$$

where $\lambda = g(\sigma), \mu = h(\tau)$ are arbitrary functions. All these expressions give the same orthogonal net of Darwin curves and generate the same rational structure. In the following Section, we prove the existence of harmonic coordinate parameters and all of our following discussion are based on the parameters.

2 Harmonic Coordinate System and Rational Structure Equation

A harmonic coordinate system is the choice of coordinate parameters such that the following equations hold

$$\begin{aligned} x'_\lambda &= y'_\mu, \\ x'_\mu &= -y'_\lambda \end{aligned} \tag{16}$$

They are the well known Cauchy-Riemann equations and the following complex function in the space (λ, μ)

$$\Psi(\lambda, \mu) = x(\lambda, \mu) + \mathrm{i}\, y(\lambda, \mu) \tag{17}$$

must be analytic. The real part $x(\lambda, \mu)$ and the imaginary part $y(\lambda, \mu)$ are determined by each other and both are harmonic functions,

$$\begin{aligned} x''_{\lambda\lambda} + x''_{\mu\mu} &= 0 \\ y''_{\lambda\lambda} + y''_{\mu\mu} &= 0 \end{aligned} \tag{18}$$

The imaginary part is usually called the conjugate of the real part (see some textbook on analytic complex functions).

Does there exist a harmonic coordinate parameter for any net of curves? We prove that the necessary and sufficient condition for the existence of harmonic coordinates is that the net of curves is orthogonal. Application of the derivative chain–rule to the composite functions (15) gives

$$\begin{aligned} x'_\sigma = x'_\lambda g'_\sigma, \quad x'_\tau = x'_\mu h'_\tau \\ y'_\sigma = y'_\lambda g'_\sigma, \quad y'_\tau = y'_\mu h'_\tau \end{aligned} \tag{19}$$

The Cauchy-Riemann equations for the harmonic coordinate parameter $(\sigma,\, \tau)$ are

$$\begin{aligned} x'_\sigma &= y'_\tau, \\ x'_\tau &= -y'_\sigma \end{aligned} \tag{20}$$

That is,

$$\begin{cases} x'_\lambda g'_\sigma - y'_\mu h'_\tau = 0, \\ y'_\lambda g'_\sigma + x'_\mu h'_\tau = 0 \end{cases} \tag{21}$$

The above linear equation system has non-zero solution $\lambda = g(\sigma), \mu = h(\tau)$ if and only if the determinant of the equation system is zero. The condition turns out to be the orthogonal condition (14). Because rational structure is always defined on orthogonal net of curves, its harmonic coordinate parameters always exist. Therefore, we from now on, assume that the curve parameters (λ, μ) are themselves harmonic. A immediate result is the following

$$P(\lambda, \mu) = \sqrt{x'^2_\lambda + y'^2_\lambda} \equiv \sqrt{x'^2_\mu + y'^2_\mu} = Q(\lambda, \mu) \tag{22}$$

(see the formulas (8)). Accordingly the rational structure equation is

$$u(\lambda)P'_\mu = v(\mu)P'_\lambda \tag{23}$$

The symbol of quantity $Q(\lambda, \mu)$ is no longer needed. Given $u(\lambda)$ and $v(\mu)$, the above rational structure equation is a first order partial differential equation whose unknown is $P(\lambda, \mu)$. Its general solution can be obtained with the standard characteristic method. The characteristic equation is

$$\frac{d\lambda}{v(\mu)} = \frac{d\mu}{-u(\lambda)} \tag{24}$$

which is an ordinary differential equation. The general solution to the ordinary equation is

$$U(\lambda) + V(\mu) = c \tag{25}$$

where c is an arbitrary constant, and $U(\lambda)$ and $V(\mu)$ are the indefinite integrals of the directional derivatives $u(\lambda)$ and $v(\mu)$, respectively

$$\begin{cases} u(\lambda) = U'_\lambda(\lambda), \\ v(\mu) = V'_\mu(\mu) \end{cases} \tag{26}$$

From now on, $U'_\lambda(\lambda)$ is simply denoted with $U'(\lambda)$ or U'. The notation applies to other quantities when confusion may be avoided. Now the general solution for the original rational structure equation (23) is

$$W(P, C) = 0 \tag{27}$$

where $W(P, C)$ is an arbitrary function of the two variables P and C, and P is replaced with the unknown $P(\lambda, \mu)$ and C is replaced with the left–hand side of the equation (25) where the arbitrary integral constant c must be isolated into the right–hand side. If we do not pay much attention to possible multi-valued functions, the general solution (27) is essentially the fact that P is a function of the single variable $C = U(\lambda) + V(\mu)$

$$P = P(C) = P(U(\lambda) + V(\mu)) \tag{28}$$

The above is our second result derived with the harmonic coordinate parameters. Note that $P(\lambda, \mu)$ and $P(C)$ are two different functions but we use the same symbol P to denote them. Which is which can be recognized with its accompanying symbol of derivatives. For simplicity we follow this convention for P and for other functions if applicable.

The formulas (9) now becomes

$$\begin{aligned} f'_\lambda &= u(\lambda)P(\lambda, \mu), \\ f'_\mu &= v(\mu)P(\lambda, \mu) \end{aligned} \tag{29}$$

Therefore,

$$u(\lambda)f'_\mu = v(\mu)f'_\lambda \tag{30}$$

Given $u(\lambda)$ and $v(\mu)$, the above first order partial differential equation is identical to the one (23). Therefore, we have

$$f = f(C) = f(U(\lambda) + V(\mu)) \tag{31}$$

It leads to

$$\begin{aligned} f'_\lambda &= u(\lambda)f'_c(C), \\ f'_\mu &= v(\mu)f'_c(C) \end{aligned} \tag{32}$$

Comparison of the above equation with the equation (29) leads to

$$P(C) = f'_c(C) \tag{33}$$

Now we understand that the original rational structure equation (23) is solved. The general solution is simply an arbitrary function of the single variable C: $P = P(C)$ or $f = f(C)$ (see the expression (28) or (31)) which is a composite function whose middle variable C is the sum of two other single–variable functions $U(\lambda)$ and $V(\mu)$

$$C = U(\lambda) + V(\mu) \tag{34}$$

However, the function expression (28) or (31) alone is not the sufficient condition for the solution of our rational structure. It is the necessary one. This is because the net of curves (2) solved from the following equation

$$P(\lambda, \mu) = \sqrt{x'^2_\lambda + x'^2_\mu} \tag{35}$$

may not be orthogonal.

3 The Necessary and Sufficient Condition for Rational Structure

The net of curves (2) is orthogonal and its parameters are harmonic if and only if the Cauchy-Riemann equations (16) hold. The equations hold if and only if $x(\lambda, \mu)$ is a harmonic function. The function $x(\lambda, \mu)$ is harmonic if and only if its derivatives

$$\Phi(\lambda, \mu) = x'_\mu + \mathrm{i}\, x'_\lambda \tag{36}$$

satisfy Cauchy-Riemann equations. The above complex function satisfies Cauchy-Riemann equations if and only if the logarithm of its modulus

$$\ln P(\lambda, \mu) = \ln \sqrt{x'^2_\lambda + x'^2_\mu} \tag{37}$$

is a harmonic function. Finally, applying harmonic operation to the above function gives the necessary and sufficient condition for the net of curves being orthogonal and its parameters being harmonic. We use the symbol L to denote the logarithm

$$L(\lambda, \mu) = \ln P(\lambda, \mu) = \ln P(C) = \ln P(U(\lambda) + V(\mu)) \tag{38}$$

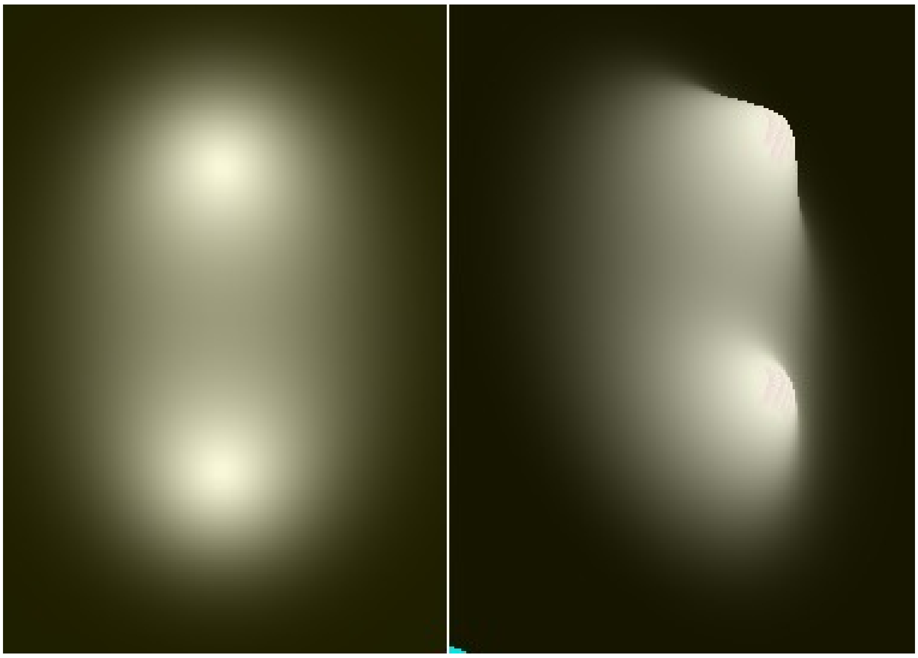

Figure 1: Left panel is the planar graph of double-breast structure, and right panel is its corresponding 3-dimensional demonstration. The graph and demonstration are based on Galaxy Anatomy graphic software.

Its derivatives are

$$\begin{array}{l} L''_{\lambda\lambda} = \frac{P''P-P'^2}{P^2}u^2(\lambda) + \frac{P'}{P}u'(\lambda), \\ L''_{\mu\mu} = \frac{P''P-P'^2}{P^2}v^2(\mu) + \frac{P'}{P}v'(\mu) \end{array} \tag{39}$$

Finally, the harmonic equation $L''_{\lambda\lambda}+L''_{\mu\mu} = 0$ leads to the following necessary and sufficient condition for the harmonic parameters (λ, μ)

$$\frac{P''_{cc}}{P'_c} - \frac{P'_c}{P} = -\frac{u'(\lambda) + v'(\mu)}{u^2(\lambda) + v^2(\mu)} \tag{40}$$

That is,

$$\frac{(\ln P)''_{cc}}{(\ln P)'_c} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{41}$$

Finally, the above equation is the necessary and sufficient condition for the harmonic parameters (λ, μ). It is also the necessary and sufficient condition for rational structure if we always remember the additional condition, i. e., the existence of the single variable function (28) whose single–variable is the formula (34)

4 Heaven Breasts: the Second Solution to Rational Structure

We know that the equation (41) is the necessary and sufficient condition for rational structure. Now we try to solve it. Rational structure (3) can be classified into two types, axi-symmetric and non-axisymmetric. Any axi-symmetric structure is rational because it has a net of orthogonal curves: the polar coordinate curves, i. e., the radial lines and the circles with center at the origin. The directional derivatives along the circles, i. e., $v(\mu)$, is zero while the directional derivatives along the radial lines, i. e., $u(\lambda)$, depends solely on the single variable $r \sim \lambda$. Therefore, any axi-symmetric structure is a rational one. Because it is axi-symmetric, the exponential disk of spiral galaxies is rational. However, it is a very special one because it has infinite nets of orthogonal proportion curves. These curves are golden spirals. Astronomical observation shows that the arms of ordinary spiral galaxies follow golden spirals. Therefore, exponential disk is called our first solution to rational structure.

From now on, we study non-axisymmetric rational structure. Now we deal with the partial differential equation (41). Firstly we try the simplest example of power function $P(C)$,

$$P = P(C) = C^k \tag{42}$$

where k is a constant. The equation (41) becomes

$$-\frac{1}{U+V} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{43}$$

which is irrelevant to the constant k. The above equation is

$$U(\lambda)V''(\mu) + V(\mu)U''(\lambda) = U'^2(\lambda) + V'^2(\mu) - U(\lambda)U''(\lambda) - V(\mu)V''(\mu) \tag{44}$$

The two variables λ and μ on the right-hand side of the above equation are separated. Accordingly, the mixed partial derivatives to the left hand side must be zero

$$U'(\lambda)V'''(\mu) + V'(\mu)U'''(\lambda) = 0 \tag{45}$$

Finally we have a general solution to the rational structure equation (41)

$$\begin{aligned} &U(\lambda) = m_0 + m_1 e^{l\lambda} + m_2 e^{-l\lambda}, \\ &V(\mu) = n_0 + n_1 \cos(l\mu) + n_2 \sin(l\mu), \\ &m_0 = -n_0, \quad 4m_1 m_2 = n_1^2 + n_2^2 \end{aligned} \tag{46}$$

where $l, m_0, m_1, m_2, n_0, n_1, n_2$ are real constants.

Now we want to solve the corresponding net of curves which is given by the equation (35). If we choose $k = 1/2$ then the equation is

$$\sqrt{U(\lambda) + V(\mu)} = \sqrt{x_\lambda'^2 + x_\mu'^2} \tag{47}$$

Amazingly, the resulting harmonic coordinate $x(\lambda, \mu)$, its conjugate, and the corresponding rational structure

$$f(C) = \int P(C)dC \tag{48}$$

are equivalent: the Heaven Breasts pattern (see Fig. 1 and the graph in [3]). Therefore, we call the above complete solution (46) the Heaven Breasts structure. It is called the second solution to rational structure. To have the breasts aligned in the horizontal direction, we need to choose $n_2 = 0$.

5 Uniqueness Equation of Rational Structure and the Third Solution

We know that the equation (41) is the necessary and sufficient condition for rational structure. Its solution is all the single-variable functions $P(C)$ whose derivatives must satisfy (41). In fact, the derivative of $P(C)$ with respect to C is still a function of C. Therefore, the left-hand side of equation (41) is still a function of C. Let us denote it to be $W(C)$

$$W(C) = \frac{P''_{cc}}{P'_c} - \frac{P'_c}{P} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{49}$$

The original equation (41) becomes

$$(U'^2(\lambda) + V'^2(\mu))W(C) = -(U''(\lambda) + V''(\mu)) \tag{50}$$

The two variables λ and μ on the right-hand side of the above equation are separated. Accordingly, the mixed partial derivatives to the left hand side of the above equation must be zero

$$\begin{aligned} 0 &= ((U'^2(\lambda) + V'^2(\mu))W(C))''_{\lambda\mu} \\ &= U'V' \ (2(U'' + V'')W'(C) + (U'^2 + V'^2)W''(C)) \end{aligned} \tag{51}$$

Therefore, the final factor in the above formula must be zero and we have

$$\frac{W''_{cc}(C)}{2W'_c(C)} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} = W(C) \tag{52}$$

Finally, we have the uniqueness equation of rational structure

$$W''_{cc}(C) - 2W(C)W'_c(C) = 0 \tag{53}$$

This is a nonlinear ordinary differential equation which rational structure must satisfy. Its all solutions we can find are

$$\begin{aligned} &W(C) = -\tfrac{1}{C}, \\ &W(C) = a\tan(aC), \ \ W(C) = -a\cot(aC), \\ &W(C) = -a\tanh(aC), \ \ W(C) = -a\coth(aC) \end{aligned} \tag{54}$$

where $a > 0$ is an arbitrary constant. The first solution in the above list is the Heaven Breasts structure (Fig. 1) as demonstrated by the equation (43). In fact, the first integral to the equation (53) is

$$W'_c(C) - W^2(C) = d \tag{55}$$

where d is an arbitrary constant. The Heaven Breasts structure corresponds to $d = 0$.

In the current paper we discuss the second solution in the above list (54)

$$W(C) = a\tan(aC) \tag{56}$$

All others are left for the future exploration. Now we try to find the integrals $U(\lambda)$ and $V(\mu)$ of the basic derivatives (12). Substitution of the above result into the left-hand of equation (41) leads to

$$a\tan(aC) = a\frac{\tan(aU) + \tan(aV)}{1 - \tan(aU)\tan(aV)} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{57}$$

That is,

$$\begin{aligned}&(U''+V'')\tan(aU)\tan(aV)-aU'^2\tan(aV)-aV'^2\tan(aU)\\&=U''+V''+aU'^2\tan(aU)+aV'^2\tan(aV)\end{aligned}\tag{58}$$

Again the two variables λ and μ on the right-hand side of the above equation are separated. Accordingly, the mixed partial derivatives to the left hand side must be zero, and the corresponding solution is

$$\begin{aligned}&aU'^2-U''\tan(aU)-\omega\tan(aU)=\iota,\\&aV'^2-V''\tan(aV)+\omega\tan(aV)=\kappa\end{aligned}\tag{59}$$

where $\omega>0,\iota,\kappa$ are all arbitrary constants. The above ordinary differential equations can not be solved with elementary functions. However, the solutions can be expressed as follows

$$\begin{aligned}&\lambda=\lambda_0+\textstyle\int^{U(\lambda)}dU/\sqrt{\omega\sin(2aU)+((\iota-a\epsilon)/2)\cos(2aU)+((\iota+a\epsilon)/2)},\\&\mu=\mu_0+\textstyle\int^{V(\mu)}dV/\sqrt{-\omega\sin(2aV)+((\kappa-a\delta)/2)\cos(2aV)+((\kappa+a\delta)/2)}\end{aligned}\tag{60}$$

where $\lambda_0,\mu_0,\epsilon,\delta$ are other integral constants. Accordingly,

$$\begin{aligned}&U'(\lambda)=\tfrac{1}{\sqrt{a}}\sqrt{\omega\sin(2aU)+((\iota-a\epsilon)/2)\cos(2aU)+((\iota+a\epsilon)/2)},\\&U''(\lambda)=\omega\cos(2aU)-((\iota-a\epsilon)/2)\sin(2aU),\\&V'(\mu)=\tfrac{1}{\sqrt{a}}\sqrt{-\omega\sin(2aV)+((\kappa-a\delta)/2)\cos(2aV)+((\kappa+a\delta)/2)},\\&V''(\mu)=-\omega\cos(2aV)-((\kappa-a\delta)/2)\sin(2aV)\end{aligned}\tag{61}$$

The above integral constants are not arbitrary, which have to meet the original equation (58). Substitution of the formulas (59) and (61) into the equation (58) leads to

$$\kappa+a\epsilon=0,\quad\iota+a\delta=0\tag{62}$$

Therefore,

$$\begin{aligned}&(\iota-a\epsilon)/2=(\kappa-a\delta/2=\iota+\kappa,\\&(\iota+a\epsilon)/2=\iota-\kappa=-(\kappa+a\delta/2\end{aligned}\tag{63}$$

Because the functions under the roots are nothing but some sinusoids, we focus on the simplest case,

$$\begin{aligned}&\iota=\kappa=0,\\&\lambda=\lambda_0+\textstyle\int^{U(\lambda)}dU/\sqrt{\omega\sin(2aU)},\\&\mu=\mu_0+\textstyle\int^{V(\mu)}dV/\sqrt{-\omega\sin(2aV)},\\&U'(\lambda)=\sqrt{\tfrac{\omega}{a}}\sqrt{\sin(2aU)},\\&U''(\lambda)=\omega\cos(2aU),\\&V'(\mu)=\sqrt{\tfrac{\omega}{a}}\sqrt{-\sin(2aV)},\\&V''(\mu)=-\omega\cos(2aV),\end{aligned}\tag{64}$$

Based on our choice, we always have

$$\begin{aligned}&\sin(2aU(\lambda))>0,\quad\sin(2aV(\mu))<0,\\&0<aU<\pi/2,\quad-\pi/2<aV<0,\\&\sin(aU)>0,\quad\cos(aU)>0,\\&\sin(aV)<0,\quad\cos(aV)>0\end{aligned}\tag{65}$$

It appears that we have solved our rational structure equations. However, our final goal is the determination of rational structure $f(x, y)$ in the physical space (x, y). But the results we have achieved are domained in the parameter space (λ, μ). The relationship between the two spaces are exactly the net of orthogonal curves (2) which is determined by $P(C)$ (see the formula (35)). Therefore, in the following Section we try to solve $x(\lambda, \mu)$ and $y(\lambda, \mu)$ based on the formula (35).

6 The Third Solution in Physical Space

The two formulas (49) and (56) combined lead to the following equation

$$\frac{P''_{cc}}{P'_c} - \frac{P'_c}{P} = a\tan(aC) \tag{66}$$

whose solution $P(C)$ is

$$P(C) = j_1 \left(\frac{\sin(aC) \pm 1}{\cos(aC)} \right)^{j_2} \tag{67}$$

where j_1, j_2 are arbitrary constants. From now on, the following is true as it is true in the above formula

$$\pm 1 \text{ in any formula, correspond to } \cos(aC) \gtrless 0, \text{ respectively} \tag{68}$$

Because the logarithm of P is harmonic (see (39) and (41)), its conjugate $\alpha(\lambda, \mu)$ must exist

$$\begin{aligned} \alpha'_\mu &= L'_\lambda = j_2 \tfrac{\cos(aC)}{\sin(aC)\pm 1} \tfrac{\cos^2(aC)+(\sin(aC)\pm 1)\sin(aC)}{\cos^2(aC)} aU'(\lambda) \\ &= \pm j_2 \sqrt{a} \tfrac{\sqrt{\omega \sin(2aU)}}{\cos(aC)}, \\ \alpha'_\lambda &= -L'_\mu = \mp j_2 \sqrt{a} \tfrac{\sqrt{-\omega \sin(2aV)}}{\cos(aC)} \end{aligned} \tag{69}$$

From now on, the following is true as it is true in the last equation of the above formula

$$\mp 1 \text{ in any formula, correspond to } \cos(aC) \gtrless 0 \text{ either} \tag{70}$$

Therefore,

$$\alpha'_\lambda U'(\lambda) = -\alpha'_\mu V'(\mu) \tag{71}$$

The solution of its characteristic equation is

$$\int \frac{d\lambda}{U'(\lambda)} - \int \frac{d\mu}{V'(\mu)} = c_1 \tag{72}$$

where c_1 is the integral constant. The above equation is identical to the following

$$\int \lambda'_U d\lambda - \int \mu'_V d\mu = c_1 \tag{73}$$

That is,

$$\int \frac{dU}{\omega \sin(2aU)} + \int \frac{dV}{\omega \sin(2aV)} = c_1 \tag{74}$$

Finally, the solution of the characteristic equation is

$$\tan(aU)\tan(aV) = c \tag{75}$$

The conjugate $\alpha(\lambda,\mu)$ to the logarithm of P must be a composite function of the product $\tan(aU)\tan(aV)$

$$\alpha(\lambda,\mu) = \alpha(\tan(aU)\tan(aV)) \tag{76}$$

Substituting the above into any equation in (69) and solving it for α, we have

$$\alpha(\lambda,\mu) = \pm j_2\sqrt{a}\left(\alpha_0 - 2\arctan\sqrt{-\tan(aU)\tan(aV)}\right) \tag{77}$$

where α_0 is an integral constant.

Finally, we have the derivatives of our net of orthogonal curves

$$\begin{aligned} x'_\mu &= P\cos\alpha \\ &= j_1\left(\tfrac{\sin(aC)\pm1}{\cos(aC)}\right)^{j_2}\cos\left(\pm j_2\sqrt{a}\left(\alpha_0 - 2\arctan\sqrt{-\tan(aU)\tan(aV)}\right)\right), \\ x'_\lambda &= P\sin\alpha \\ &= j_1\left(\tfrac{\sin(aC)\pm1}{\cos(aC)}\right)^{j_2}\sin\left(\pm j_2\sqrt{a}\left(\alpha_0 - 2\arctan\sqrt{-\tan(aU)\tan(aV)}\right)\right) \end{aligned} \tag{78}$$

In the formula (56), we made a special choice of $W(C)$. In the first equation of (64), we made the simplest choice of the sinusoids for the functions $u(\lambda) = U'(\lambda)$ and $v(\mu) = V'(\mu)$. Because this paper is a preliminary study on rational structure, we make further choice for the formulas x'_μ and x'_λ

$$\begin{aligned} &j_1 = 1,\quad j_2 = 1,\quad \alpha_0 = 0,\quad a = 1, \\ &x'_\mu = P\cos\alpha = \tfrac{\sin C\pm1}{\cos C}\cos\left(2\arctan\sqrt{-\tan U\tan V}\right), \\ &x'_\lambda = P\sin\alpha = \mp\tfrac{\sin C\pm1}{\cos C}\sin\left(2\arctan\sqrt{-\tan U\tan V}\right) \end{aligned} \tag{79}$$

Based on our choice (65), we have

$$\begin{aligned} x'_\mu &= -\tfrac{\cos(U-V)}{\sin(U+V)\mp1}, \\ x'_\lambda &= \pm\tfrac{\sqrt{-\sin(2U)\sin(2V)}}{\sin(U+V)\mp1} \end{aligned} \tag{80}$$

It is straightforward to verify that $x(\lambda,\mu)$ is harmonic: $x''_{\lambda\lambda} + x''_{\mu\mu} \equiv 0$.

As the above equations indicate, x itself must be the function of two variables U and V. We want to calculate the function. Because $x'_\lambda = x'_U U'_\lambda$ and $x'_\mu = x'_V V'_\mu$, we have

$$\begin{aligned} x'_U &= \pm\tfrac{1}{\sqrt{\omega}}\tfrac{\sqrt{-\sin(2V)}}{\sin(U+V)\mp1}, \\ x'_V &= -\tfrac{1}{\sqrt{\omega}}\tfrac{\cos(U-V)}{(\sin(U+V)\mp1)\sqrt{-\sin(2V)}} \end{aligned} \tag{81}$$

It is straightforward to show that $x''_{UV} = x''_{VU}$, therefore, our necessary and sufficient condition for rational structure, i. e. the equation (41), is consistent.

Now we integrate the above equations for the solution $x(U,V)$. The integration of the first equation in the above equations is simple

$$\begin{aligned} x(U,V) &= \tfrac{2}{\sqrt{\omega}}\tfrac{\sqrt{-\sin(2V)}}{\tan((U+V)/2)\mp 1} + F(V) \\ &= \tfrac{2}{\sqrt{\omega}}\tfrac{(1+\cos(U+V))\sqrt{-\sin(2V)}}{\sin(U+V)\mp 1\mp\cos(U+V)} + F(V) \end{aligned} \tag{82}$$

where $F(V)$ is a function of the single variable V only, and is determined by the second equation. That is,

$$\begin{aligned} &F'(V) = \\ &\tfrac{1}{\sqrt{\omega}}\left(-\tfrac{\cos(U-V)}{(\sin(U+V)\mp 1)\sqrt{-\sin(2V)}} \mp \tfrac{\sqrt{-\sin(2V)}}{\sin(U+V)\mp 1} + 2\tfrac{(1+\cos(U+V))\cos(2V)}{(\sin(U+V)\mp 1\mp\cos(U+V))\sqrt{-\sin(2V)}}\right) \end{aligned} \tag{83}$$

The right-hand side of the above equation must be independent of the variable U and is to be calculated in the following. Fully employing the formula $\frac{1}{\zeta\pm\varsigma} = \frac{\zeta\mp\varsigma}{\zeta^2-\varsigma^2}$ for the denominators in the above equation, we have

$$\begin{aligned} &F'(V) = -\tfrac{1}{\sqrt{\omega}} \\ &\tfrac{\pm\cos(\xi-\eta)\cos^2\xi+\cos(\xi-\eta)\sin\xi\cos\xi\pm\cos(\xi-\eta)\cos\xi+\sin(\xi-\eta)\mp\cos\eta-\cos\eta\sin\xi\pm\sin(\xi-\eta)\sin\xi}{\cos^2\xi\sqrt{-\sin(\xi-\eta)}} \end{aligned} \tag{84}$$

where

$$\begin{aligned} &\xi = U+V, \quad \eta = U-V, \\ &2U = \xi+\eta, \quad 2V = \xi-\eta \end{aligned} \tag{85}$$

The numerator is

$$\begin{aligned} &\pm\cos(\xi-\eta)\cos^2\xi + \cos(\xi-\eta)\sin\xi\cos\xi \pm \cos^2\xi\cos\eta \pm \sin\xi\sin\eta\cos\xi \\ &+\sin\xi\cos\eta - \cos\xi\sin\eta \mp \cos\eta - \cos\eta\sin\xi \pm \sin^2\xi\cos\eta \mp \cos\xi\sin\eta\sin\xi \\ &= \pm\cos(\xi-\eta)\cos^2\xi + \cos(\xi-\eta)\sin\xi\cos\xi - \cos\xi\sin\eta \end{aligned} \tag{86}$$

Therefore,

$$\begin{aligned} F'(V) &= -\tfrac{1}{\sqrt{\omega}}\tfrac{\pm\cos(\xi-\eta)\cos\xi+\cos(\xi-\eta)\sin\xi-\sin\eta}{\cos\xi\sqrt{-\sin(\xi-\eta)}} \\ &= -\tfrac{1}{\sqrt{\omega}}\tfrac{\pm\cos(\xi-\eta)\cos\xi+\cos\xi\cos\eta\sin\xi-\cos^2\xi\sin\eta}{\cos\xi\sqrt{-\sin(\xi-\eta)}} \\ &= -\tfrac{1}{\sqrt{\omega}}\tfrac{\pm\cos(\xi-\eta)+\sin(\xi-\eta)}{\sqrt{-\sin(\xi-\eta)}} \\ &= -\tfrac{1}{\sqrt{\omega}}\tfrac{\pm\cos(2V)+\sin(2V)}{\sqrt{-\sin(2V)}} \end{aligned} \tag{87}$$

which demonstrates that the formula (83) is independent of the variable U. Finally, we have

$$F(V) = \frac{1}{\sqrt{\omega}}\left(\pm\sqrt{-\sin(2V)} + \int\sqrt{-\sin(2V)}dV\right) \tag{88}$$

We have obtained the function $x(U,V)$ (see the formula (82)). Now we need calculate its conjugate $y(U,V)$ to complete the harmonic coordinates (2)

$$\begin{aligned} &y'_\lambda = -x'_\mu = \tfrac{\cos(U-V)}{\sin(U+V)\mp 1}, \\ &y'_\mu = x'_\lambda = \pm\tfrac{\sqrt{-\sin(2U)\sin(2V)}}{\sin(U+V)\mp 1} \end{aligned} \tag{89}$$

That is,

$$\begin{aligned} y'_U &= \tfrac{1}{\sqrt{\omega}}\tfrac{\cos(U-V)}{(\sin(U+V)\mp1)\sqrt{\sin(2U)}}, \\ y'_V &= \pm\tfrac{1}{\sqrt{\omega}}\tfrac{\sqrt{\sin(2U)}}{\sin(U+V)\mp1} \end{aligned} \tag{90}$$

It is straightforward to verify that $y''_{UV} = y''_{VU}$. Therefore, we proceed to integrate the above equations in the same way as we did for $x(U,V)$. The integration of the second equation in the above equations is simple

$$\begin{aligned} y(U,V) &= \tfrac{2}{\sqrt{\omega}}\tfrac{\sqrt{\sin(2U)}}{\tan((U+V)/2)\mp1} + G(U) \\ &= \tfrac{2}{\sqrt{\omega}}\tfrac{(1+\cos(U+V))\sqrt{\sin(2U)}}{\sin(U+V)\mp1\mp\cos(U+V)} + G(U) \end{aligned} \tag{91}$$

where $G(U)$ is a function of the single variable U only, and is determined by the first equation. That is,

$$\begin{aligned} &G'(U) = \\ &\tfrac{1}{\sqrt{\omega}}\left(\tfrac{\cos(U-V)}{(\sin(U+V)\mp1)\sqrt{\sin(2U)}} \mp \tfrac{\sqrt{\sin(2U)}}{\sin(U+V)\mp1} - 2\tfrac{(1+\cos(U+V))\cos(2U)}{(\sin(U+V)\mp1\mp\cos(U+V))\sqrt{\sin(2U)}}\right) \end{aligned} \tag{92}$$

Similarly we have

$$G'(U) = \frac{1}{\sqrt{\omega}}\frac{\pm\cos(2U)+\sin(2U)}{\sqrt{\sin(2U)}} \tag{93}$$

Its integration is

$$G(U) = \frac{1}{\sqrt{\omega}}\left(\pm\sqrt{\sin(2U)} + \int\sqrt{\sin(2U)}dU\right) \tag{94}$$

Finally we complete our harmonic coordinate (2)

$$\begin{aligned} x(U,V) &= \tfrac{2}{\sqrt{\omega}}\tfrac{(1+\cos(U+V))\sqrt{-\sin(2V)}}{\sin(U+V)\mp1\mp\cos(U+V)} + F(V), \\ y(U,V) &= \tfrac{2}{\sqrt{\omega}}\tfrac{(1+\cos(U+V))\sqrt{\sin(2U)}}{\sin(U+V)\mp1\mp\cos(U+V)} + G(U) \end{aligned} \tag{95}$$

where

$$\begin{aligned} \lambda &= \lambda_0 + \tfrac{1}{\sqrt{\omega}}\textstyle\int^U dU/\sqrt{\sin(2U)}, \\ \mu &= \mu_0 + \tfrac{1}{\sqrt{\omega}}\textstyle\int^V dV/\sqrt{-\sin(2V)} \end{aligned} \tag{96}$$

and λ_0, μ_0 are integral constants.

By now we have obtained the third solution to rational structure. Our formulas involve sign symbols ±1 or ∓1. However, we have not obtained two solutions in this paper. Instead we have obtained only one solution whose formulas take the upper sign if $\cos(U+V) > 0$ at the corresponding point (x,y) in the physical space or the lower sign if $\cos(U+V) < 0$ at the corresponding point (x,y).

7 The Graph of the Third Solution

We are ready to draw the graph of the third solution to rational structure. Based on the results in the last Section, we do not need the the variables λ, μ to draw our graph.

What we need is the middle variables U, V. With the formula (95) we have obtained the function between the point (x, y) in physical space and the parameter point (U, V). However, we need its inverse function to draw the graph of $\rho(x, y)$

$$\begin{aligned} \rho(x,y) &= \rho_0 \exp(f(x,y)) \\ &= \rho_0 \exp\left(\int P(C)dC\right) \\ &= \rho_0 \exp\left(\int \frac{\sin(C)\pm 1}{\cos(C)}dC\right) \\ &= \rho_0 \exp\left(\ln \frac{1}{1\mp\sin(C)}\right) \\ &= \rho_0 \frac{1}{1\mp\sin(C)} \\ &= \rho_0 \frac{1}{1\mp\sin(U+V)} \end{aligned} \tag{97}$$

The graph is left to be the future work.

References

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