A NEW EXACT SOLUTION OF EINSTEIN'S EQUATIONS

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September 20, 2013

Abstract. This article presents a new exact solution of Einstein's equations with cosmological constant, which includes de Sitter's metric as a special case. The generalized solution admits a nonzero stress energy momentum tensor. The second section is concerned with a transformation of the line element into a spherical symmetric but anisotropic form.

1. The line element and Einstein's equations

In the following, the coordinates are $\{t, q, \theta, \phi\}$ and $d\sigma^2 = dq^2 + q^2 d\theta^2 + q^2 \sin^2(\theta)\, d\phi^2$. Consider the ansatz given in [1]:

$$ds^2 = \left[\frac{1 - La^2(t)\, q^2}{1 + La^2(t)\, q^2}\right]^2 c^2 dt^2 - a^2(t) \left[\frac{1}{1 + La^2(t)\, q^2}\right]^2 d\sigma^2 \tag{1.1}$$

L is a constant. Its physical unit coincides with the unit of the cosmological constant Λ.

Theorem 1. *Exact solution of Einstein's equations*
Metric (1.1) is an exact solution of Einstein's equations with nonzero cosmological constant, if the stress energy tensor has the form

$$\begin{aligned} T_t^t &= \frac{c^2}{8\pi\gamma}\left[3H^2 - c^2\Lambda + 12c^2 L\right] \\ T_r^r = T_\theta^\theta = T_\phi^\phi &= c^2\rho + \frac{c^2}{4\pi\gamma} \cdot \frac{1 + La^2q^2}{1 - La^2q^2}\dot{H} \end{aligned} \tag{1.2}$$

where $H := \dot{a}/a$.

Proof. Einstein's equations $R_{ik} - \frac{1}{2}R\, g_{ik} - \Lambda g_{ik} = \frac{8\pi\gamma}{c^4}T_{ik}$ for the interval (1.1) together with a stress energy tensor of the form

$$T_t^t = c^2\rho; \quad T_r^r = T_\theta^\theta = T_\phi^\phi = -p; \quad \text{and} \quad T_k^i = 0 \text{ for } i \neq k \tag{1.3}$$

reduce to:

$$\frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \Lambda + 12L = \frac{8\pi\gamma}{c^2}\rho \tag{1.4}$$

$$\frac{2\frac{\ddot{a}}{a}\left(1 + La^2q^2\right) + \left(1 - 5La^2q^2\right)\left(\frac{\dot{a}}{a}\right)^2}{c^2\left(1 - La^2q^2\right)} - \Lambda + 12L = -\frac{8\pi\gamma}{c^4}p \tag{1.5}$$

The corresponding empty space equations and the calculation of Einstein's tensor for the metric (1.1) are given in [1]. Since $H := \dot{a}/a$ it is $\dot{H} + H^2 = \ddot{a}/a$ and the above equations (1.4) and (1.5) can be rearranged to:

$$\begin{aligned} \rho &= \frac{1}{8\pi\gamma}\left[3H^2 - c^2\Lambda + 12c^2 L\right] \\ p &= -c^2\rho - \frac{c^2}{4\pi\gamma} \cdot \frac{1 + La^2q^2}{1 - La^2q^2}\dot{H} \end{aligned} \tag{1.6}$$

Obviously, the interval (1.1) is an exact solution of Einstein's equations if the functions ρ and p are given by (1.6). Correspondingly, the stress energy tensor (1.3) takes the form (1.2). □

The special case $\rho = p = 0$ was already considered in [1]: Let $r_\Lambda := \sqrt{3/\Lambda}$ and $L \leq (2r_\Lambda)^{-2}$, the interval (1.1) is an exact empty space solution if

$$a(t) = a_0 \exp\left[\left(r_\Lambda^{-2} - 4L\right)^{1/2} ct\right] \tag{1.7}$$

where a_0 is a constant of integration.

2. Coordinate transformation

The interval (1.1) can be transformed into a metric which has the form

$$ds^2 = g_{tt}dt^2 + 2g_{tr}dtdr + g_{rr}dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta)\, d\phi^2. \tag{2.1}$$

Now the coordinates are $\{t, r, \theta, \phi\}$. Comparing the $g_{\theta\theta}$ components of (1.1) and (2.1) leads to

$$r = \frac{aq}{1 + La^2q^2}. \tag{2.2}$$

This equation can be rearranged to get the transformation for the $q-$coordinate:

$$q = \frac{1}{2Lra}\left(1 \pm \sqrt{1 - 4Lr^2}\right)$$

Theorem 2. *Coordinate transformation*
Let $H := \dot{a}/a$. With the transformation $q = \frac{1}{2Lra}(1 \pm \xi)$ where $\xi := \sqrt{1 - 4Lr^2}$ metric (1.1) takes the form

$$ds^2 = \left(c^2\xi^2 - H^2r^2\right)dt^2 - \frac{dr^2}{\xi^2} \mp \frac{2Hr}{\xi}dtdr - r^2 d\theta^2 - r^2 \sin^2(\theta)\, d\phi^2. \tag{2.3}$$

Proof. The components g_{tt}, g_{tr} and g_{rr} of metric (2.1) are determined by using $q = \frac{1}{2Lra}(1 \pm \xi)$ in (1.1). We receive

$$\frac{\partial q}{\partial t} = -\frac{\dot{a}}{a} \cdot \frac{1}{2Lra}\left(1 \pm \sqrt{1 - 4Lr^2}\right) = -Hq$$

and

$$\frac{\partial q}{\partial r} = -\frac{1}{r}q \mp \frac{2}{a\sqrt{1 - 4Lr^2}} = -\frac{q}{r} \mp \frac{2}{a\xi}.$$

Accordingly, it is

$$dq = -Hq\, dt - \left(\frac{q}{r} \pm \frac{2}{a\xi}\right) dr. \tag{2.4}$$

Equation (2.2) directly leads to

$$\frac{a}{1 + La^2q^2} = \frac{r}{q} \tag{2.5}$$

and together with (2.4) the g_{qq} component of (1.1) transforms into

$$g_{qq}dq^2 = -a^2\left[\frac{1}{1 + La^2q^2}\right]^2 dq^2 = -\frac{r^2}{q^2}\left[-Hq\, dt - \left(\frac{q}{r} \pm \frac{2}{a\xi}\right) dr\right]^2 = -\left[-Hr\, dt - \left(1 \pm \frac{2r}{aq\xi}\right) dr\right]^2. \tag{2.6}$$

The latter expression can be simplified, it is

$$1 \pm \frac{2r}{aq\xi} = 1 \pm \frac{4Lr^2}{(1 \pm \xi)\,\xi} = \frac{\xi \pm \xi^2 \pm 4Lr^2}{(1 \pm \xi)\,\xi} = \frac{\xi \pm \left(1 - 4Lr^2\right) \pm 4Lr^2}{(1 \pm \xi)\,\xi} = \frac{\xi \pm 1}{(1 \pm \xi)\,\xi} = \pm\frac{1}{\xi}.$$

Hence, equation (2.6) reads:

$$g_{qq}dq^2 = -\left[-Hr\, dt \mp \frac{1}{\xi}dr\right]^2 = -\left(H^2r^2dt^2 + \frac{1}{\xi^2}dr^2 \pm \frac{2Hr}{\xi}dtdr\right) \tag{2.7}$$

Analogously, we now determine the g_{tt} component of (1.1). From $q = \frac{1}{2Lra}(1 \pm \xi)$ we get

$$q^2 = \left(\frac{1 \pm \xi}{2Lra}\right)^2 = \frac{1 \pm 2\xi + 1 - 4Lr^2}{4L^2r^2a^2} = \frac{1}{Lra} \cdot \frac{1 \pm \xi}{2Lra} - \frac{1}{La^2} = \frac{1}{La}\left(\frac{q}{r} - \frac{1}{a}\right)$$

and therewith $La^2q^2 = \frac{aq}{r} - 1$. Correspondingly, it is

$$\frac{1-La^2q^2}{1+La^2q^2} = \frac{2-\frac{aq}{r}}{\frac{aq}{r}} = \frac{2r}{aq} - 1 = \frac{4Lr^2}{1\pm\xi} - 1 = \frac{4Lr^2-1\mp\xi}{1\pm\xi} = \frac{-\xi^2\mp\xi}{1\pm\xi} = -\xi\frac{\xi\pm1}{1\pm\xi} = \mp\xi$$

and the g_{tt} component of (1.1) transforms as

$$g_{tt} = \left[\frac{1-La^2q^2}{1+La^2q^2}\right]^2 c^2 = c^2\left(\pm\xi\right)^2 = c^2\xi^2 \tag{2.8}$$

It is clear from equation (2.2) that $g_{\theta\theta} = -r^2$ and $g_{\phi\phi} = -r^2\sin^2\theta$. Finally, with (2.8) and (2.7) it remains

$$ds^2 = c^2\xi^2dt^2 - \left(H^2r^2dt^2 + \frac{1}{\xi^2}dr^2 \pm \frac{2Hr}{\xi}dtdr\right) - r^2d\theta^2 - r^2\sin^2(\theta)\,d\phi^2$$

and we get metric (2.3). □

2.1. **The vacuum solution.**

If $a(t)$ is given by (1.7) the $H-$term reduces to

$$H = \left(r_\Lambda^{-2} - 4L\right)^{1/2} c.$$

Thus it is $H^2r^2 = c^2\left(r^2/r_\Lambda^2 - 4Lr^2\right)$, and the line element (2.3) reads:

$$ds^2 = \left(1 - \frac{r^2}{r_\Lambda^2}\right)c^2dt^2 - \frac{dr^2}{1-4Lr^2} \mp 2c\left(\frac{\frac{r^2}{r_\Lambda^2} - 4Lr^2}{1-4Lr^2}\right)^{\frac{1}{2}} dtdr - r^2d\theta^2 - r^2\sin^2(\theta)\,d\phi^2 \tag{2.9}$$

Metric (2.9) is an exact solution of $R_{ik} - \frac{1}{2}R\,g_{ik} - \Lambda g_{ik} = 0$, Einsteins vacuum equations with nonzero cosmological constant. In case of $4L = r_\Lambda^{-2}$ metric (2.9) reduces to the line element

$$ds^2 = \left[1 - \left(\frac{r}{r_\Lambda}\right)^2\right]c^2dt^2 - \frac{1}{1-\left(\frac{r}{r_\Lambda}\right)^2}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\,d\phi^2. \tag{2.10}$$

which represents the de Sitter[1] spacetime.

3. Conclusion

The astrophysical relevance of the general solution given by (1.1) and (1.6) is not clear. It can be regarded as a modification or generalization of the de Sitter metric. In any case, a new exact solution of Einstein's equations with cosmological constant is of mathematical interest.

References

[1] T. Günther: Matching of local and global geometry in our universe, URN: urn:nbn:de:hbz:6-46339386471, URL: http://nbn-resolving.de/urn:nbn:de:hbz:6-46339386471, 2013

[1] With respect to the coordinates $\{\bar{t},\bar{r},\bar{\theta},\bar{\phi}\}$ de Sitter's interval is given by

$$ds^2 = c^2d\bar{t}^2 - a_0^2\exp\left[\frac{2c\bar{t}}{r_\Lambda}\right]\left(d\bar{r}^2 + \bar{r}^2d\bar{\theta}^2 + \bar{r}^2\sin\bar{\theta}\,d\bar{\phi}^2\right) \tag{2.11}$$

where $r_\Lambda = \sqrt{3/\Lambda}$, and a_0 is a constant of integration. Using the coordinate transformation

$$\bar{t} = t + \frac{r_\Lambda}{2c}\ln\left[1 - \left(\frac{r}{r_\Lambda}\right)^2\right], \quad \bar{r} = \frac{r\exp\left(-\frac{c}{r_\Lambda}t\right)}{a_0\sqrt{1-\left(\frac{r}{r_\Lambda}\right)^2}}, \quad \bar{\theta} = \theta, \quad \bar{\phi} = \phi \tag{2.12}$$

de Sitter's metric (2.11) transforms into the static line element (2.10).