The geodesic precession as a 3-D Schouten precession plus a gravitational Thomas precession.

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Abstract

The Gravity Probe B (GP-B) experiment measured the geodetic precession due to parallel transport in a curved space-time metric, as predicted by de Sitter, Fokker and Schiff. Schiff included the Thomas precession in his treatment and argued that it should be zero in a free fall orbit. We review the existing interpretations regarding the relation between the Thomas precession and the geodetic precession for a gyroscope in a free fall orbit. Schiff and Parker had contradictory views on the status of the Thomas precession in a free fall orbit, a contradiction that continues to exist in the literature. In the second part of this paper we derive the geodetic precession as a global Thomas Precession by use of the Equivalent Principle and some elements of hyperbolic geometry, a derivation that allows the treatment of GP–B physics in between SR and GR courses.

I. REVIEW OF EXISTING INTERPRETATIONS OF THE RELATION BETWEEN THE THOMAS PRECESSION AND THE GEODESIC PRECESSION

A. Parallel transport of the spin vector of a gyroscope

In flat Euclidian three space with a Newtonian time the parallel transport of a vector along a closed curve is rotation free, meaning that after one complete rotation the vector will have an unchanged orientation in space. In Special Relativity, a velocity vector in parallel transport along a circular path will have changed its orientation after one loop with an angle called the Thomas rotation angle. The same goes for the spin angular momentum connected to a gyroscope in circular rotation in Special Relativity. The measurable effect is then called the Thomas precession, acquired by the spin orientation of the gyroscope. Both cases basically deal with parallel transport of a velocity vector in a three dimensional velocity vector space or rapidity space. In General Relativity the deviation from its original orientation of the spin angular momentum of a gyroscope in a gravitational orbit is more complicated and the connected parallel transport of this vector has its setting in four dimensional curved space-time. The deviation from parallel transport can be split in a part that is quite analogous to the Thomas precession but three times as big, called the geodesic precession, and a part that is caused by the fact that the central mass has an angular momentum of its own and this rotating mass is dragging the surrounding space, including the orbiting gyroscope, with it. This dragging effect is called the Lense–Thirring effect. The magnitude of the geodesic precession, or the deviation angle from its original orientation despite the parallel transport procedure, reflects the curvature factor of space-time as caused by mass. The different interpretations existing in the literature as to the connection between the geodesic precession and the Thomas precession is the main subject of this paper. The attempts to grasp parallel transport in curved environments are at the core of understanding space-time. Pauli stated that: *The concept of parallel displacement of a vector has turned out to be more and more fundamental to the geometrical basis of the tensor calculus in Riemannian space. [...] Hessenberg and Levi-Civita found a geometrical, intuitive, interpretation of the curvature tensor by starting from the concept of the parallel displacement of a vector. [...] If, however, a vector undergoes parallel displacement along a closed curve, one obtains a vector which is different from the original vector. This fact can be used for the definition of the* *curvature tensor.*[1]

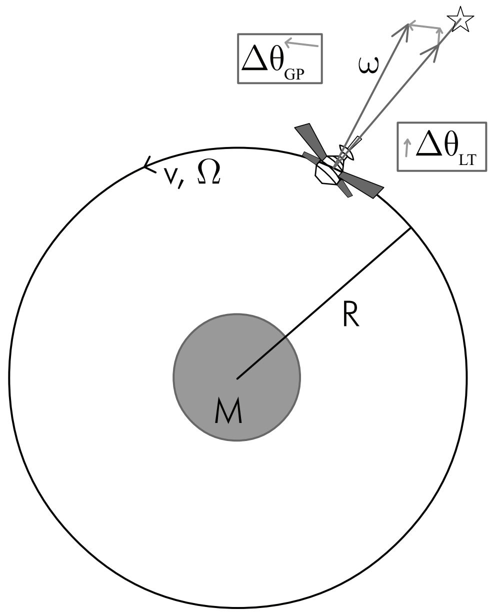

FIG. 1. The Gravity Probe B experiment. The star is at infinity so its rays of light define the Euclidian parallel direction on the orbit. The gyro spin vector ω is at the beginning aligned with the starlight and parallel transported along the orbit but nevertheless deviates from the Euclidian parallel direction after rotation. The deviation angle $\Delta\theta_{GP}$ reflects the curvature of space-time and $\Delta\theta_{LT}$ the dragging of space-time.

B. The Pugh-Schiff proposal leading to the GP-B experiment

In 1916 de Sitter[2] derived the geodetic precession of a spinning satellite in the orbit around a mass M as an effect of curved space-time on the spin axis of rotation of the satellite[3]. The calculations of de Sitter regarding the geodetic precession were extended by Schouten[4] in 1918 and two years later again by Kramers[5] and finally adjusted by Fokker[6]. In 1929 Fokker[7] presented the geodesic precession angle as

$$\frac{3}{2} \cdot \frac{m}{r} \cdot 2\pi, \tag{1}$$

using $G = 1$ and $c = 1$, with $r = R$ and $m = M$ as in Fig.(1).

In 1959-60 Pugh[8] and Schiff[9] independently proposed to test the geodetic precession using a gyroscope in an orbiting satellite. Pugh primarily focussed on measuring the Lense–Thirring precession, with the de Sitter–Fokker precession as an important secondary effect in the experiment. In his 1959 proposal he did not mention the Thomas precession. According to the analysis of Schiff, the total relativistic precession Ω_R experienced by a gyroscope in an orbiting satellite had to be a superposition of the Thomas precession, the geodetic precession and the Lense–Thirring precession,

$$\boldsymbol{\Omega}_R = \boldsymbol{\Omega}_T + \boldsymbol{\Omega}_G + \boldsymbol{\Omega}_{LT}, \tag{2}$$

with

$$\boldsymbol{\Omega}_T + \boldsymbol{\Omega}_G = \frac{1}{2} \cdot \frac{\boldsymbol{F} \times \boldsymbol{v}}{mc^2} + \frac{3}{2} \cdot \frac{GM}{Rc^2} \cdot \boldsymbol{\Omega}, \tag{3}$$

$\boldsymbol{F}$ as an external non-gravitational force and $\boldsymbol{v}$ and m as the velocity and mass of the satellite. Schiff argued that he was the first to derive the total relativistic precession including all three terms. In his derivation he started with an observer on earth, derived the gyroscope's precession using General Relativity and then performed a coordinate transformation in order to get the precession from the perspective of a comoving observer on the satellite, with Eqs.(2) and (3) as the result[10].

Schiff noticed that if $\boldsymbol{v}$ was of constant speed and $\boldsymbol{F}$ the gravitational force, then the resulting Thomas precession was one third of the geodetic precession and of the same sign. If we omit the Lense-Thirring precession, Eq. (3) should then become

$$\boldsymbol{\Omega}_R = \boldsymbol{\Omega}_T + \boldsymbol{\Omega}_G = \frac{1}{2}\alpha \cdot \boldsymbol{\Omega} + \frac{3}{2}\alpha \cdot \boldsymbol{\Omega} = \frac{4}{2}\alpha \cdot \boldsymbol{\Omega}, \tag{4}$$

a result discussed by Schiff for the case of a quasi–uniform motion, when M and R are very large but $g = GM/R^2$ remains constant[11]. The result of Eq. (4) was avoided by Schiff by the demand that the force $\boldsymbol{F}$ in the Thomas precession factor Ω_T should be an external non-gravitational force only. In Schiff's analysis, a satellite in a free fall orbit had no external forces working on it, which meant that it didn't experience any boost in the perspective of a comoving observer and without such a succession of boosts there couldn't be a Thomas precession. In his own words:

> *If a spinning particle is in free fall, as in a satellite, then $\boldsymbol{F} = 0$. For an orbit in the earth's equatorial plane, for example,*
>
> $$\boldsymbol{\Omega} = (3GM/2rc^2)\boldsymbol{\omega}_0 - (2MGR^2/5c^2r^3)\boldsymbol{\omega}$$
>
> *where $\boldsymbol{\omega}_0 = (\boldsymbol{r} \times \boldsymbol{v})/r^2$ is the instantaneous orbital angular velocity vector of the particle.*[9]

If we compare the force of gravity to the electric force, a clear difference arises. Given a satellite and a non rotating central mass with only electric forces working between the two, then the spin of the satellite acquires a relativistic precession equal to the Thomas precession Ω_T. A comoving observer can measure the electric force and use it to calculate the Thomas precession. When the electric force is replaced by the gravitational force, the Thomas precession is zero and only the Geodetic precession is present as relativistic effect. At least, according to Schiff. Because of the principle of equivalence, the comoving observer cannot measure the force of gravity and hence he cannot determine the existence of a Thomas precession. Schiff's assumption comes down to the statement that all forces except gravity produce a Thomas precession as relativistic precession and that gravity produces a relativistic precession not by force but through a four dimensional space-time curvature of the metric, the geodetic precession. For the experiment Schiff proposed, and given his interpretation of Ω_R, the relativistic precession of a gyroscope in an orbiting satellite consists of two factors, the geodetic and the Lense–Thirring precessions.

Eventually, the Pugh-Schiff proposal lead to the GP-B experiment. The orbit of the GP-B satellite was chosen such that Ω_{LT} and Ω_G were independent of each other due to their perpendicular spin axes of rotation. For the GP-B experiment, the predicted geodetic precession was $\Omega_G = 6.6$ arcs/year, given Schiff's Ω_R interpretation. The results of the GP-B experiment was in accordance with Schiff's prediction. A relativistic precession of $\Omega_R = \Omega_G = 6.6018 \pm 0.0183$ arcs/year was measured, were a precession of $\Omega_R = \Omega_G = 6.6061$ arcs/year was predicted, so the predicted geodetic effect was confirmed to better than 0.5 percent[12].

C. The Schiff–Weinberg interpretation.

Since Schiff's proposal, different interpretations of the geodetic precessions and the role of the Thomas precession therein have been proposed by several physicists. We will review the three most significant interpretations, the Schiff–Weinberg interpretation, the Fokker–Parker interpretation and the Schwinger interpretation. All three lines of interpretations have the same outcome for a GP-B like experiment, i.e. the de Sitter–Fokker prediction.

But we start with two possible interpretations, possible from a logical point of view but contradicted by the GP-B outcome. If space-time in a field of gravity was Minkowski flat and gravity was a force just like the electric force, then the outcome of the GP-B experiment should have been $\Omega_R = \Omega_T = 2.2$ arcs/year. This option has been ruled out by the outcome of the GP-B experiment. Based on this result, either the force of gravity is more complicated than a pure electric Coulomb force or space-time is not Minkowski flat, or both apply simultaneously. This means that GP-B has ruled out the option that the field of gravity is a pure scalar field in Minkowski flat space–time.

Another logical possibility might be that the Thomas precession is a pure kinematic relativistic effect that is always present, regardless of gravity as correctly described by the Einstein Equations. In this logical situation, the Thomas precession depends on the orbit only, independent of forces, free fall and curvature. Then for the GP-B experiment an extra $\Omega_T = 2.2$ arcs/year had to be added and a total $\Omega_R = \Omega_G + \Omega_T = 8.8$ arcs/year should be measured. This option has also been ruled out by the GP-B experiment.

This means that if the geodesic precession is correctly *and exclusively* described by the Fokker–Schiff expression, and exists in nature as such, then the Thomas precession in a free fall orbit has to be zero, such is the outcome of GP-B. This is the Schiff interpretation, an interpretation that can also be found in the textbook *Gravitation*, by Misner, Thorne and Wheeler, who state that *The Thomas precession comes into play for a gyroscope on the earth, but not for a gyroscope in a freely moving satelite. [] The general–relativistic term is caused by the motion of the gyroscope through the earth's curved, static space–time geometry.*[13]

There is another interpretation, formulated by Steven Weinberg, very similar to Schiff's interpretation, but the underlying physics is slightly different. According to Weinberg and in his own words, the geodesic precession or the 3/2 term *is essentially just the Thomas Precession caused by gravitation*[14]. Where Schiff was considering an ad-hoc mathematical replacement of the Thomas precession by the geodesic precession, Weinberg focusses on the dynamic, causal continuity instead. The resulting Schiff–Weinberg interpretation is still common and used in most of the graduate textbooks on General Relativity, often in combination with the PPN formalism where the 3/2 term is written as $(\gamma + 1/2)$. We collected a few relevant examples from the literature.

Robertson and Noonan in 1968: *The precession due to the geodesic effect may be contrasted with the Thomas precession as follows. The geodesic precession depends on the metric and appears even in a particle following geodesic motion. The Thomas precession depends* *on the particle's absolute acceleration and thus vanishes for a particle in geodesic motion.*[15]

Barker and O'Connel in 1970: *It is possible to have Ω_T essentially zero by putting the gyroscope in a satellite.*[18]

Lämmerzahl and Neugebauer in 2001: *... with $\boldsymbol{\Omega} = \boldsymbol{v} \times (-\frac{1}{2}\boldsymbol{a} + \frac{3}{2}\boldsymbol{\nabla} U) + \boldsymbol{\nabla} \times \boldsymbol{h}$. The first term $\boldsymbol{v} \times \boldsymbol{a}$ is a special relativistic term, called the Thomas precession which is known from atomic physics. It describes the precession of the spin due to inertial forces. Thus, the second term, $\boldsymbol{v} \times \boldsymbol{\nabla} U$, is a gravity induced Thomas precession, the so called de Sitter precession or geodetic precession. Note that only the Newtonian potential enters this term. The last term is purely post-Newtonian and describes the Lense-Thirring effect.*[17]

McGlinn in 2003: *This is referred to as the de Sitter precession. It is not the analog of the Thomas precession – rather, it is the precession of freely falling frames, not frames that are accelerated, as is the case of the Thomas precession.*[20]

Ryder in 2009: *For geodesic motion $\mathbf{F} = 0$; there is no Thomas precession. On a Newtonian view the gravitational force is $\mathbf{F} = \frac{GMm}{r3}\mathbf{r}$, so the de Sitter precession could be described as being like Thomas precession due to the gravitational force, but with an extra factor of 3. In General Relativity, however, a particle (satellite, gyroscope) in geodesic motion has no absolute acceleration, so no Thomas precession. On the other hand, there is a precession – the geodetic precession – given by $\Omega_{deSitter}$.*[21]

Ohanian and Ruffine in 2013: *Warning: The agreement between parallel transport and gyroscope transport holds only along a geodesic, that is, the gyroscope has to be in free fall. Along a nongeodesic worldline, the equation of motion of the spin is given by Fermi-Walker transport, not parallel transport. Along a nongeodesic worldline in flat spacetime (that is, a trajectory with acceleration), this gives rise to a Thomas precession. [...] The geodesic precession has sometimes been described as analogous to the Thomas precession of special relativity. But this analogy is somewhat misleading: The Thomas precession in the flat spacetime of special relativity results from nongeodesic motion, that is, it results from accelerated motion brought about by the push of some force (without torque) acting on a gyroscope or on a spinning particle, such as an electron. To find a true analogue of the Thomas precession in curved spacetime, we would have to examine the behavior of a gyroscope when some extra, non gravitational force makes it deviate from geodesic motion. The Thomas precession would then arise as an extra contribution to the precession, in excess of the contribution from parallel transport.*[22]

The consequential either–or split between an inertial Thomas precession and a gravitational geodetic precession, with the Thomas precession being zero in free fall orbits when external forces are absent, is what we call the Schiff–Weinberg interpretation. (See also:[16],[19].)

D. The Fokker–Parker interpretation.

The second interpretation can be called the Fokker–Parker interpretation. It is connected to the derivation of geodesic precession by Schouten in 1918[4], which resulted in a precession angle of $\frac{GM}{Rc^2} \cdot 2\pi$, so two third of the actual precession. In 1920, Fokker corrected the derivation of Schouten: *The problem should be put as one of four-dimensional geometry; it is a problem of mechanics, and not a problem of three-dimensional geometry. If this be done properly, then the result is that we are to expect a precession one and a half times the precession foreseen by Schouten.*[6] In 1921 Schouten explained the one and a half difference between his derivation and Fokker's derivation as caused by the use of the spatial part of the linear element of Schwarzschild in his own case and the complete linear element of Schwarzschild in a four dimensional derivation by Fokker. In the words of Schouten[23]: *Now we can show that this difference is caused by the fact, that the fourdimensional problem can be reduced then and only then to a threedimensional one, when the square of the velocity is of order $\frac{G^2M^2}{R^2}$, the square of the real occurring velocity in general being of order $\frac{GM}{R}$.*

In 1963 Fokker commented on these papers in an AIP interview:[24] *The first idea was given by (Schouten), but he made a mistake in his calculation. The idea was very good, however; he said when you have a curvature of space and you go around and you have it parallel to itself, and you make a circuit; then you will have a change of direction, you see. And that struck me very much, and I spoke of it with Lorentz. [...] I found the thought that was mistaken, because in making the circumference he only had looked at the curvature of the surface and he had forgotten what we call now the precession of Thomas. And if you add the precession of Thomas to the curvature precession then you get the right value, you see, and I had the right value.* So in 1920 Fokker explained the difference between Schouten's geodesic precession angle $\frac{2}{2} \cdot \frac{GM}{Rc^2} \cdot 2\pi$ and his $\frac{3}{2} \cdot \frac{GM}{Rc^2} \cdot 2\pi$ result as a difference between working in curved 3-space versus curved 4-space-time. Then in 1963 this same difference was explained as adding or not adding the gravitational Thomas precession angle $\frac{1}{2} \cdot \frac{GM}{Rc^2} \cdot 2\pi$ to the curved 3-space result of Schouten.

In 1968 Parker had the same idea, independent of Fokker, but also motivated by the paper of Schouten. Parkers hypothesis stated that *the actual motion of the spin is just the sum of the effect of the curvature of the physical three dimensional space in which the particle moves, plus the effect of the gravitational field strength in the absence of any spatial curvature.*[25] Parker added: *In 1918, Schouten assumed that in the absence of any other causes of precession, the spin axis of a planet in a circular orbit around the sun will move by parallel transfer in the three-dimensional Schwarzschild space. This assumption yielded two-thirds of the full general relativistic precession. Schouten would have obtained the full value if he had realized that there was an additional effect which would remain even if the three-space were flat. However, the additional effect, as we shall see, is a consequence of the Thomas precession which was not discovered until 1926, eight years after Schouten's work.*

In his calculations Parker added the inertial Thomas precession as caused by an external force and he didn't include the Lense–Thirring frame dragging effect. In our symbolism, the total relativistic precession is then given by Parker as $\boldsymbol{\Omega}_R = \boldsymbol{\Omega}_{Ti} + \boldsymbol{\Omega}_{Tg} + \boldsymbol{\Omega}_C$, and

$$\boldsymbol{\Omega}_R = \frac{1}{2}\cdot\frac{\boldsymbol{F}\times\boldsymbol{v}}{mc^2} + \frac{1}{2}\alpha\cdot\boldsymbol{\omega}_0 + \frac{2}{2}\alpha\cdot\boldsymbol{\omega}_0 = \frac{1}{2}\cdot\frac{\boldsymbol{a}\times\boldsymbol{v}}{c^2} + \frac{3}{2}\alpha\cdot\boldsymbol{\omega}_0, \tag{5}$$

with $\boldsymbol{\Omega}_C$ as the effect of 3-space curvature on the precession of an orbiting gyroscope or the Schouten precession, $\boldsymbol{\Omega}_{Ti}$ as the inertial Thomas precession caused by an external force and $\boldsymbol{\Omega}_{Tg}$ as the weak field, Minkowski space-time gravitational Thomas precession. His end result is equivalent to the Schiff interpretation and in the absence of any external force it equals the geodetic precession of de Sitter and Fokker.

Since the 1969 paper of Parker in the American Journal of Physics, the linear splitting of the geodesic precession in a Schouten precession and a Thomas precession remained popular. In 1972, Fischer wrote:[26] *The Thomas precession is a kinematical effect which depends only upon the velocity and acceleration of the object. Since it does not matter what caused the acceleration the effect has interesting applications to electrons accelerated by the electric field in atoms, neutrons, and protons accelerated by nuclear forces, and earth satellites accelerated by gravity.* The Parker interpretation also appeared in the paper by Shapiro and others published in 1988. We quote Shapiro:[27] *The de Sitter precession may be thought of as having contributions from two sources: The first is the effect of mass on the curvature of space, which results in locally measured angles differing from those measured with respect to the fixed stars. The second source, which contributes half as much as the* *first, is the gravitational analog of the spin-orbit coupling of an electron in an atom.* We translate this into the mathematical equation

$$\mathbf{\Omega}_R = \mathbf{\Omega}_S + \mathbf{\Omega}_T = \frac{2}{2}\alpha \cdot \boldsymbol{\omega}_0 + \frac{1}{2}\alpha \cdot \boldsymbol{\omega}_0 = \frac{3}{2}\alpha \cdot \boldsymbol{\omega}_0 = \mathbf{\Omega}_G, \tag{6}$$

with $\mathbf{\Omega}_S$ as the effect of curvature on the precession of an orbiting gyroscope and $\mathbf{\Omega}_T$ as the spin-orbit term.

In 1988, Everitt referred to the spin-orbit Thomas precession term, as it was also mentioned in the work of Thorne, *as an effect of the lateral motion of the gyroscope throuth the radial gradient in the time dimension of the Schwarzschild metric.* Everitt, following Thorne, wondered if this split in $\mathbf{\Omega}_S$ and $\mathbf{\Omega}_T$ followed the 3 space plus 1 time split in relativity. Regarding this $3+1$ split in the relativistic precession, Everitt added: *An interesting investigation not yet attempted would be to see whether the time gradient part of the gyro precession can be derived from the equivalence principle by an argument analogous to Einstein's argument for light deflection. Schiff would have said no.*[28]

Rindler and Perlick wrote in 1990: *It can be shown fairly simply that two-thirds of the precession can be ascribed to the spatial geometry of the Schwarzschild metric, while one-third is essentially due to Thomas precession; however, the latter is now in the forward rather than the retrograde sense, for it is now the frame of the field that Thomas-precesses around the gyroscope, which itself is free, i.e. unaccelerated.*[29] Rindler stated in his 2001 textbook that in the post–Newtonian approximation: *The total effect, geometric and Thomas, gives the well–known de Sitter precession.*[30] In the same textbook, Rindler also gives a direct calculation of the de Sitter precession via rotating coordinates and then there is no split in two different terms, but just a direct calculation of the geodetic precession in the 4-D Schwarzschild metric, with the remark by Rindler that in flat space–time the same method results in the Thomas precession.

In 2007 Wortel, Malin and Semon interpreted the outcome of the GP-B experiment, in the line of Parker, as an experimental confirmation of the Thomas precession.[31] From the paper of Wortel et. al. we select two quotes. *The Stanford–NASA satellite Gravity Probe B (GP–B), launched in 2004, contains four gyroscopes predicted to precess, in part, due to Thomas precession.* [...] *Consequently, a measurement of the net precession in the satellites orbital plane should detect both Thomas and geodetic precession.* In their paper, these authors explicitly refer to the GP–B experiment as a confirmation of the Thomas precession analogous to its original appearance in quantum physics where the electron with spin orbits the nucleus of a hydrogen atom.

It is our opinion that the most accurate, non–confusing way to describe the Fokker–Parker interpretation of the geodesic precession is to portrait it as a linear splitting of the geodesic precession in a *Schouten precession* and a *Thomas precession.* In this interpretation, the Schouten precession is caused by parallel transport of a gyroscope in curved 3–space in a Schwarzschild metric and the Thomas precession is caused by parallel transport of a gyroscope in a Minkowski metric in which a Newtonian force of gravity exists. The gravitational Thomas precession is somehow connected to the time aspect in the full 4-D space-time Schwarzschild metric, but, as was mentioned by Everett, this connection has not been clarified yet.

E. The Schwinger interpretation and gravitomagnetism I.

A prominent line of interpretations follows the gravitomagnetic analogy of Schwinger.[32] According to Schwinger, $\boldsymbol{\Omega}_R = \boldsymbol{\Omega}_{SO} + \boldsymbol{\Omega}_T + \boldsymbol{\Omega}_{LT}$ with $\boldsymbol{\Omega}_{SO}$ as the spin–orbit coupling term and $\boldsymbol{\Omega}_{LT}$ as derived from the analogy between relativistic electrodynamics and General Relativity. Schwinger calculated the $\boldsymbol{\Omega}_{LT}$ from the analogous situation in electromagnetics where a rotating uniformly charged spherical shell produces a constant magnetic field in its interior and used the correspondences $e \to 2m$, $Q \to 2M$ and $1/(4\pi) \to -G$ to derive the 'outer' Lense–Thirring effect. Then he calculated the spin–orbit coupling precession as being *twice* the Schouten precession. Schwinger derived the spin–orbit coupling term not by using electromagnetic analogs but by using the interaction stress–energy tensor terms for an orbiting satellite. His derivation was not based on parallel transport in a 3–space Schwarzschild metric but on manipulations of the stress energy tensor for a spinning satellite and thus his spin–orbit precession term is not a Schouten precession. If we assume $\boldsymbol{\Omega}_{LT} = 0$, then Swinger stated that

$$\boldsymbol{\Omega}_R = \boldsymbol{\Omega}_{SO} + \boldsymbol{\Omega}_T = \frac{4}{2}\alpha \cdot \boldsymbol{\omega}_0 - \frac{1}{2}\alpha \cdot \boldsymbol{\omega}_0 = \frac{3}{2}\alpha \cdot \boldsymbol{\omega}_0 = \boldsymbol{\Omega}_G. \tag{7}$$

Thus according to Schwinger's 1974 interpretation, the Thomas precession of a satellite in a free fall orbit is not zero but equal to the expected electric analogue. By adding a spin–orbit interaction term of twice the Schouten precession to the negative gravitational Thomas precession, Schwinger recovered the quantitative result obtained by Schiff.

In 1994, Bini, Carini, Jantzen and Wilkins wrote a paper on the Thomas precession in post-Newtonian gravitoelectromagnetism, in which they reported of a split of the geodesic precession in a gravitomagnetic part equal to twice the Schouten precession and a gravitational Thomas precession, which added up as in Eq.(7). Where Schwinger gave a gravitomagnetic derivation of the Lense–Thirring effect, Bini et.al. had a gravitomagnetic derivation of a term equal to twice the Schouten precession.[33]

Holstein and Vető also extended the approach to what they call the spin–orbit interaction term which for both resulted in twice the Schouten precession term. By subtracting the Thomas precession term they arrived at the correct geodesic precession, as did Schwinger. The approaches of Holstein and Vető to arrive at twice the Schouten precession are very different. Holstein connects his derivation to the regular General Relativistic textbook derivation by using the geodesic equation. This equation is then interpreted as the analogue of the Lorentz Force Law, an analogue which is used to arrive at a first order approximation of the geodesic equation much like the Lorentz Force Law. In a similar manner he goes from the General Relativistic spin–field expression to the gravitomagnetic expression for spin in a gravitomagnetic field. This leads to the gravitomagnetic spin–orbit precession as being twice the Schouten precession. And as with Schwinger, Eq.(7) then leads to the correct geodesic precession.[34]

Vető used the gravitomagnetic Biot-Savart law to determine the induced gravitomagnetic field experienced in the satellite due to the orbiting of the Earth and thus arrived at the angular velocity of the gravitomagnetic precession of the gyroscope's spin axis as being twice the Schouten precession.[35] He then argued that since the satellite is orbiting the Earth, the gyroscope is also undergoing the Thomas precession. This Thomas precession also takes place in the orbit plane but opposite the orbital velocity. As a result Vető had $\mathbf{\Omega}_G = \mathbf{\Omega}_{GM} + \mathbf{\Omega}_T$ as in Eq.(7) with $\mathbf{\Omega}_{GM} = \mathbf{\Omega}_{SO} = 2\mathbf{\Omega}_S$.

Both Holstein, in the American Journal of Physics, and Vető, in the European Journal of Physics, argue that their versions of the gravitomagnetic approach as simplifications of the complete General Relativistic derivation are useful for didactic purposes, because they do not need complex General Relativistic mathematics, as the geodesic equation and the GR spin equation involving Christoffel symbols, to arrive at the GP–B results. This may be true, but the fact that they both arrive at twice the Schouten precession, and then subtract the Thomas precession in order to arrive at the correct geodesic precession might have a confusing effect on students when later on will they be confronted with the Fokker–Parker interpretation of the geodesic precession.

F. The PPN approach, Thorne and gravitomagnetism II.

The parametrized post-Newtonian (PPN) formalism was developed by Nordtvedt, Will, Thorne and other physicists for high precision test of relativistic theories of gravity.[37] Thorne and Will listed the Schiff proposal and the group of Fairbank at Standford University working on Schiff's test of GR as one of the possible new experimental precision tests of relativistic gravity that made it necessary to improve the PPN formalism of their time. According to Will and Thorne, the PPN formalism considers all theories of gravity which are compatible with Special Relativity, are compatible with the local equivalence principle and agree with Newtonian gravity in the solar system to the then available accuracy. All theories of this type have to be metric theories, theories that can be put in Riemannian geometric form.[36]

The parameterized post-Newtonian formalism or PPN formalism is a calculation tool that expresses the Einstein equations in terms of the lowest-order deviations from Newton's law of universal gravitation and is applicable to weak fields and slowly moving objects. It assigns to each theory of gravity a set of ten experimentally measurable quantities, the PPN parameters. The values of the PPN parameters of a theory are closely linked to its physical properties such as strength of curvature and drag coefficient.

In the case of the geodesic precession, only the γ parameter is relevant because it determines the strength of the curvature of space–time. In terms of the PPN formalism the geodesic precession is given by Hartle as[16]

$$\mathbf{\Omega}_G = (\gamma + \frac{1}{2}) \cdot \frac{GM}{Rc^2} \cdot \mathbf{\Omega}. \tag{8}$$

Hartle added that a measurement of the geodesic precession thus determines the PPN parameter γ and tests if the value of γ equals exactly 1, as is predicted by general relativity.

The $(\gamma + \frac{1}{2})$ formulation of the PPN formalism as applied to the geodesic precession fits both the Schiff–Weinberg interpretation as the Fokker–Parker interpretation perfectly. It can be looked upon as one single variable, with the GR prediction giving the value 3/2 for the strength of space-time curvature. Or it can be seen as expressing the split between the

Schouten precession connected to the curvature of 3-space strength parameter γ, with GR prediction $\gamma = 2/2$, and the Thomas precession of Special Relativity with the already known and quantum mechanically tested value 1/2.

Thorne starts with a fully non-linear general relativity tensor theory derivation of the total precession of a gyroscope orbiting a rotating earth and then applies a PPN linearization by splitting the resulting precession tensor in three parts. With the appropriate approximation he gets two precession terms from the diagonal parts of the tensor and one precession term from the off diagonal terms. The off diagonal terms give the Lense–Thirring precession $\mathbf{\Omega}_{LT}$ and Thorne calles it the gravitomagnetic effect or the magnetic analogue part of the precession, $\mathbf{\Omega}_{GM}$. The diagonal terms are split in a space-like part to give the Schouten precession $\mathbf{\Omega}_S$ and Thorne calles it the Schwarzschild curved 3-space metric effect, $\mathbf{\Omega}_C$. Another split of the diagonal term result in a time-like part to give the Thomas precession $\mathbf{\Omega}_T$ and this part is called the gravitoelectric effect or electric analogue effect but also the spin–orbit part of the precession , $\mathbf{\Omega}_{SO}$.

Thorne's approach starts with the Weinberg point of view of a fully non-linear Riemannian metric approach of General Relativity but then he switches to Applied General Relativity, for which he needs a usable linearization of the resulting precession in curved space-time. This results in a Fokker–Parker formulation in a mixed curved–gravito–electro–magnetic nomenclature. In Thorne's words: *I shall introduce you to some unusual but powerful viewpoints about general relativistic gravity: (i) the split of the spacetime metric $g_{\alpha\beta}$ and its associated forces into a "gravitoelectric field" $\mathbf{g}$, a "gravitomagnetic field" $\mathbf{H}$, and a space curvature (not spacetime curvature) with metric g_{ik}.*[38] The splitting of the diagonal terms in a space-like part and a time-like part can be done in many different ways, one of which we already mentioned as a result reported by Bini et.al.[33] The discussion regarding the way how to linearize the diagonal of the GR tensor in a space-like and a time-like part belongs on a post-doc level. We just categorize the resulting precession-interpretations.

Although Thorne arrives at the Fokker–Parker split of the geodesic precession, he is not saying that there is a Thomas precession connected to a gyroscope in a geodesic, free fall orbit.[38] He doesn't fall back on Minkowski space-time and a Special Relativity calculation to arrive at the Thomas precession part of the geodesic precession, as Fokker and Parker did. Thorne starts with the diagonal terms in the resulting GR precession tensor and with some approximations and linearization of the result arrives at a term that can be called the gravitoelectric time-like part of the precession, comparable to the Thomas precession due to a Newtonian force of gravity in a Minkowski space-time. The practical use in astrophysics justifies his method and the resulting expressions. Thorne starts with the full GR position as did Weinberg, and at the end he has a linearized result analog to Parker's.

It is the resemblance of the results in GR after linearization that motivates Thorne to use the electromagnetic analogy, whereas Schwinger went the other way and used the resemblance of the electromagnetic calculations to GR expressions to arrive at the analogy. This might be the reason why Thorne ends with the Schouten precession and Schwinger (and Holstein and Vető) with twice the Schouten precession in their linear splitting of the geodesic precession in two terms.

G. Three interpretations of the geodesic precession's relation to the Thomas precession and the Schouten precession

So basically there are at the moment three lines of interpretations of the relativistic precession of a gyroscope in a free fall orbit around a non rotating mass. First the interpretation in the line of Schiff and Weinberg, in which the geodetic precession, derived in a four dimensional Schwarzschild space-time metric, replaces the Thomas precession that is derived in a four dimensional Minkowski space-time metric. Second the interpretation in the line of Fokker–Parker, which follows the $3+1$ space-time split as a split in a Schouten precession due to 3-space Schwarzschild curvature and a Thomas precession due to Newtonian gravity in a flat space-time Minkowski metric on a non-geodesic classical orbit. And third the interpretation in the line of Swinger in which the geodesic precession is split in a term that equals twice the Schouten precession, from which is subtracted the second term, the (gravitoelectric) Thomas precession, to give the total geodesic precession.

Because all three interpretations predict the same outcome, GP–B cannot verify the one and falsify the other. But given the history of the geodesic effect and also from a didactic point of view, we think that in educating the GP–B physics it is less confusing for the students if the Schwinger–Holstein–Vető split of the geodesic precession in twice the Schouten precession and an inverse Thomas precession is avoided at the level in between Special Relativity and General Relativity. Presenting both the Schiff–Weinberg interpretation and the Fokker–Parker interpretation introduces already enough controversy and discussion regarding the status of the Thomas precession in a free fall orbit. And from an applied GR or PPN point of view, both the Schiff–Weinberg and the Fokker–Parker interpretations fit well into the PPN formalism and formula for the geodesic precession. This supports the Thorne–Will reasoning that the PPN formalism is not restricted to one single theory of gravity, a strict interpretation of Einstein's GR, but includes a broad range of metric theories of gravity. So in teaching GP–B physics to students in between SR and GR courses, the Thomas precession, the Schouten precession and the full geodesic precession can be presented in combination with the PPN formalism, the applied GR linearization method with $\mathbf{\Omega}_G = \mathbf{\Omega}_S + \mathbf{\Omega}_T$ and with a discussion regarding the problematic status of the Thomas precession in a geodesic orbit.

As for the gravitomagnetic vocabulary, in the case of the Thomas precession it is an obvious analogy that already exists in the classical treatment of the planetary model and in the resemblance between the laws of Newton and Coulomb. When applied to the Schouten precession, as Bini et.al. and Vető, its leads to twice the Schouten precession, a result that is best avoided at the level inbetween SR and GR. Schwinger, Holstein and Thorne all three do not use the gravitomagnetic analogy for the curved 3-space part of the precession. The only additional value of the gravitomagnetic analogy is found in the case of the Lense–Thirring precession, presented as a spin–spin coupling effect with an important analogy in the hyperfine structure of quantum mechanical treatment of the hydrogen atom. We believe that in treating the physics of the GP–B precession, it is important to stress the fact that the Schouten precession has no gravitomagnetic analogy. This means that for parallel transport in a 3-space Schwarzschild curved metric there exists thus far no viable analogy in Maxwell's theory of electromagnetism.

II. THE GEODESIC PRECESSION DERIVED FROM SPECIAL RELATIVITY AND THE EQUIVALENCE PRINCIPLE

A. Motivations for a new derivation of the geodesic precession

In the following sections we will give a derivation of the geodetic precession that we will position in the line of the Weinberg interpretation, according to which the geodetic precession is just the gravitational version of the Thomas precession. We will use only

Special Relativity and the Equivalence Principle to derive the geodetic precession. Although we place our approach, involving the equivalence principle, in the line of the Schiff–Weinberg interpretation, according to Schiff, the geodetic precession and the equivalence principle were mutually exclusive. In 1960, Schiff stated that: *if the precession of the gyroscope axis is to be significant, the gravitational field cannot be regarded as uniform, and the equivalence principle is not applicable.*[11]

We will argue that our derivation is not in conflict with Schiff's assertion because we use a different geodesic trajectory for our observer. Schiff's statement applied to an observer comoving with the gyroscope in the satellite. Our observer will be in a free fall from infinity towards the surface of the planet and will pass by the orbiting satellite in which the gyroscope is placed. Schiff used one observer on the planet and one on the satellite. We will use a third observer, one on a geodesic that connects the distant star with the observer on the planet by a straight line. In Fig.(4) Schiff's geodesic is the trajectory 2 and our geodesic is trajectory 1.

In our derivation we need hyperbolic geometry as the metric of rapidity space in Special Relativity to derive the gravitational Thomas precession. Our SR approach is not new but a concise presentation doesn't exist in the literature so we will present the hyperbolic SR way to the Thomas precession first. Our subsequent derivation of the geodesic precession based on SR and the Equivalence principle has the advantage that it can be presented at the undergraduate level as an introduction to GR. And the derivation will allow us to sketch a new perspective on the Schouten precession and on the status of the Thomas precession in a free fall orbit.

B. Thomas precession and hyperbolic geometry

In 1925 Uhlenbeck and Goudsmit, also from Leiden University in the Netherlands as were de Sitter, Kramers and Fokker, introduced the concept of electron-spin. With this idea of electron-spin and its precession in the orbit around the nucleus they managed to explain the doublet terms in the Hydrogen atom's spectral lines in the Röntgen region and also the a-normal Zeeman-effect.[40] But they didn't manage to explain the factor 2 difference in the magnitude of the coupling of their electron spin to its intrinsic magnetic momentum needed to explain the correct width of the splitting of spectral lines in the a-normal Zeeman-effect.[41] They send their results to Bohr, who discussed it with Kramers, who was working in Kopenhagen on his Ph.D. under Bohr's direction. While Bohr and Kramers were arguing about the idea of spin, Thomas, who was there too at the time, joined the discussion: *I being a reasonably brash young man in the presence of Bohr said, "Why doesn't someone work it out relativistically." Kramers who had known of the earlier work on the motion of the moon by De Sitter said to me "It would be a very small relativistic correction. You can work it out, I won't."*[42] Thomas did work out the idea of a relativistic precession of the orbit of the electron, the Thomas precession, that produced effects that had to be added to the precession of the electron–spin in its own rest–system.[43] He showed that this extra orbital precession of the electron as a gyroscope was a consequence of relativistic velocity addition applied to rotations, two successive Lorentz boosts added up to one Lorentz boost and a rotation. Thomas explained the factor-two difference in the coupling constant α in the Uhlenbeck and Goudsmit approach as a consequence of the kinematics involved, according to which the relativistic precession of the orbit of the electron had to be added to the precession of the spin in the electrons own reference frame.[44]

The old quantum theory[45] as embodied by Bohr and Sommerfeld around 1924 was the context in which the discovery of electron spin and relativistic precession of the spin axis of the electron in a circular orbit and the relativistic precession of this orbit itself took place. Sommerfeld's book "Atombau und Spektrallinien", translated as "Atomic Structure and Spectral Lines", expressed the old approach which was centered around the model of electrons orbiting a nucleus analogous to the planetary system but subjected to Bohr's restrictive quantum postulates.[47] Part of the problem of the fine structures was solved by the innovative work of Uhlenbeck, Goudsmit and Thomas, who were operating in the context of the Bohr–Sommerfeld approach. In the words of Pais: *the discovery of spin, made after Heisenberg had already published the first paper on quantum mechanics, is an advance in the spirit of the old quantum theory, that wonderfully bizarre mixture of classical reasoning supplemented by ad hoc quantum rules.*[46] In the new Quantum Mechanics of Heisenberg, Schrödinger, Born, Pauli and Dirac, the atomic theory based on the model of the semi-classically orbiting electron became outdated and so did the model of the electron with an internal structure of a spinning gyroscope. Outdated as it might be in Quantum Mechanics, it is this model that seems to be one of the few existing close connections between spin in Quantum Mechanics and spin in General Relativity.

There was another line of thought that afterwards merged with Thomas' theory on the precessing electron spin in orbit around a nucleus, the one of hyperbolic geometry as applied to relativistic velocities. This line started before Thomas' work on the spinning electron, and even before the work of de Sitter and Fokker on the geodetic precession. Mathematicians and physicians like Sommerfeld, Robb, Varičak, Borel, Föppl and Daniell worked on the hyperbolic formulation of Special Relativity as it was already nascent in Minkowski's approach.[48] In 1911 Robb introduced the rapidity φ as connected to the velocity v.[49] Given this velocity v, we can define γ as

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{9}$$

and β as $\beta = \frac{v}{c}$. The rapidity φ is then defined through $\beta = \tanh \varphi$ or through

$$\gamma = \cosh \varphi. \tag{10}$$

Robb and Varičak worked out relativistic velocity addition using rapidities. Given relativistic velocities v_1 and v_2, we can add them through the rapidities φ_1 and φ_2 according to the rule formulated in 1912 by Varičak[50]

$$\cosh(\psi) = \cosh(\varphi_1)\cosh(\varphi_2) + \sinh(\varphi_1)\sinh(\varphi_2)\cos(\alpha), \tag{11}$$

with the angle α as the angle between the two velocity vectors. If the two velocities are perpendicular to each other, then $\cos(\alpha) = 0$ and the addition of the rapidities reduces to

$$\cosh(\psi) = \cosh(\varphi_1)\cosh(\varphi_2). \tag{12}$$

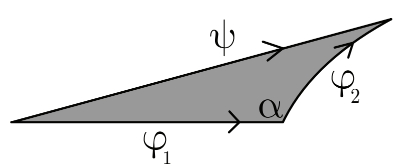

FIG. 2. The hyperbolic rapidity triangle.

In 1913 Borel formulated what was afterwards called the Thomas rotation and the Thomas–Wigner rotation angle.[51] Borel stated that if the velocity vector of observer A described a closed path then the axis of A remained parallel for A but rotated with an angle equal to the enclosed area for an observer who at all times moved at the initial and final velocity of observer A. In Fig.(2), this means that if we have for the rapidities as vectors $\boldsymbol{\varphi}_A = \boldsymbol{\varphi_1} + \boldsymbol{\varphi_2} - \boldsymbol{\psi}$ then the rotation angle of $\boldsymbol{\varphi}_A$ equals the gray area of Fig.(2).

In the formulation of Thomas, if we have a local inertial frame of reference undergoing two subsequent Lorentz boosts, then the result is one Lorentz boost and a rotation. In terms of rapidities, if the first boost is represented by he rapidity φ_1 and the second boost by he rapidity φ_2, then the resulting Lorentz boost can be represented by the rapidity ψ. The rotation acquired by $\boldsymbol{\psi} = \boldsymbol{\varphi_1} + \boldsymbol{\varphi_2}$ is what we now call the Thomas-Wigner rotation angle ϑ_{TW}. This angle is equal to the area of the rapidity triangle on the Poincaré disk, the gray area in Fig.(2).

Borel's description fits nicely with the approach of Rhodes and Semon towards the Thomas–Wigner rotation angle and the Thomas precession, especially Rhodes and Semon's clear and graphic representation of hyperbolic geometry as applied on the Poincaré rapidity disk and depicted in fig. 7, 8 and 9 in the Rhodes–Semon paper.[52] The Thomas precession angle was described by Borel as follows: *If point A describes a closed curve or, for greater clarity, a polygon whose sides are very small arcs of great circles, we can define a closer and closer correspondence between the directions that will be called parallel. We know that when A will have returned to the starting point, the axes, at each moment supposed to be parallel to the axes of the neighboring point, have rotated in reality by an angle equal to the surface of the spherical polygon.* Borel described the rapidity disk version of parallel transport of a vector along a closed path in a curved metric. This description nicely fits the graphic representation of Rhodes and Semon's *Fig.(12), Polygonal approximations to curved paths in rapidity space.*

Our representation of the Thomas precession is given in Fig.(3). If φ_1 is the rapidity connected to an orbiting velocity v_{orbit} and φ_2 is the rapidity connected to the change in velocity dv_{orbit}, then the resulting rapidity ψ has an equal magnitude but a slightly different direction than φ_1. After one orbit in real space, the rapidity vector on the Poincaré disk has also closed a circle on the hyperbolic disk. The total rotation or total Thomas-Wigner rotation angle of the orbiting object is simply the summation of all the infinitesimal Thomas-Wigner rotation angles. Geometrically, this is equal to the surface of the disk with the rapidity ψ as radius, see Fig.(3). Now, in hyperbolic geometry the area A of a disk with radius ψ is given by[53]

$$A = (\cosh\psi - 1)2\pi. \tag{13}$$

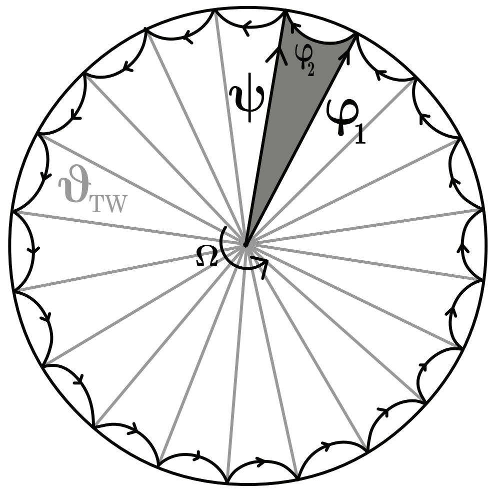

FIG. 3. A segment of the hyperbolic rapidity circle. The global Thomas precession angle equals the area of the circle on the hyperbolic disk, which is just the summation over all the local Thomas–Wigner rotation angles.

We have $A = \Sigma\vartheta_{TW} = \vartheta_T$ and if we divide the angles ϑ_T and 2π by the rotation time T we get the angular velocities Ω. The direction of the Thomas precession is opposite to the angular momentum of the orbiting object. The Thomas precession results as

$$\boldsymbol{\Omega}_T = -(\cosh\psi - 1)\boldsymbol{\Omega}. \tag{14}$$

If we go back to the velocity given in γ, we get the Thomas precession as

$$\boldsymbol{\Omega}_T = -(\gamma - 1)\boldsymbol{\Omega}. \tag{15}$$

The second order in β approximation results in the more familiar formulations of the Thomas precession

$$\boldsymbol{\Omega}_T = -\frac{1}{2}\frac{v^2}{c^2}\boldsymbol{\Omega} = -\frac{1}{2}\frac{\boldsymbol{F}\times\boldsymbol{v}}{mc^2} = -\frac{1}{2}\frac{\boldsymbol{a}\times\boldsymbol{v}}{c^2}. \tag{16}$$

The Thomas rotation angle $(\cosh\psi - 1)2\pi$ was already derived by Föppl and Daniell in their 1913 paper[54], whereas Borel only gave a graphic description of this rotation angle. The Föppl–Daniell formula equals the one derived in 1926 by Thomas. In the physics literature, the precession has been named after Thomas because he could solve a physical enigma with it, a correct calculation of the anomalous Zeeman effect, whereas Borel, Föppl and Daniell before him remained on a mathematical, slightly kinematic, level, without connecting it to a concrete physical application. Since the time of Thomas' derivation of the electron spin precession, many derivations and expressions of the Thomas precession have been published.[55]

In our view, the hyperbolic approach is among the more comprehensible and elegant. The hyperbolic formulation of the Thomas precession of Eq.(14) can also be found in Fokker's 1960 book on relativity, a book in which he did not yet connect the Thomas precession to the geodesic precession as in the interview three years later.[39]

C. The geodetic precession from Special Relativity and the Equivalence Principle

Given a gravitational mass M and the Newtonian potential, then zero gravity exists in infinity from M. A space ship called Elevator with mass m is in a free fall towards the planet with mass M. The free fall started at infinity, with $v_\infty = 0$. At a distance R from the center of the planet, the space ship has gained kinetic energy equal to the loss of potential energy. Classically, with low velocities, we have

$$v^2 = \frac{2GM}{R}. \tag{17}$$

According to the Equivalence Principle, the space ship is an inertial system so the laws of special relativity apply to it. This form of the Equivalence Principle equals the one Everitt refers to in his paper as used by Schiff, Eriksson and others. See ref.[33] in Everitt's Overview[28]. Our approach is also analyzed in detail by Rindler in reaction to Schiff's derivation of the bending of light round a mass point[56]. We use the part of Schiff's approach that was approved by Rindler and avoid the crucial point of Rindler's objection, the premature and ad hoc use of a curved 3-space metric.

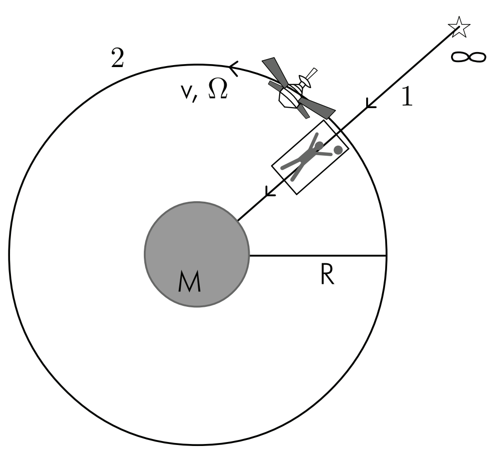

FIG. 4. Escape pod GP–B launched from the Elevator

The space ship Elevator has a transparent floor and the astronauts observe a massive object M accelerating towards it. Suppose the Elevator itself cannot avoid collision but the ship's astronauts can use an escape pod to fly into safety. The captain has to stay behind because we need him as an inertial observer of the events. The escape pod, named GP–B, is launched in two successive boosts. The first boost gives the pod an escape velocity v_{esc} relative to the space ship and radially away from the colliding planet. This first boost compensates the velocity of the space ship and results for a small moment in an absence of velocity between escape pod and planet with mass M. The second boost is perpendicular to the first one and gives the pod an orbiting velocity v_{orb}. Classically, we have for the escape velocity for an object at rest at a distance R from a planet with mass M

$$v_{esc}^2 = \frac{2GM}{R} = -2\phi \tag{18}$$

and for the orbiting velocity, due to the virial theorem,

$$v_{orb}^2 = \frac{GM}{R} = -\phi. \tag{19}$$

The two boosts can be delivered to the escape pod in such a way that both boosts still happen within the local inertial area of the space ship Elevator. The captain of the Elevator may apply the rules of Special Relativity regarding the addition of two successive Lorentz boosts. If we use the rapidities φ_{esc} and φ_{orb}, then the total rapidity of escape pod GP–B is ψ.

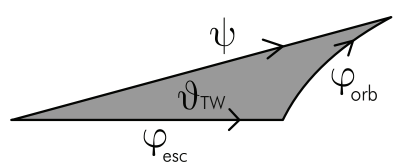

FIG. 5. The escape pod rapidity triangle.

Observers on the planet or at rest in infinity, but all on the same radial line that connects the space ship to the center of the planet, should observe the same total velocity of the escape pod v_{total} and its related rapidity ψ with

$$\cosh(\psi) = \cosh(\varphi_{esc}) \cosh(\varphi_{orb}). \tag{20}$$

The escape pod has been given a total velocity v_{total} and rapidity ψ, not only in the perspective of the captain of the about to crash space ship, but also in the perspectives of the outside observers at infinity and at the planets surface, as long as they interpret it as the velocity given to the escape pod by the space ship and relative to the space ship during the small window of its launch by two successive boosts. This velocity enabled the escape pod to acquire a stable orbit around the planet. The launch events and their respective interpretations happened in the local inertial area of the space ship. The fact that the escape pod acquires a stable orbit is a global event and can only be observed by the external observers at infinity and at the surface of the planet.

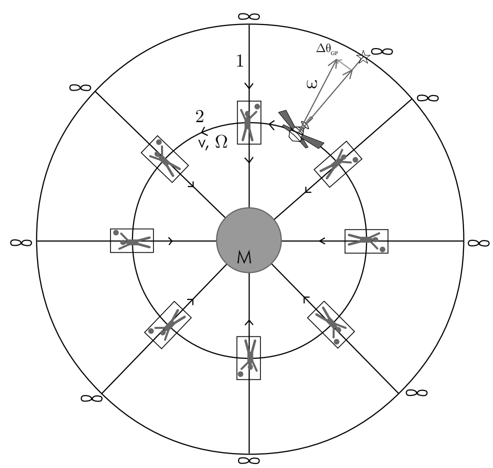

FIG. 6. Escape pod GP–B as seen from a large number of Elevators

We will go global by adding more and more space ships of the Elevator type in free fall from infinity as in Fig.(6). Assume that the escape pod is in a stable orbit around the planet with mass M. Let the velocity situation at launch be represented by the hyperbolic triangle A in Fig.(7). Let there be an identical space ship in free fall from rest at infinity towards the planet. This space ship B sees the escape pod closely passing by with rapidity ψ as in triangle B, so after one quarter of the pod's orbit. During the fly by of the pod relative to the space ship B, the pod can be observed in the local inertial area of the space ship. The captain of ship B will observe a pod with rapidity ψ if the time interval of the fly by observation is such that the relative acceleration between escape pod and space ship can be ignored. If captain B observes this rapidity, then should captains C and D, and all other passing by similar free fall space ships on there way to crash on the surface of the planet. They all see the same rapidity, but under a different angle. This rapidity changes direction continuously and when the escape pod has made one revolution, so does the rapidity ψ.

Where a non–inertial observer on the planet will see an escape pod with velocity v_{orb} and

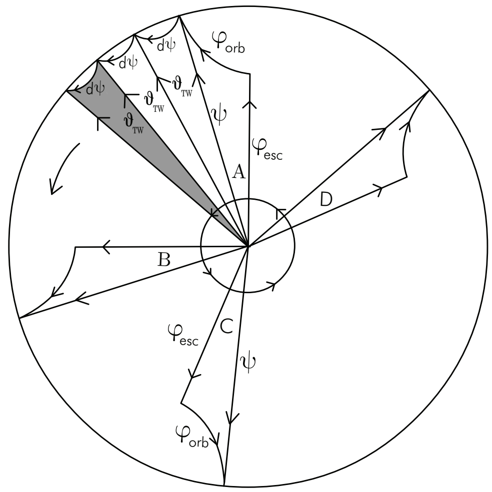

FIG. 7. The hyperbolic rapidity circle for the global escape pod as seen from a sequence of passing free falling space ships. The area swept by ψ represents the global Thomas precession.

connected rapidity φ_{orb}, the inertial observers in the space ships will witness an escape pod with rapidity ψ. We can construct the trajectory of the escape pod on the basis of a large number of inertial observers on the free fall space ships, thus forming a polygon around the planet with radius R and polygon sides dr. On the rapidity disk we get a similar figure with as radius the rapidity ψ and rapidity changes $d\psi$. This way of going global on the basis of local inertial frames is the usual procedure in treating accelerations in the context of Special Relativity. In our procedure, the use of inertial space ships in free fall from infinity and initially at rest connects all inertial observations to the same zero field potential of ϕ and to the same value of v_{esc}. On the rapidity disk the global situation as reconstructed from our inertial observers is given in Fig.(7).

If we compare Fig.(7) with Fig.(3), it will be clear that, in the perspective of the free fall geodesic reference frames, the escape pod experiences a Thomas precession. From the perspective of our polygon–like connected free fall captains, the Thomas precession angle is represented by the area swept by the escape pods rapidity ψ on the hyperbolic disk during one orbit, with $A = \Sigma\vartheta_{TW} = \vartheta_T$. The connected global Thomas precession acquired by the escape pod is given by the formula

$$\Omega_T = (\cosh\psi - 1)\Omega = (\cosh(\varphi_{esc})\cosh(\varphi_{orb}) - 1)\Omega, \tag{21}$$

which can be written as

$$\Omega_T = (\gamma_{esc}\gamma_{orb} - 1)\Omega. \tag{22}$$

If we combine Eq. (9) and Eq. (18) we get

$$\gamma_{esc} = \frac{1}{\sqrt{1+\frac{2\phi}{c^2}}} \tag{23}$$

and Eq. (9) and Eq. (19) gives

$$\gamma_{orb} = \frac{1}{\sqrt{1+\frac{\phi}{c^2}}}. \tag{24}$$

We can insert Eq. (23) and Eq. (24) in Eq. (22) to get

$$\Omega_T = (\frac{1}{\sqrt{1+\frac{2\phi}{c^2}}}\frac{1}{\sqrt{1+\frac{\phi}{c^2}}} - 1)\Omega \tag{25}$$

and

$$\Omega_T = (\frac{1}{\sqrt{1+\frac{3\phi}{c^2}+\frac{2\phi^2}{c^4}}} - 1)\Omega. \tag{26}$$

If we neglect the second order in ϕ/c^2 term, we get

$$\Omega_T = (\frac{1}{\sqrt{1+\frac{3\phi}{c^2}}} - 1)\Omega. \tag{27}$$

In a first order in ϕ/c^2 Taylor expansion the result is

$$\Omega_T = -\frac{3\phi}{2c^2}\Omega \tag{28}$$

or

$$\Omega_T = \frac{3}{2}\cdot\frac{GM}{Rc^2}\cdot\Omega = \frac{3}{2}\alpha\cdot\Omega, \tag{29}$$

a result that equals the geodetic precession Ω_G as it has been measured by GP-B.

The crucial point in our reconstruction of the global orbit on the basis of local observations is that all local inertial systems, our space ships, started from a gravity free position at infinity and with the same velocity $v_\infty = 0$, let's say they all came from a circle defined by the GP–B guide star IM Pegasi (HR 8703). The fact that the global circular orbit of GP–B is acceleration connected is countered by having many space ships, with the polygon approximation that the change from one space ship to the next can be treated as rectilinear or acceleration free. In the limit of an infinite number of Elevators we thus realize parallel transport of the gyroscope angular momentum axis from one Elevator to the next and thus from one GP–B position to the next. When the GP–B with its gyroscope is back to its initial launching velocity, the gyroscope axis will have rotated with an angle equal to the enclosed area on the rapidity disk. In hyperbolic kinematics the rapidity disk metric determines the rotation angle and the precession, with the space-time metric as secondary or non-relevant for the precession.

If we go back to Rindler's analysis of Schiff's use of the equivalence principle to derive the bending of light round a mass point[56], we can specify that we did not use a space-time metric that we borrowed from General Relativity to derive the geodesic angle of rotation in the way Schiff did, at least did according to Rindler, to derive the angle of the bending of light around a mass point. We used a rapidity metric, the one specified by hyperbolic special relativity. Of course we started with a Newtonian flat space metric by using the virial theorem to determine the orbiting velocity

$$v_{orb}^2 = \frac{GM}{R} = -\phi$$

as an initial condition. But the virial theorem itself is a statement about energies, not about the space-time metric so it might well be more general than the Newtonian context we borrowed it from and we can look at it as a good lowest approximation of the orbiting velocity of the satellite in whatever space-time metric.

The interesting result is that we have to conclude that the space-time metric around a gravitating mass is curved in such a way that a comoving observer on the orbiting geodesic of the satellite's gyroscope will calculate the same precession, but for him based on the analysis of the motion through this curved space-time metric. In the perspective of an observer comoving in the rest frame of the gyroscope in free fall, there can be no outside observable force and no relative motion of the gyroscope so this observer has to place the gyroscope all around on the origin of his rapidity disk, resulting in a zero surface and thus a zero precession angle. But he would still measure the geodesic precession and thus he would be forced to look for a cause outside the realm of Special Relativity and the method of parallel transport of the gyroscope's velocity vector on the rapidity disk with a hyperbolic metric.

III. CONFRONTATION WITH THE EXISTING INTERPRETATIONS

Schiff's either or interpretation of the relation between the geodetic precession and the Thomas precession was already attenuated by Weinberg's interpretation of the geodetic precession as a gravity caused Thomas precession. Our derivation of the geodetic precession has just that form, a gravity caused Thomas precession. But the either or interpretation is also present in our derivation, because we choose the perspective of an inertial frame in free fall from an at rest initial position in infinity, without excluding the perspective of an comoving observer in the satellite, moving side by side with the gyroscope. If this comoving observer wants to describe the same gravity caused precession as explained from his position, he will need a space-time metric theory of gravity as for example General Relativity provides to explain the precession of the gyroscope. From the perspective of a comoving observer using the rapidity disk method of SR, the de Sitter–Fokker theory is the whole thing and the Thomas precession on his geodesic motion in a free fall orbit must be absent. So in a certain sense, the comoving metric field perspective and the infinity inertial perspective are complementary either–or perspectives, where only one comoving observer is needed beside the gyroscope and an infinite number of observers on a crash course are needed to make our free fall perspective accurate enough for a first order approximation (of parallel transport).

As for the Fokker–Parker interpretation, where the relativistic precession is build from two different sources, first a curved 3-space caused Schouten precession and second a classical Thomas precession caused by a spin–orbit interaction as

$$\boldsymbol{\Omega}_G = \boldsymbol{\Omega}_S + \boldsymbol{\Omega}_T = \frac{2}{2}\alpha \cdot \boldsymbol{\omega}_0 + \frac{1}{2}\alpha \cdot \boldsymbol{\omega}_0 = \frac{3}{2}\alpha \cdot \boldsymbol{\omega}_0, \tag{30}$$

it is possible to interpret our derivation in this line of thought. For this we go back to the words of Schouten, when he analyzed the difference between his result and Fokker's result[23]: *Now we can show that this difference is caused by the fact, that the fourdimensional problem can be reduced then and only then to a threedimensional one, when the square of the velocity is of order* $\frac{G^2M^2}{R^2}$*, the square of the real occurring velocity in general being of order* $\frac{GM}{R}$*.*

If we were able to reduce the square of the orbital velocity in Fig.(7) slowly from $v_{orb}^2 \sim \phi$ to $v_{orb}^2 \sim \phi^2$ without changing the radius of the orbit and the geodesic character of its orbit, then Eq.(25) would not lead to the geodesic precession but to the Schouten precession, exactly as Schouten did get in his General Relativistic analysis of the problem. Such a slow reduction of the orbital velocity without changing the geodesic or free fall character of the satellite's motion is of course physically impossible, due to the virial theorem.

But the fact that if we apply the same change as Schouten did in his GR analysis we get the same outcome in our SR derivation, means that we can confirm Schouten's GR analysis based on just SR and the Equivalence Principle. On the other hand, if, in our analysis, we go from gravity to electricity, then we as an uncharged neutral observer can stay stationary at a distance R from the charged sphere in the center while a charged satellite is passing by with orbital velocity. In that case Eq.(25) with a zero free fall velocity would lead to the Thomas precession. By manipulating theoretically the free fall or escape velocity and the orbital velocity we can arrive at either the Schouten precession or the Thomas precession. This motivates the same split of the result of our derivation in a Schouten precession and a Thomas precession, as long as it is interpreted in the line of Thorne and the PPN formalism as a split at the end of analysis useful for applied physics and not interpreted as a linear split with its origin at the beginning of the causal chain of events. Our relativistic precession addition follows relativistic rapidity addition, and that on its turn is analogous to relativistic velocity addition. The last one is clearly not a simple linear superposition of velocities. Thus, the assumption that a total relativistic precession can be given as a simple superposition of two independent relativistic precessions looks like a highly optimistic approach. Linearization at the end is possible, whereas it is highly problematic at the beginning of the derivation.

The Schwinger interpretation seems without connection to our derivation. Nowhere do we arrive at a precession that is twice the Schouten precession. This does in no way mean that one single interpretation can be given a higher rank on the interpretation scale. All interpretations give a correct GP–B prediction and thus cannot be falsified.

In our opinion, the Schiff–Weinberg interpretation seems best appropriate for those who engage in a full course of General Relativity. The discussion how the precession tensor of GR is best split in a space-like term and a time-like term belongs in our view at a post GR course level. Our Special Relativity plus Equivalence Principle derivation seems appropriate for those students who have no working knowledge of General Relativity but a good training in Special Relativity and are given a basis knowledge of hyperbolic geometry. The various gravitoelectromagnetic approaches are best kept for the post-doc level, due to their plurality.[57] We believe that he Schwinger interpretation, with $2\Omega_s$, is best discussed in that post-doc context.

Our derivation has the advantage of introducing the students to the concept of parallel transport in curved metrics in a way that can be visualized. The rapidity space approach is equivalent to Sommerfeld's approach using a sphere with a complex radius, which in turn has most of its properties in common with ordinary spheres.[58] On an ordinary sphere one can easily demonstrate the fact that parallel transport of a vector produces a change in direction. From a real 3-space sphere you go to a complex 3-space sphere and from there to the hyperbolic rapidity disk.

Our analysis using the rapidity disk showed that in free fall the precession angle during parallel transport must lead to the conclusion that for a comoving observer in orbital geodesic the only way to explain the parallel transport precession angle is by assuming a curved space-time metric. Hence General Relativity.

The Fokker–Parker interpretation and the PPN formalism follow naturally from our derivation and should be part of the treatment. Discussion of the Fokker–Parker interpretation with its unavoidable but problematic linearization of the geodesic precession in a 3-space Schouten effect and a time-like Thomas effect seems useful in showing the intrinsic fusion of space and time in General Relativity into a space-time quadruple. Our derivation put in PPN form $\Omega_G = \Omega_S + \Omega_T = (\gamma + 1/2)\Omega_T$ can be interpreted as a GR independent first order approximation SR plus EP derivation of the space-time curvature parameter γ, together with the remark that the full PPN formalism there are ten parameters due to the fact that a symmetric tensor in the full theory of relativity has ten independent variables. Our derivation of the geodesic precession based on hyperbolic relativity and the equivalence principle has in our view its natural place in between a course in Special Relativity and General Relativity, either at the end of the first or at the beginning of the latter.

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