Some results in classical mechanics for the case of a variable mass.

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Abstract.

In most, if not all, introductions to classical mechanics, the mass is assumed to be constant. Usually this is mentioned and often attention is drawn to such systems as rocket motion to indicate that, in practice, the mass is not always a constant. In truth, many students actually meet a varying mass for the first time when introduced to the Special Theory of Relativity. However, varying masses do occur in nature when relativistic effects are not important. Here an attempt is made to draw together some common results of classical mechanics with a variable mass taken into account. Particular attention will be drawn to a perceived change in the expression for the kinetic energy and to crucial changes in the basic form of Lagrange's equations of motion.

Introduction.

It was at a PIRT meeting in London in the 1990's that Ifirst met Ruggero Santilli and discussed the well-known Lagrange equations of motion and the derivation normally presented to undergraduates. Since I had always presented the general form in undergraduate lectures before proceeding to discuss the special case concerning conservative fields, I was a little surprised to find that he found a great many were unaware of the fact that the special case is just that and regarded Lagrange's equations of motion as being the form depending on the so-called Lagrangian, which is the difference between the kinetic and potential energies of a system and, hence, purely relevant to the case of conservative fields of force. However, that common particularisation is not the only one being, at least tacitly, fed to students in classical mechanics. In most, if not all, introductions to classical mechanics, the mass is assumed to be constant. Frequently, when Newton's famous Second Law is introduced, attention is drawn to such systems as rocket motion to indicate that, in practice, the mass is not always a constant but often this is as far as discussion of variable mass systems goes. In truth, many students actually meet a varying mass for the first time when introduced to the Special Theory of Relativity. However, varying masses do occur in nature when relativistic effects are not important. Here an attempt is made to draw together some common results of classical mechanics with a variable mass taken into account. Again, the traditional approach to the derivation of Lagrange's equations of motion is considered and extended. However, it should be noted that the other means of deriving these equations contain the same flaws for variable mass situations as are encountered here. Again, in addition, the derivation via Hamilton's Principle relies on use of the Lagrangian function which, in turn, is dependent on the potential energy and so ensures that attention is immediately restricted to conservative systems as well. It is possibly because variable mass systems are encountered so rarely by most that these restrictions on the use of Lagrange's equations of motion go unnoticed. However, these are important points to be noted on those relatively rare occasions when a variable mass does come into play. Particular attention will be drawn to a perceived change in the expression for the kinetic energy and to crucial changes in the basic form of Lagrange's equations of motion but first the situation pertaining to rotating frames of reference will be considered before proceeding to the main two topics to be considered here.

Rotating frames of reference.

Consider the well-known situation concerning rotating frames of reference. Suppose a vector *A* has components a, b, c with respect to a set of axes along which the unit vectors are *i, j, k*. Then

$$A = ai + bj + ck$$

and

$$\frac{d\boldsymbol{A}}{dt} = \frac{da}{dt}\boldsymbol{i} + a\frac{d\boldsymbol{i}}{dt} + \frac{db}{dt}\boldsymbol{j} + b\frac{d\boldsymbol{j}}{dt} + \frac{dc}{dt}\boldsymbol{k} + c\frac{d\boldsymbol{k}}{dt}.$$

But, as is well-known

$$\frac{d\boldsymbol{i}}{dt} = \boldsymbol{\omega}\text{x}\boldsymbol{i}, \text{ etc.}$$

Hence, finally,

$$\frac{d\boldsymbol{A}}{dt} = \frac{da}{dt}\boldsymbol{i} + \frac{db}{dt}\boldsymbol{j} + \frac{dc}{dt}\boldsymbol{k} + a\boldsymbol{\omega}\text{x}\boldsymbol{i} + b\boldsymbol{\omega}\text{x}\boldsymbol{j} + c\boldsymbol{\omega}\text{x}\boldsymbol{k}$$

$$= \left(\frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt}\right) + \boldsymbol{\omega}\text{x}(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})$$

$$= \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{\omega}\text{x}\boldsymbol{A} = \dot{\boldsymbol{A}} + \boldsymbol{\omega}\text{x}\boldsymbol{A}.$$

The notation in the final equations might be termed a convention since a partial derivative in the usual sense is not being discussed. Also, the dot refers to differentiation with respect to time of the components of the vectors *A*.

In this case of variable mass, the equation symbolising Newton's second law becomes

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) = \frac{dm}{dt}\boldsymbol{v} + m\boldsymbol{a},$$

where *F* represents the force acting, *v* the velocity and *a* the acceleration.

But, if a rotating frame is involved, this becomes

$$\boldsymbol{F} = \frac{\partial}{\partial t}(m\boldsymbol{v}) + \boldsymbol{\omega}\text{x}(m\boldsymbol{v})$$

that is

$$\boldsymbol{F} = \dot{m}\boldsymbol{v} + m\dot{\boldsymbol{v}} + \boldsymbol{\omega}\text{x}[m(\dot{\boldsymbol{r}} + \boldsymbol{\omega}\text{x}\boldsymbol{r})]$$
$$= \dot{m}(\dot{\boldsymbol{r}} + \boldsymbol{\omega}\text{x}\boldsymbol{r}) + m(\ddot{\boldsymbol{r}} + \dot{\boldsymbol{\omega}}\text{x}\boldsymbol{r} + \boldsymbol{\omega}\text{x}\dot{\boldsymbol{r}}) + \boldsymbol{\omega}\text{x}[m(\dot{\boldsymbol{r}} + \boldsymbol{\omega}\text{x}\boldsymbol{r})]$$
$$= \dot{m}(\dot{\boldsymbol{r}} + \boldsymbol{\omega}\text{x}\boldsymbol{r}) + m\big(\ddot{\boldsymbol{r}} + \dot{\boldsymbol{\omega}}\text{x}\boldsymbol{r} + \boldsymbol{2}\boldsymbol{\omega}\text{x}\dot{\boldsymbol{r}} + \boldsymbol{\omega}\text{x}(\boldsymbol{\omega}\text{x}\boldsymbol{r})\big)$$

or

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} - \dot{m}\dot{\boldsymbol{r}} - m\boldsymbol{\omega}\text{x}(\boldsymbol{\omega}\text{x}\boldsymbol{r}) - \dot{m}(\boldsymbol{w}\text{x}\boldsymbol{r}) - m(\dot{\boldsymbol{\omega}}\text{x}\boldsymbol{r}) - 2m(\boldsymbol{\omega}\text{x}\dot{\boldsymbol{r}})$$

Here, once again, the dot refers to differentiation with respect to time of the components of the various vectors.

Kinetic energy.

In many approaches to classical mechanics, the starting point is to assume a constant acceleration so that

$$\frac{d^2s}{dt^2} = a,$$

where *s* is distance, *t* time, and *a* the constant acceleration.
Integrating once and assuming the velocity, *v*, initially has the value *u*, we get

$$v = \frac{ds}{dt} = u + at.$$

Integrating a second time and assuming *s* is zero initially, we get

$$s = ut + \tfrac{1}{2}ft^2.$$

These are very well-known straightforward equations but are included here, together with their derivation, to highlight the fact that they do not depend in any way on the mass of an object. In fact, all assumptions made are clearly stated.

Finally, if *s* is eliminated between the latter two equations

$$v^2 = u^2 + 2as$$

results and it follows from this equation that, if an object is brought to rest ($v = 0$) by a retardation *a* then this equation gives

$$0 = u^2 - 2as \;\Rightarrow\; \tfrac{1}{2}u^2 = as$$

If this equation is multiplied throughout by *m*, one obtains

$$\tfrac{1}{2}mu^2 = mas.$$

With *a* taken to be *g*, the acceleration due to gravity, this equation takes on the popular interpretation of kinetic energy being equivalent to potential energy. Here the interpretation of the right-hand side follows from considering Newton's Second Law with the mass assumed constant; that is

$$\text{Force} = P = \tfrac{d}{dt}(mu) = ma.$$

Then, since work done is force and *x* distance, *mas* is a work term and is regarded as being equal here to the kinetic energy. This is how the notion is introduced in textbooks but, since the acceleration is taken to be constant, it follows that the force, *P*, must be constant also.

If the mass is assumed variable, then Newton's Second Law takes the form

$$P = \frac{d}{dt}(mu) = ma + v\frac{dm}{dt}.$$

The definition of the kinetic energy of a body is usually taken to be the energy the body possesses by virtue of its motion and is measured by the amount of work which it does in coming to rest. The introduction of the second term on the right-hand side of the above equation to take account of the fact that the mass is assumed to vary with time obviously can have no bearing on the actual definition but, if you consider the work done in a displacement, *s*, in this case, the work done might, at first sight, seem to be given by

$$\text{work done} = Ps = mas + vs\tfrac{dm}{dt} = \tfrac{1}{2}mu^2 + us\tfrac{dm}{dt} = \tfrac{1}{2}mu^2 + us,$$

and so, in this case, there would appear to be an extra term appearing in the expression for the kinetic energy. However, this last step is incorrect because, in this case of varying mass, the force is ***not*** constant. In this case, the final step must be replaced by

$$\text{work done} = \int_1^2 Pds = \int_1^2 \left(m\frac{dv}{dt} + \frac{dm}{dt}v\right) ds = \int_1^2 \left(m\frac{dv}{dt}\frac{ds}{dt} + \frac{dm}{dt}\frac{ds}{dt}\frac{ds}{dt}\right) dt$$

$$= \int_1^2 \left(mv\frac{dv}{dt}dt + v^2 dm\right)$$

$$= \int_1^2 mv\frac{dv}{dt}dt + [mv^2]_1^2 - \int_1^2 m2v\frac{dv}{dt}dt$$

$$= [mv^2]_1^2 - \int_1^2 mv\frac{dv}{dt}dt,$$

where it is assumed that the work being done is between states 1 and 2. Note also that this expression reduces to the familiar one for kinetic energy when the mass, *m*, is supposed constant. In accordance with what has preceded it in this section, this final result is derived using scalar quantities at all points; the slight generalisation using vector quantities is a trivial extension. It should be noted that attention has already been drawn to this very point by one of us (D.A.) in an article entitled *Neo-Newtonian Theory* [1].

Again, it should be remembered that, here, all that has been assumed about the variability of the mass is that it varies with time. No mention is made of how! The said variation could be through being a direct function of time; through being a function of varying position coordinates or varying velocity components or varying acceleration components, etc. The exact form of this dependence doesn't matter at this juncture. However, when it comes to deriving Lagrange's equations of motion, the exact form of the dependence does assume genuine significance.

Modified Lagrange Equations.

For the sake of this discussion it will be assumed that the mass does not depend on either position or velocity; that is, *m* is independent of both q and $\dot{q}$ in all that follows.

To follow, but generalise, the basic outline in *Synge and Griffith* [2], suppose (x, y, z) are the Cartesian coordinates of a typical particle of a system and suppose we have a holonomic system of n degrees of freedom described by generalised coordinates q_i, $i = 1, 2, \ldots, n$. Then,

$$dx = \sum_{i=1}^{n} \frac{\partial x}{\partial q_i} dq_i, \quad \dot{x} = \sum_{i=1}^{n} \frac{\partial x}{\partial q_1} \dot{q}$$

with similar equations for both $\dot{y}$ and $\dot{z}$.
From the second equation, it is seen immediately that

$$\frac{\partial \dot{x}}{\partial \dot{q}_i} = \frac{\partial x}{\partial q_i}.$$

It is straightforward to show that the operators $\frac{d}{dt}$ and $\frac{\partial}{\partial q_i}$ commute. Then

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{x}^2\right) = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i}\right) = \ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i} + \dot{x}\frac{d}{dt}\left(\frac{\partial \dot{x}}{\partial \dot{q}_i}\right)$$

$$= \ddot{x}\frac{\partial x}{\partial q_i} + \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_i}\right) = \ddot{x}\frac{\partial x}{\partial q_i} + \dot{x}\frac{\partial}{\partial q_i}(\dot{x})$$

$$= \ddot{x}\frac{\partial x}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{1}{2}\dot{x}^2\right)$$

That is

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{x}^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}\dot{x}^2\right) = \ddot{x}\frac{\partial x}{\partial q_i} \qquad *$$

with similar equations for *y* and *z.*

The next step is to multiply these equations by *m*, sum over all particles of the system and add the three resulting equations together. However, here *m* is not constant and so it cannot be taken inside the differentiation signs with impunity. Let us suppose now that *m* is independent of both position and velocity. If that is the case, it may be taken inside all differentiation signs except *d/dt*. Hence, if we carry out the above sequence of procedures but note this final point, after some algebra one arrives at

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}m\dot{x}^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}m\dot{x}^2\right) = m\ddot{x}\frac{\partial x}{\partial q_i} + \frac{dm}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{x}^2\right)$$

again with similar equations for *y* and *z.*

Note that the right-hand side of this equation may be written
$$m\ddot{x}\frac{\partial x}{\partial q_i} + \dot{m}\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i} = m\ddot{x}\frac{\partial x}{\partial q_i} + \dot{m}\dot{x}\frac{\partial x}{\partial q_i} = (m\ddot{x} + \dot{m}\dot{x})\frac{\partial x}{\partial q_i}.$$
Now note that, for variable mass
$$\text{Force} = m(\ddot{x}, \ddot{y}, \ddot{z}) + \dot{m}(\dot{x}, \dot{y}, \dot{z}) = (m\ddot{x} + \dot{m}\dot{x}, m\ddot{y} + \dot{m}\dot{y}, m\ddot{z} + \dot{m}\dot{z})$$
Therefore, the *x*- component of force is
$$m\ddot{x} + \dot{m}\dot{x}$$
It follows immediately that, as in the case of constant mass, the right-hand side of these modified Lagrange equations is equal to the coefficient of dq_i in the equations of virtual work. This follows because
$$\delta W = X\delta x + Y\delta y + Z\delta z = X\sum_{j=1}^{n}\frac{\partial x}{\partial q_i}\delta q_i + Y\sum_{j=1}^{n}\frac{\partial y}{\partial q_i}\delta q_i + Z\sum_{j=1}^{n}\frac{\partial z}{\partial q_i}\delta q_i$$
where *X*, *Y*, *Z* are the components of the force; that is
$$(X, Y, Z) \equiv (m\ddot{x} + \dot{m}\dot{x}, m\ddot{y} + \dot{m}\dot{y}, m\ddot{z} + \dot{m}\dot{z}).$$
Hence, it follows that, as far as the case of a mass varying with time, but not with either position or velocity, the formal form of Lagrange's equations of motion remains unaltered. It is just the form of the components of force that change. However, it should be noted also that the expression being differentiated in both terms on the left-hand side of these modified Lagrange equations is still $\frac{1}{2}m\dot{x}^2$ or $\frac{1}{2}mv^2$; it seems this part of the equations remains unaltered. However, given that the starting point for these considerations was an examination of a derivative of $\frac{1}{2}\dot{x}^2$, this is possibly not too surprising. Again, as in the usual derivations of Lagrange's equations of motion, the system concerned is assumed holonomic. The only difference here is that the mass is assumed variable and to be dependent on the time *t*.

It might be noted also that this form of the Lagrange equations will hold as long as the mass remains independent of both position and velocity; that is, it may depend on acceleration or on even higher time derivatives of position. This latter point follows because the only partial derivatives appearing in the derivation are those with respect to position and velocity. It is also important to note at this point that no mention has been made of potential energy; that concept simply hasn't been introduced nor has it been needed. As with the usual derivation of Lagrange's equations of motion, the right hand side is dependent on the generalised components of force; the introduction of potential energy and, hence, the Lagrangian function defined by $L = T + V$, where T is kinetic energy and V potential energy, is a later development in the theory which restricts attention from then on to conservative fields of force.

Generalisation to the cases when the mass depends on position and/or velocity.

Now suppose the mass depends on the velocity as well; in other words the mass is dependent on $\dot{q}$. In this case, it is seen immediately that
$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\frac{\partial m}{\partial \dot{q}} + m\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}}\right)$$
$$= x\ddot{x}\frac{\partial m}{\partial \dot{q}} + \frac{1}{2}\dot{x}^2\frac{d}{dt}\frac{\partial m}{\partial \dot{q}} + \dot{m}\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\dot{x}^2\right) + \dot{x}\frac{\partial \dot{x}}{\partial \dot{q}}\frac{dm}{dt}.$$
In this case, if equation * is multiplied throughout by *m*, the resulting equation may be written
$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{x}^2\right) - \frac{\partial}{\partial q}\left(\frac{1}{2}m\dot{x}^2\right) = (m\ddot{x} + \dot{m}\dot{x})\frac{\partial x}{\partial q} + \dot{x}\ddot{x}\frac{\partial m}{\partial \dot{q}} + \frac{1}{2}\dot{x}^2\frac{d}{dt}\frac{\partial m}{\partial \dot{q}}$$
or
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = (m\ddot{x} + \dot{m}\dot{x})\frac{\partial x}{\partial q} + \dot{x}\ddot{x}\frac{\partial m}{\partial \dot{q}} + \frac{T}{m}\frac{d}{dt}\frac{\partial m}{\partial \dot{q}}$$
where, as previously $T = m\dot{x}^2/2$.
Hence, as soon as the mass becomes dependent on the velocity, two more terms appear on the right-hand side of the relevant Lagrange equations of motion and the equations become more complicated.

Considering this is the situation when the mass is dependent on the velocity, it raises questions about the apparent use of the more familiar orthodox form of Lagrange's equations of motion in some texts concerned with Special Relativity where the mass is velocity-dependent.

If, further, the mass depends on position q, then

$$\frac{\partial}{\partial q}\left(\frac{1}{2}m\dot{x}^2\right) = m\frac{\partial}{\partial q}\left(\frac{1}{2}\dot{x}^2\right) + \frac{1}{2}\dot{x}^2\frac{\partial m}{\partial q} = m\frac{\partial}{\partial q}\left(\frac{1}{2}\dot{x}^2\right) + \frac{T}{m}\frac{\partial m}{\partial q}$$

and, in this case, the final form of the Lagrange equations of motion is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = (m\ddot{x} + \dot{m}\dot{x})\frac{\partial x}{\partial q} + \dot{x}\ddot{x}\frac{\partial m}{\partial \dot{q}} + \frac{T}{m}\left(\frac{d}{dt}\frac{\partial m}{\partial \dot{q}} + \frac{\partial m}{\partial q}\right).$$

Concluding remarks.

Some of the above results may be familiar to some readers but they are not, as far as we are aware, to be found commonly in existing mechanics texts; neither are these extensions to familiar theory taught in many undergraduate courses. In general, it seems that, at most, only a cursory mention is made of variable mass situations in most courses and then usually only reference to rocket motion. Here various applicable results have been derived and, in the case of the examination of the Lagrange equations of motion, a distinction has had to be drawn among various individual cases with the resulting equations becoming more complicated if the mass is dependent on velocity and/or position. The case when the mass is independent of velocity and position is interesting since it is possible to interpret the resulting right-hand side of the equations in a manner similar to that for the constant mass situation. It should be noted also that, here, attention is not restricted to conservative forces and so a potential energy appears nowhere. Attention was drawn to this point re the usual derivation of the Lagrange equations of motion presented to most undergraduates in the introduction and it is no less valid a point here. It is possibly worth noting that the derivation of the Lagrange equations presented here is, in common with most derivations, for a purely mechanical system. Any extension to other systems and situations would need to be addressed afresh, starting from fundamental principles and seeing if the approach could be adapted successfully.

References.

[1] D. A. Allen Jr., *Neo-Newtonian Theory* – preprint.

[2] J.L.Synge and B.A.Griffith, 1959, *Principles of Mechanics* (McGraw-Hill, New York)

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