Some Further Problems with Analytical Mechanics.

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Abstract.

Here two derivations of Hamilton's equations of motion are examined and found to depend crucially on both the system being conservative and the mass being constant. It is speculated that these derivations might be extended to the case where, as well as the system being conservative, the mass varies purely with time. However, at this stage, other generalisations seem unlikely and, if that is so, the usefulness of the Hamilton approach in mechanics would appear to be limited.

Introduction.

Modern analytical mechanics is closely associated with the names of two men; Joseph-Louis Lagrange (1736 – 1813), an Italian who worked mainly in Germany and France and William Rowan Hamilton (1805 – 1865) an Irishman. Both made highly significant contributions to our knowledge in both mathematics and physics but both are irrevocably connected with analytical advances in Newtonian mechanics which are taught to virtually all present day mathematics and physics undergraduates as well as to many in other disciplines. However, as has been shown in an article in a recent book [1], some of the results associated with the name of Lagrange as expounded popularly are not completely general. It should be noted that this is not necessarily due to any error on the part of Lagrange but could well be due to the modern form of exposition employed to introduce the results. Again, the dates of the two men are included to show quite clearly that Hamilton's work followed that of Lagrange. This may be obvious to some and/or trivial to others but later comments will illustrate the need to make this point at the very outset because it does lend some credence to the thought that Hamilton's work may have been inspired by the earlier Lagrange work and that it always has depended, at least to some extent, on this earlier work. In fact, this whole investigation raises the question of whether, or not, Hamilton's can be derived without any reference to Lagrange's work and his results.

The present position.

It seems there are two basic ways of deriving the equations of motion in the form due to Hamilton and both depend to differing degrees on the earlier work of Lagrange and his equations of motion. These two approaches will now be considered separately:

Method 1.

The first method relies from the outset on use of the Lagrange equations of motion in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \tag{1}$$

where $L = L(\dot{q}_i, q_i), i = 1,2, \ldots$. The time t could also be included as an independent variable but that is not done here as it neither adds to, not detracts from, the discussion.
Hence, from the very beginning of this derivation, attention is restricted to conservative systems since introduction of the Lagrangian L implies the existence of a potential energy function because L is defined as equalling the difference between the kinetic and potential energies of a system. Again, as was shown in the relevant article in reference [1], the derivation of these Lagrange equations also restricts attention to cases where the mass is a constant. This latter is an interesting point for another reason. The first step in this derivation of Hamilton's equations of motion is to put

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \tag{2}$$

where p_i is referred to as the momentum canonically conjugate to q_i.
Using this latter equation, (1) is seen to become

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{3}$$

and these equations are used to express the p_i's as functions of the $\dot{q}_i$ and the q_i. Hence, the $\dot{q}_i$'s may be eliminated in favour of the p_i's and q_i's; that is, the generalised velocities may be expressed in terms of the generalised coordinates and momenta.
Then the Hamiltonian function

$$H = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i) \qquad (4)$$

Is introduced and regarded as a function of the p_i and q_i, imagining the $\dot{q}_i$ to have been eliminated using (2).

Since the Hamiltonian function depends only on the p_i and the q_i, it follows that

$$dH = \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right)$$

But, from (4), it is seen that

$$dH = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL$$

$$= \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i)$$

Comparing the two expressions for dH leads to the familiar Hamilton equations of motion:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \text{ and } \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

This then is a familiar derivation of the Hamilton equations of motion but it is seen that a number of restricting assumptions have been made in achieving this. Firstly, the system is very clearly assumed to be conservative. Secondly, and possibly more importantly, because the Lagrange equations of motion are used, any restrictions imposed during their derivation must be carried over the affect the Hamilton equations of motion too. Hence, these Hamilton equations too are valid only for a fixed value of the mass. This, incidentally, does raise the minor query of why using the momenta instead of the velocities is regarded as being so important since the two would seem to bear a very simple relationship to one another.

Method 2.

The second common method for deriving Hamilton's equations of motion relies on results from the Calculus of Variations. In this derivation, it is noted first that Lagrange's equations of motion in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

may be regarded as the necessary and sufficient conditions for the integral

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i) dt$$

to be stationary; that is

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0.$$

This is, of course, a statement of Hamilton's Principle.
Then, since the Hamiltonian, H, is defined by

$$H = \sum_i p_i \dot{q}_i - L,$$

it follows that

$$S = \int_{t_1}^{t_2} \left\{ \sum_i p_i \dot{q}_i - H(p_i, q_i) \right\}$$

and

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right) dt$$

which, after integrating the second term by parts and some rearranging of terms, gives

$$\delta S = \int_{t_1}^{t_2} \sum_i \left\{ \delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) + \delta q_i \left(-\dot{p}_i - \frac{\partial H}{\partial q_i} \right) \right\}$$

which equals zero by Hamilton's Principle.
Since the variations $\delta p_i, \delta q_i$ are arbitrary, Hamilton's equations of motion follow.

Once again, though, it is seen that the derivation depends crucially on the form of Lagrange's equation which applies specifically to conservative systems and only in the case where the mass is a definite constant.

Comments.

The above are the two commonly adopted means for deriving Hamilton's equations of motion and, although both are well-known, they have been included in full here to illustrate just how crucial is the dependence on the earlier Lagrange equations of motion for both. Considering the discussion of the situation when the mass varies in reference [1], it seems unlikely that the derivation of Hamilton's equations will be able to be extended to that more general case. The one possible exception to this is the case where the mass simply varies with time and the general form of Lagrange's equations remains

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_{i,}$$

where T is the usual form for the kinetic energy (that is $\frac{1}{2}mv^2$) and Q_i are the so-called generalised components of force. In this particular case, for a conservative system

$$Q_i = -\frac{\partial V}{\partial q_i}$$

and the common form of Lagrange's equations would appear to be retained but with a different expression for the Q_i.

Reference.

[1] Dennis P. Allen & Jeremy Dunning-Davies, 2013, *Neo-Newtonian Mechanics With Extension To Relativistic Velocities: Part 1: Non-Radiative Effects:* chap.6**;** and at http://viXra.org/abs/1309.0210