Graphs with Contributions to Fundamental Groups

Linfan MAO

Chinese Academy of Mathematics and System Science, Beijing 100190, P.R.China

Beijing Institute of Architectural Engineering, Beijing 100044, P.R.China

E-mail: maolinfan@163.com

Beijing Normal University

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Contents

§1. What is a Combinatorial Space?

1.1 Topological spaces with fundamental groups

Examples of Topological Space:

(1) Real numbers $\mathbb{R}$. Complex numbers $\mathbb{C}$.

(2) Euclidean space $\mathbf{R}^n$, Spheres $\mathbf{S}^n$ for $n \geq 1$;

(3) Product of spaces, such as $\mathbf{S}^2 \times \mathbf{S}^{n-2}$ for $n \geq 4$.

Definition 1.1 Topological space, Hausdorff space, Open or closed sets, Open neighborhood, Cover, Basis, Compact space, ..., in [1]-[3] following.

[1] John M.Lee, *Introduction to Topological Manifolds*, Springer-Verlag New York, Inc., 2000.

[2] W.S.Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, etc.(1977).

[3] Munkres J.R., *Topology* (2nd edition), Prentice Hall, Inc, 2000.

Definition 1.2 *Let S be a topological space and $I = [0,1] \subset \mathbf{R}$. An arc a in S is a continuous mapping $a : I \to S$ with initial point $a(0)$ and end point $a(1)$, and S is called arcwise connected if every two points in S can be joined by an arc in S. An arc $a : I \to S$ is a loop based at p if $a(0) = a(1) = p \in S$. A degenerated loop $\mathbf{e} : I \to x \in S$, i.e., mapping each element in I to a point x, usually called a point loop.*

Example Let G be a planar 2-connected graph on $\mathbf{R}^2$ and S is a topological space consisting of points on each $e \in E(G)$. Then S is a arcwise connected space by definition. For a circuit C in G, we choose any point p on C. Then C is a loop $\mathbf{e}_p$ in S based at p, such as those shown in Fig.1.1.

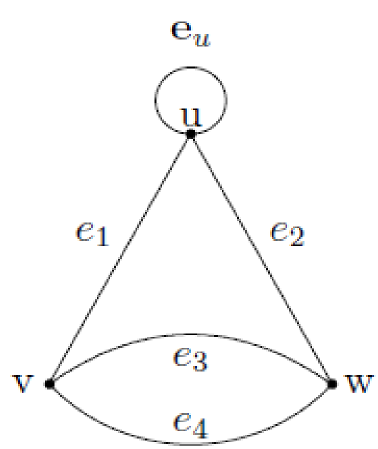


Fig.1.1

Definition 1.3 *Let a and b be two arcs in a topological space S with $a(1) = b(0)$. A product mapping $a \cdot b$ of a with b is defined by*

$$a \cdot b(t) = \begin{cases} a(2t), & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ b(2t-1), & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

and an inverse mapping $\overline{a} = a(1-t)$ by a.

Definition 1.4 *Let S be a topological space and $a, b : I \to S$ two arcs with $a(0) = b(0)$ and $a(1) = b(1)$. If there exists a continuous mapping*

$$H : I \times I \to S$$

such that $H(t,0) = a(t)$, $H(t,1) = b(t)$ for $\forall t \in I$, then a and b are said homotopic, denoted by $a \simeq b$ and H a homotopic mapping from a to b.

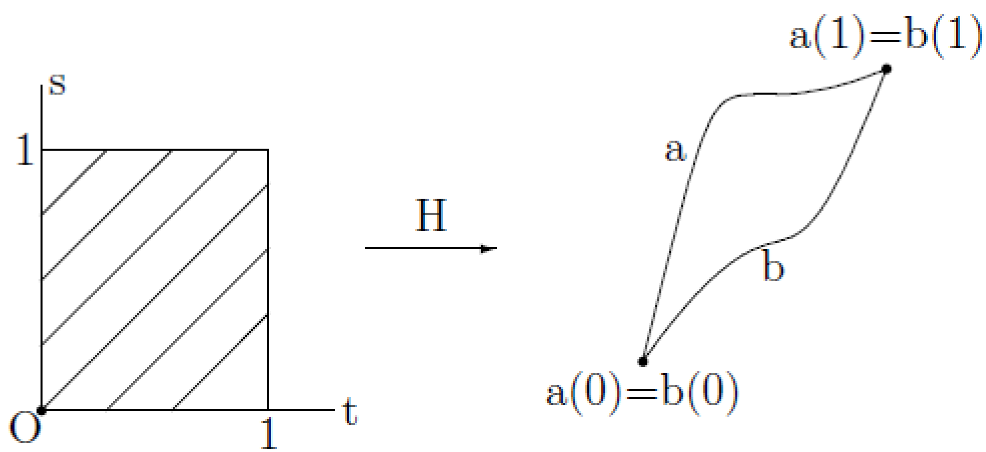


Fig.1.2

Theorem 1.1 *The homotopic $\simeq$ is an equivalent relation, i.e, all arcs homotopic to an arc a is an equivalent arc class, denoted by $[a]$.*

Definition 1.5 *For a topological space S and $x_0 \in S$, let $\pi_1(S, x_0)$ be a set consisting of equivalent classes of loops based at x_0. Define an operation $\circ$ in $\pi_1(S, x_0)$ by*

$$[a] \circ [b] = [a \cdot b] \ \ \textit{and} \ \ [a]^{-1} = [a^{-1}].$$

Theorem 1.2 *$\pi_1(S, x_0)$ is a group.*

Example: (1) $\pi_1(\mathbf{R}^n, x_0), x_0 \in \mathbf{R}^n$ and $\pi_1(\mathbf{S}^n, y_0), y_0 \in \mathbf{S}^n$ is trivial for $n \geq 2$;
(2) $\pi_1(\mathbf{S}, y_0) \cong Z$ and $\pi_1(T^2, z_0) \cong Z^2$, $z_0 \in T^2$.

1.2 Smarandache multi-space with geometry

Definition 1.6 *A rule on a set Σ is a mapping*

$$\underbrace{\Sigma \times \Sigma \cdots \times \Sigma}_{n} \rightarrow \Sigma$$

for some integers n. A mathematical system is a pair $(\Sigma; \mathcal{R})$, where Σ is a set consisting mathematical objects, infinite or finite and $\mathcal{R}$ is a collection of rules on Σ by logic providing all these resultants are still in Σ, i.e., elements in Σ is closed under rules in $\mathcal{R}$.

Definition 1.7 *A rule in a mathematical system* $(\Sigma; \mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ, *i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule in* $\mathcal{R}$. *Particularly, if all systems in* $(\Sigma; \mathcal{R})$ *is a geometrical space, such a Smarandache system is called Smarandache geometry.*

[4] F.Smarandache, *Paradoxist mathematics*, *Collected Papers*, Vol.II, 5-28, University of Kishinev Press, 1997.

[5] H.Iseri, *Smarandache Manifolds*, American Research Press, Rehoboth, NM, 2002.

[6] L.F.Mao, *Smarandache Multi-Space Theory*, Hexis. Phoenix, USA 2006.

Example: Consider a geometry induced from a Euclidean planar geometry by planar maps. Let a complete graph K_4 be embedded in a Euclidean plane $\mathbf{R}^2$, where points $1, 2$ are elliptic, 3 is Euclidean but the point 4 is hyperbolic. Then all lines in the field A do not intersect with L, but each line passing through the point 4 will intersect with the line L. Therefore, (M, μ) is a Smarandache geometry by denial the axiom (E5) with these axioms (E5), (L5) and (R5).

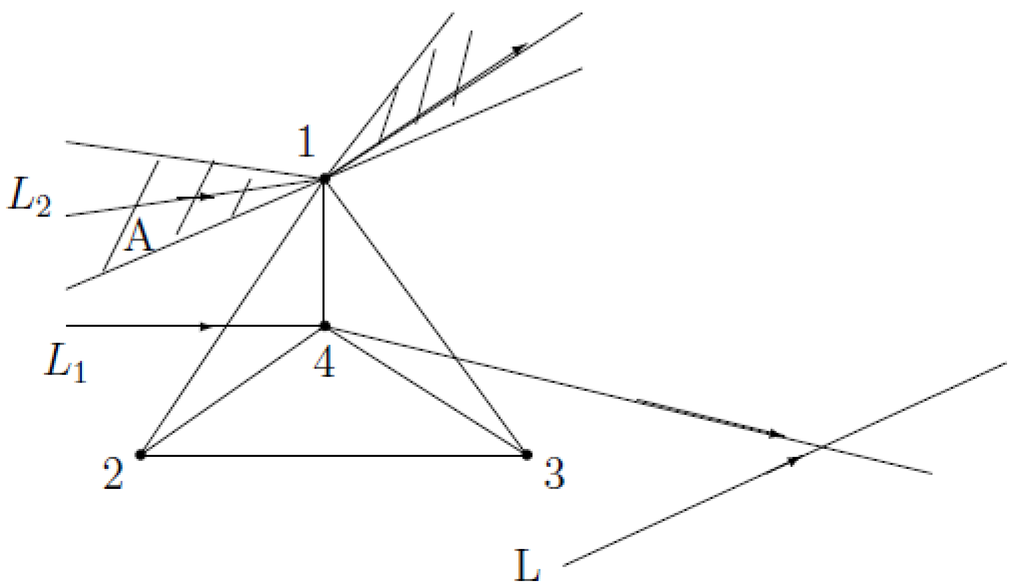


Fig.1.3

Definition 1.8 *For an integer* $m \geq 2$, *let* $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ *be* m *mathematical systems different two by two. A Smarandache multi-space is a pair* $(\widetilde{\Sigma}; \widetilde{\mathcal{R}})$ *with* $\widetilde{\Sigma} = \bigcup\limits_{i=1}^{m} \Sigma_i$ *and* $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$.

Definition 1.9 *A combinatorial system* $\mathscr{C}_G$ *is a union of mathematical systems* $(\Sigma_1; \mathcal{R}_1)$,$(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ *for an integer* m, *i.e.*, $\mathscr{C}_G = (\bigcup\limits_{i=1}^{m} \Sigma_i; \bigcup\limits_{i=1}^{m} \mathcal{R}_i)$ *with an underlying connected graph structure* G, *where*

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

CC Conjecture(2005, MAO) *A mathematical science can be reconstructed from or made by combinatorialization.*

(i) There is a combinatorial structure and finite rules for a classical mathematical system, which means one can make combinatorialization for all classical mathematical subjects.

(ii) One can generalizes a classical mathematical system by this combinatorial notion such that it is a particular case in this generalization.

(iii) One can make one combination of different branches in mathematics and find new results after then.

(iv) One can understand our WORLD by this combinatorial notion, establish combinatorial models for it and then find its behavior, and so on.

[7] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[8] L.F.Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1(2007), 1-19.

1.3 Combinatorial manifolds with topology

An *n-dimensional manifold* is a second countable Hausdorff space such that each point has an open neighborhood homomorphic to a Euclidean space $\mathbf{R}^n$ of dimension n, abbreviated to *n-manifold*.

Loosely speaking, a combinatorial manifold is a combination of finite manifolds, such as those shown in Fig.1.4.

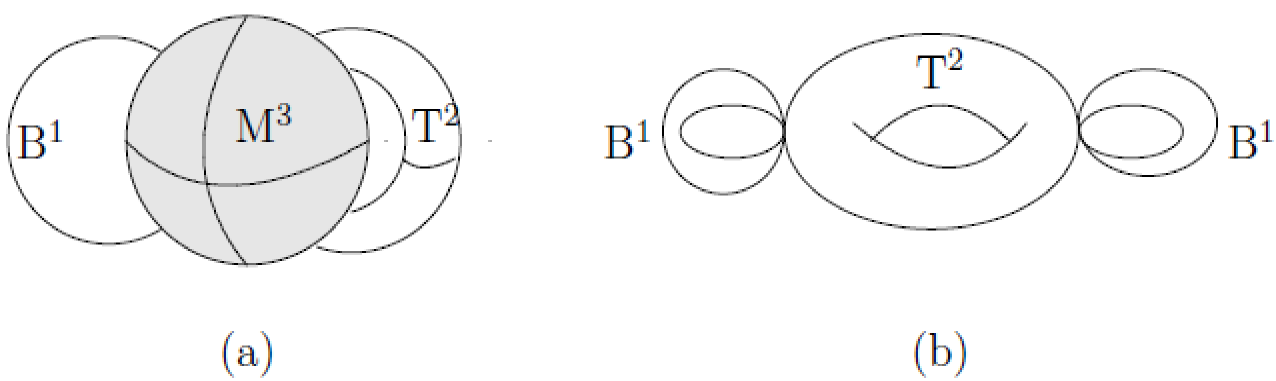


Fig.1.4

Definition 1.10 *A combinatorial Euclidean space is a combinatorial system* $\mathscr{C}_G$ *of Euclidean spaces* $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ *underlying a connected graph* G *defined by*

$$V(G) = \{\mathbf{R}^{n_1}, \mathbf{R}^{n_2}, \cdots, \mathbf{R}^{n_m}\},$$

$$E(G) = \{ \ (\mathbf{R}^{n_i}, \mathbf{R}^{n_j}) \mid \mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} \neq \emptyset, 1 \leq i, j \leq m\},$$

denoted by $\mathscr{E}_G(n_1, \cdots, n_m)$ *and abbreviated to* $\mathscr{E}_G(r)$ *if* $n_1 = \cdots = n_m = r$, *which enables us to view an Euclidean space* $\mathbf{R}^n$ *for* $n \geq 4$.

Definition 1.11 *A combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ *is the combinatorial Euclidean space* $\mathscr{E}_{K_m}(n_1, \cdots, n_m)$ *of* $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ *such that*

$$\mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} = \bigcap_{k=1}^{m} \mathbf{R}^{n_k}.$$

for any integers i, j, $1 \le i \ne j \le m$.

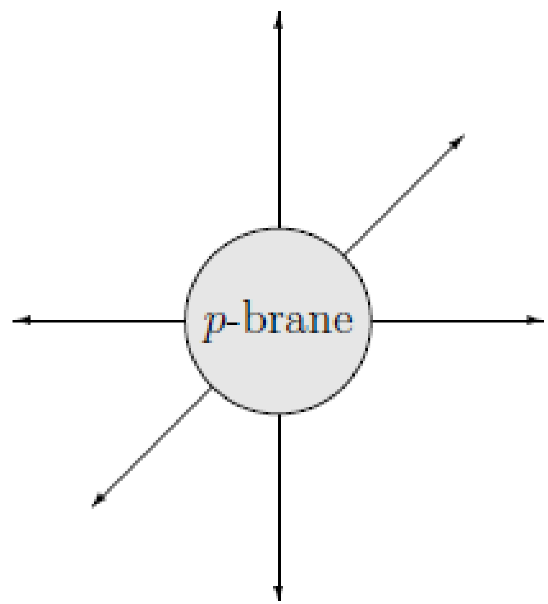


Fig.1.5

Definition 1.12 *For a given integer sequence* $0 < n_1 < n_2 < \cdots < n_m$, $m \geq 1$, *a combinatorial manifold* $\widetilde{M}$ *is a Hausdorff space such that for any point* $p \in \widetilde{M}$, *there is a local chart* (U_p, φ_p) *of* p, *i.e., an open neighborhood* U_p *of* p *in* $\widetilde{M}$ *and a homoeomorphism* $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, *a combinatorial fan-space with*

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\},$$
$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\},$$

denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *or* $\widetilde{M}$ *on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

A combinatorial manifold $\widetilde{M}$ *is finite if it is just combined by finite manifolds with an underlying combinatorial structure* G *without one manifold contained in the union of others.*

Question:

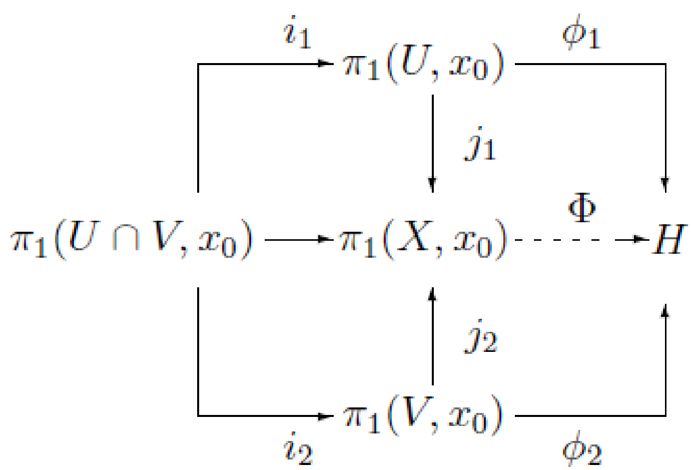
Can we find the fundamental groups of finitely combinatorial manifolds?

§2 Classical Seifert-Van Kampen Theorem with Applications

Theorem 2.1(Seifert and Van-Kampen) *Let* $X = U \cup V$ *with* U, V *open subsets and let* X, U, V, $U \cap V$ *be non-empty arcwise-connected with* $x_0 \in U \cap V$ *and* H *a group. If there are homomorphisms*

$$\phi_1 : \pi_1(U, x_0) \to H \quad \textit{and} \quad \phi_2 : \pi_1(V, x_0) \to H$$

and



with $\phi_1 \cdot i_1 = \phi_2 \cdot i_2$, *where* $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$, $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, $j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$ *and* $j_2 : \pi_1(V, x_0) \to \pi_1(X, x_0)$ *are homomorphisms induced by inclusion mappings, then there exists a unique homomorphism* $\Phi : \pi_1(X, x_0) \to H$ *such that* $\Phi \cdot j_1 = \phi_1$ *and* $\Phi \cdot j_2 = \phi_2$.

Theorem 2.2(Seifert and Van-Kampen theorem, classical version) *Let spaces* X, U, V *and* x_0 *be in Theorem* 2.1. *If*

$$j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$$

is an extension homomorphism of j_1 *and* j_2, *then* j *is an epimorphism with kernel* $\mathrm{Ker} j$ *generated by* $i_1^{-1}(g)i_2(g)$, $g \in \pi_1(U \cap V, x_0)$, *i.e.,*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\langle i_1^{-1}(g) \cdot i_2(g) |\ g \in \pi_1(U \cap V, x_0)\rangle^N}.$$

Corollary 2.1 *Let spaces* X, U, V *and* x_0 *be in Theorem* 2.1. *If* $U \cap V$ *is simply connected, then*

$$\pi_1(X) = \pi_1(U, x_0) * \pi_1(V, x_0).$$

Application: Let $B_n = \bigcup_{i=1}^{n} S_i^1$ be a bouquet shown in Fig.2.1 with $v_i \in S_i^1$, $W_i = S_i^1 - \{v_i\}$ for $1 \leq i \leq n$ and

$$U = S_1^1 \bigcup W_2 \bigcup \cdots \bigcup W_n \quad \text{and} \quad V = W_1 \bigcup S_2^1 \bigcup \cdots \bigcup S_n^1.$$

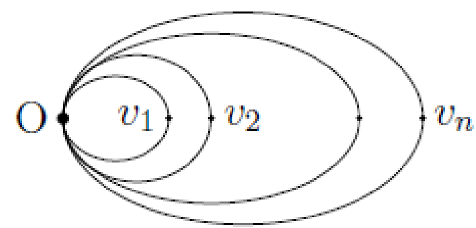


Fig.2.1

Then $U \bigcap V = S_{1.n}$, an arcwise-connected star. Whence,

$$\pi_1(B_n, O) = \pi_1(U, O) * \pi_1(V, O) \cong \pi_1(B_{n-1}, O) * \left\langle S_1^1 \right\rangle.$$

By induction, we easily get that

$$\pi_1(B_n, O) = \left\langle S_i^1, \ 1 \leq i \leq n \right\rangle.$$

§3. Dimensional Graphs

3.1 What is a dimensional graph?

A *topological graph* $\mathscr{T}[G]$ of a graph G is a 1-dimensional graph in a topological space.

Definition 3.1 *A topological graph $\mathscr{T}[G]$ is a pair (X, X^0) of a Hausdorff space X with its a subset X^0 such that*

(1) *X^0 is discrete, closed subspaces of X;*

(2) *$X - X^0$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open interval $(0, 1)$;*

(3) *the boundary $\overline{e}_i - e_i$ of e_i consists of one or two points. If $\overline{e}_i - e_i$ consists of two points, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $([0,1], (0,1))$; if $\overline{e}_i - e_i$ consists of one point, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(S^1, S^1 - \{1\})$;*

(4) *a subset $A \subset \mathscr{T}[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \leq i \leq m$.*

Theorem 3.1([2]) *Any tree is contractible.*

Theorem 3.2([2]) *Let T_{span} be a spanning tree in the topological graph $\mathscr{T}[G]$, $\{e_\lambda : \lambda \in \Lambda\}$ the set of edges of $\mathscr{T}[G]$ not in T_{span} and $\alpha_\lambda = A_\lambda e_\lambda B_\lambda \in \pi(\mathscr{T}[G], v_0)$ a loop associated with $e_\lambda = a_\lambda b_\lambda$ for $\forall \lambda \in \Lambda$, where $v_0 \in \mathscr{T}[G]$ and A_λ, B_λ are unique paths from v_0 to a_λ or from b_λ to v_0 in T_{span}. Then*

$$\pi(\mathscr{T}[G], v_0) = \langle \alpha_\lambda | \lambda \in \Lambda \rangle .$$

Definition 3.2 *An n-dimensional graph $\widetilde{M}^n[G]$ is a combinatorial Euclidean space $\mathscr{E}_G(n)$ of $\mathbf{R}^n_\mu, \mu \in \Lambda$ underlying a combinatorial structure G such that*

(1) *$V(G)$ is discrete consisting of B^n, i.e., $\forall v \in V(G)$ is an open ball B^n_v;*

(2) *$\widetilde{M}^n[G] \setminus V(\widetilde{M}^n[G])$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open ball B^n;*

(3) *the boundary $\overline{e}_i - e_i$ of e_i consists of one or two B^n and each pair $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(\overline{B}^n, B^n)$;*

(4) *a subset $A \subset \widetilde{M}^n[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \leq i \leq m$.*

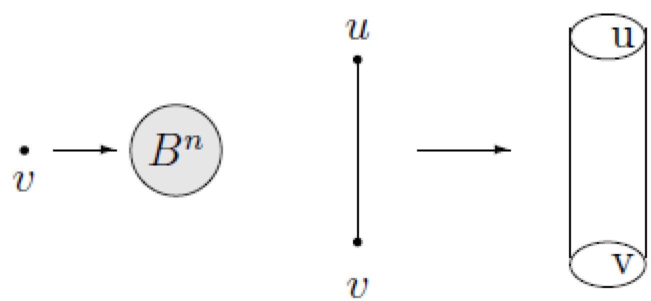


Fig.2.2

3.2 Fundamental groups of dimensional graphs

Theorem 3.3 *For any integer* $n \geq 1$, $\mathscr{T}_0[G]$ *is a deformation retract of* $\widetilde{M}^n[G]$.

Sketch of Proof If $n = 1$, then $\widetilde{M}^n[G] = \mathscr{T}_0[G]$ is itself a topological graph. So we assume $n \geq 2$.

For $n \geq 2$, let $f(\overline{x}, t) = (1-t)\overline{x} + t\overline{x}_0$ be a mapping $f : \widetilde{M}^n[G] \times I \to \widetilde{M}^n[G]$ for $\forall \overline{x} \in \widetilde{M}^n[G]_1, t \in I$, where $\overline{x}_0 = O_v$ if $\overline{x} \in B_v^n$, and $\overline{x}_0 = p(\overline{x})$ if $\overline{x} \in e_i$, where $p : uv \to e_{uv}$ a projection for $1 \leq i \leq m$, such as those shown in Fig.2.3.

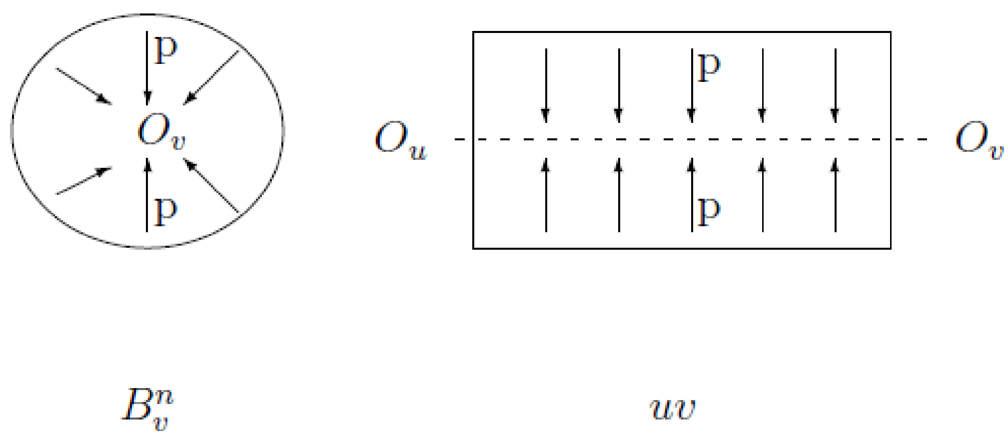


Fig.2.3

Then f is such a deformation retract. $\square$

§4. Generalized Seifert-Van Kampen Theorem

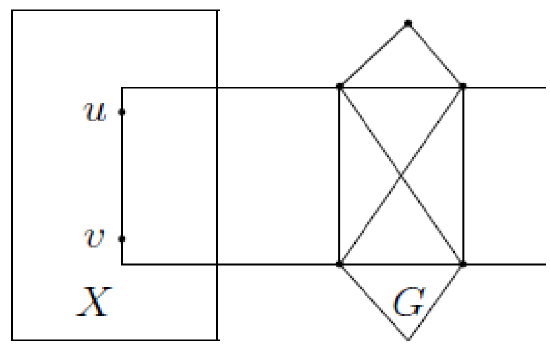
4.1 Topological space attached graphs

Definition 4.1 *A topological space X attached with a graph G is a space $X \odot G$ such that*

$$X \bigcap G \neq \emptyset, \quad G \not\subset X$$

and there are semi-edges $e^+ \in (X \bigcap G) \setminus G$, $e^+ \in G \setminus X$.

An example for $X \odot G$ can be found in Fig.4.1.



$X \odot G$

Fig.4.1

Theorem 4.1 *Let X be arc-connected space, G a graph and H the subgraph $X \cap G$ in $X \odot G$. Then for $x_0 \in X \cap G$,*

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\langle i_1^{-1}(\alpha_{e_\lambda}) i_2(\alpha_{e_\lambda}) | \ e_\lambda \in E(H) \setminus T_{span} \rangle^N},$$

where $i_1 : \pi_1(H, x_0) \to X$, $i_2 : \pi_1(H, x_0) \to G$ are homomorphisms induced by inclusion mappings, T_{span} is a spanning tree in H, $\alpha_\lambda = A_\lambda e_\lambda B_\lambda$ is a loop associated with an edge $e_\lambda = a_\lambda b_\lambda \in H \setminus T_{span}$, $x_0 \in G$ and A_λ, B_λ are unique paths from x_0 to a_λ or from b_λ to x_0 in T_{span}.

Sketch of Proof Let $U = X$ and $V = G$. Applying the Seifert-Van Kampen theorem, we get that

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\langle i_1^{-1}(g) i_2(g) | \ g \in \pi_1(X \cap G, x_0) \rangle},$$

Applying Theorem 3.2, We finally get the following conclusion,

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\langle i_1^{-1}(\alpha_{e_\lambda}) i_2(\alpha_{e_\lambda}) | \ e_\lambda \in E(H) \setminus T_{span} \rangle^N}$$

□

Corollary 4.1 *Let X be arc-connected space, G a graph. If $X \cap G$ in $X \odot G$ is a tree, then*

$$\pi_1(X \odot G, x_0) \cong \pi_1(X, x_0) * \pi_1(G, x_0).$$

Particularly, if G is graphs shown in Fig.2.2 following

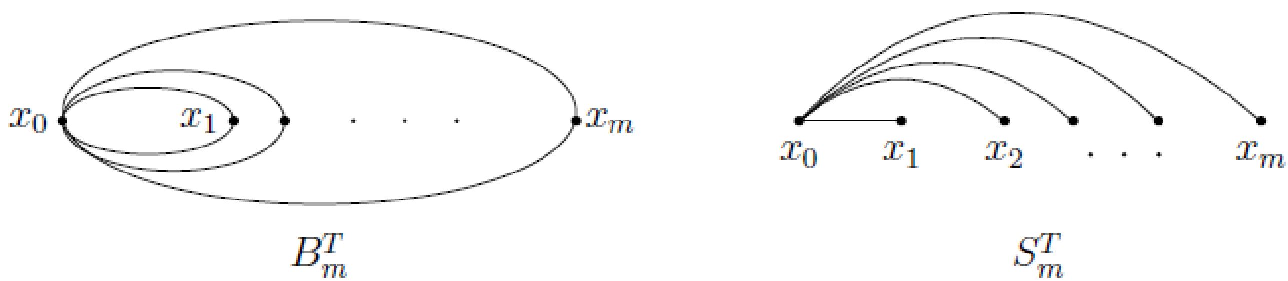


Fig.4.2

and $X \cap G = K_{1,m}$, Then

$$\pi_1(X \odot B_m^T, x_0) \cong \pi_1(X, x_0) * \langle L_i | 1 \le i \le m \rangle,$$

where L_i is the loop of parallel edges (x_0, x_i) in B_m^T for $1 \le i \le m-1$ and

$$\pi_1(X \odot S_m^T, x_0) \cong \pi_1(X, x_0).$$

Theorem 4.2 *Let $\mathscr{X}_m \odot G$ be a topological space consisting of m arcwise-connected spaces $X_1, X_2, \cdots, X_m$, $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq m$ attached with a graph G, $V(G) = \{x_0, x_1, \cdots, x_{l-1}\}$, $m \leq l$ such that $X_i \cap G = \{x_i\}$ for $0 \leq i \leq l-1$. Then*

$$\begin{aligned} \pi_1(\mathscr{X}_m \odot G, x_0) &\cong \left(\prod_{i=1}^{m} \pi_1(X_i^*, x_0)\right) * \pi_1(G, x_0) \\ &\cong \left(\prod_{i=1}^{m} \pi_1(X_i, x_i)\right) * \pi_1(G, x_0), \end{aligned}$$

where $X_i^* = X_i \bigcup (x_0, x_i)$ with $X_i \cap (x_0, x_i) = \{x_i\}$ for $(x_0, x_i) \in E(G)$, integers $1 \leq i \leq m$.

Sketch of Proof The proof is by induction on m with Theorem 4.1 and the Seifert-Van Kampen theorem. $\square$

Corollary 4.2 *Let G be the graph B_m^T or a star S_m^T. Then*

$$\begin{aligned}\pi_1(\mathscr{X}_m \odot B_m^T, x_0) &\cong \left(\prod_{i=1}^{m} \pi_1(X_i^*, x_0)\right) * \pi_1(B_m^T, x_0) \\ &\cong \left(\prod_{i=1}^{m} \pi_1(X_i, x_{i-1})\right) * \langle L_i | 1 \leq i \leq m \rangle,\end{aligned}$$

where L_i is the loop of parallel edges (x_0, x_i) in B_m^T for integers $1 \leq i \leq m$ and

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) \cong \prod_{i=1}^{m} \pi_1(X_i^*, x_0).$$

Corollary 4.3 Let $X = \mathscr{X}_m \odot G$ be a topological space with a simply-connected space X_i for any integer i, $1 \leq i \leq m$ and $x_0 \in X \cap G$. Then we know that

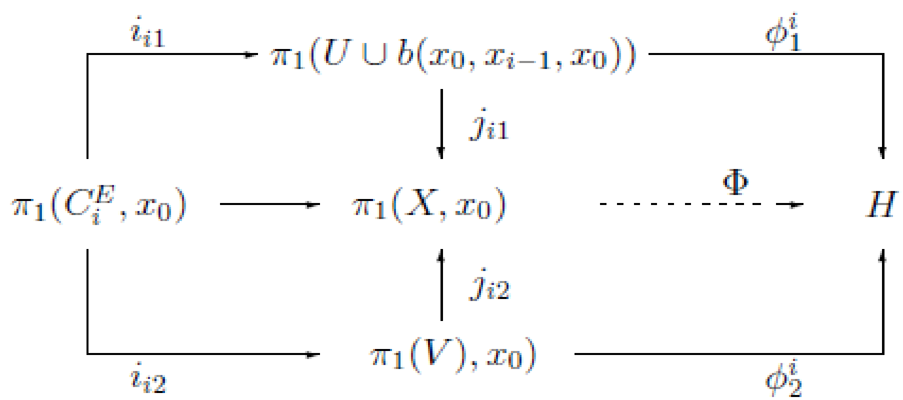
$$\pi_1(X, x_0) \cong \pi_1(G, x_0).$$

4.2 Generalized Seifert-Van Kampen theorem

Theorem 4.3 *Let $X = U \cup V$, $U, V \subset X$ be open subsets and X, U, V arcwise-connected and let $C_1, C_2, \cdots, C_m$ be arcwise-connected components in $U \cap V$ for an integer $m \geq 1$, $x_{i-1} \in C_i$, $b(x_0, x_{i-1}) \subset V$ an arc : $I \to X$ with $b(0) = x_0, b(1) = x_{i-1}$ and $b(x_0, x_{i-1}) \cap U = \{x_0, x_{i-1}\}$, $C_i^E = C_i \bigcup b(x_0, x_{i-1})$ for any integer i, $1 \leq i \leq m$, H a group and there are homomorphisms*

$$\phi_1^i : \pi_1(U \bigcup b(x_0, x_{i-1}), x_0) \to H, \quad \phi_2^i : \pi_1(V, x_0) \to H$$

such that



with $\phi_1^i \cdot i_{i1} = \phi_2^i \cdot i_{i2}$, where $i_{i1} : \pi_1(C_i^E, x_0) \to \pi_1(U \cup b(x_0, x_{i-1}), x_0)$, i_{i2} : $\pi_1(C_i^E, x_0) \to \pi_1(V, x_0)$ and $j_{i1} : \pi_1(U \cup b(x_0, x_{i-1}, x_0)) \to \pi_1(X, x_0)$, $j_{i2} : \pi_1(V, x_0)) \to \pi_1(X, x_0)$ are homomorphisms induced by inclusion mappings, then there exists a unique homomorphism $\Phi : \ \pi_1(X, x_0) \to H$ such that $\Phi \cdot j_{i1} = \phi_1^i$ and $\Phi \cdot j_{i2} = \phi_2^i$ for integers $1 \leq i \leq m$.

Sketch of Proof Define $U^E = U \bigcup \{ b(x_0, x_i) \mid 1 \leq i \leq m-1\}$. Then we get that $X = U^E \cup V$, $U^E, V \subset X$ are still opened with an arcwise-connected intersection $U^E \cap V = \mathscr{X}_m \odot S_m^T$, where S_m^T is a graph formed by arcs $b(x_0, x_{i-1})$, $1 \leq i \leq m$.

Notice that $\mathscr{X}_m \odot Sm^T = \bigcup_{i=1}^{m} C_i^E$ and $C_i^E \bigcap C_j^E = \{x_0\}$. Therefore, we get that

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) = \bigotimes^{m} \pi_1(C_i^E, x_0).$$

This fact enables us knowing that there is a unique m-tuple $(h_1, h_2, \cdots, h_m)$, $h_i \in \pi_1(C_i^E, x_{i-1})$, $1 \leq i \leq m$ such that

$$\mathscr{I} = \prod_{i=1}^{m} h_i$$

for $\forall \mathscr{I} \in \pi_1(\mathscr{X}_m \odot S_m^T, x_0)$ and inclusion maps

$$i_1^E : \pi_1(\mathscr{X}_m \odot S_m^T, x_0) \to \pi_1(U^E, x_0),$$

$$i_2^E : \pi_1(\mathscr{X}_m \odot S_m^T, x_0) \to \pi_1(V, x_0),$$

$$j_1^E : \pi_1(U^E, x_0) \to \pi_1(X, x_0),$$

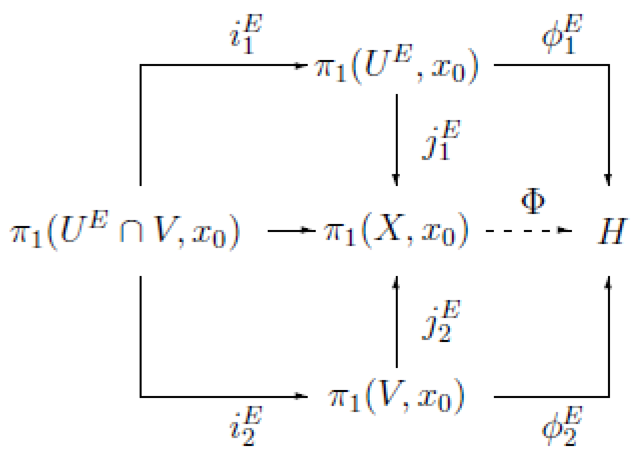
$$j_2^E : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

with $i_1^E|_{\pi_1(C_i^E, x_0)} = i_{i1}$, $i_2^E|_{\pi_1(C_i^E, x_0)} = i_{i2}$, $j_1^E|_{\pi_1(U \cup b(x_0, x_{i-1}, x_0))} = j_{i1}$ and $j_2^E|_{\pi_1(V, x_0)} = j_{i2}$ for integers $1 \leq i \leq m$.

Define ϕ_1^E and ϕ_2^E by

$$\phi_1^E(\mathscr{I}) = \prod_{i=1}^{m} \phi_1^i(i_{i1}(h_i)), \quad \phi_2^E(\mathscr{I}) = \prod_{i=1}^{m} \phi_2^i(i_{i2}(h_i)).$$

Then the following diagram



is commutative. Applying Theorem 2.1, we get the conclusion. □

Theorem 4.4 *Let* X, U, V, C_i^E, $b(x_0, x_{i-1})$ *be arcwise-connected spaces for any integer* i, $1 \leq i \leq m$ *as in Theorem* 3.1, $U^E = U \bigcup \{\, b(x_0, x_i) \mid 1 \leq i \leq m-1\}$ *and* B_m^T *a graph formed by arcs* $a(x_0, x_{i-1})$, $b(x_0, x_{i-1})$, $1 \leq i \leq m$, *where* $a(x_0, x_{i-1}) \subset U$ *is an arc* : $I \to X$ *with* $a(0) = x_0, a(1) = x_{i-1}$ *and* $a(x_0, x_{i-1}) \cap V = \{x_0, x_{i-1}\}$. *Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2(g) | \ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0) \ \right\rangle^N},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ *and* $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ *are homomorphisms induced by inclusion mappings.*

Sketch of Proof By the proof of Theorem 4.3 we have known that there are homomorphisms ϕ_1^E and ϕ_2^E such that $\phi_1^E \cdot i_1^E = \phi_2^E \cdot i_2^E$. Applying Theorem 2.2, we get that

$$\pi_1(X, x_0) \cong \frac{\pi_1(U^E, x_0) * \pi_1(V, x_0)}{\langle (i_1^E)^{-1}(\mathscr{I}) \cdot i_2^E(\mathscr{I}) | \mathscr{I} \in \pi_1(U^E \cap V, x_0) \rangle^N}.$$

Notice that $U^E \cap V^E = \mathscr{X}_m \odot S_m^T$ and

$$\pi_1(U^E, x_0) \cong \pi_1(U, x_0) * \pi_1(B_m^T, x_0).$$

We finally get that

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) | \ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0) \right\rangle^N}. \qquad \square$$

Theorem 4.5 *Let* X, U, V, $C_1, C_2, \cdots, C_m$ *be arcwise-connected spaces,* $b(x_0, x_{i-1})$ *arcs for any integer* i, $1 \leq i \leq m$ *as in Theorem* 3.1, $U^E = U \bigcup \{ b(x_0, x_{i-1}) \mid 1 \leq i \leq m\}$ *and* B_m^T *a graph formed by arcs* $a(x_0, x_{i-1})$, $b(x_0, x_{i-1})$, $1 \leq i \leq m$. *Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) | \ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0) \right\rangle^N},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \rightarrow \pi_1(U^E, x_0)$ *and* $i_2^E : \pi_1(U^E \cap V, x_0) \rightarrow \pi_1(V, x_0)$ *are homomorphisms induced by inclusion mappings.*

Corollary 4.4 *Let* $X = U \cup V$, $U, V \subset X$ *be open subsets and* X, U, V *and* $U \cap V$ *arcwise-connected. Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\langle i_1^{-1}(g) \cdot i_2(g) | \ g \in \pi_1(U \cap V, x_0) \rangle^N},$$

where $i_1 : \pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0)$ *and* $i_2 : \pi_1(U \cap V, x_0) \rightarrow \pi_1(V, x_0)$ *are homomorphisms induced by inclusion mappings.*

Corollary 4.5 *Let X, U, V, C_i, $a(x_0, x_i)$, $b(x_0, x_i)$ for integers i, $1 \le i \le m$ be as in Theorem 3.1. If each C_i is simply-connected, then*

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0).$$

Corollary 4.6 *Let X, U, V, C_i, $a(x_0, x_i)$, $b(x_0, x_i)$ for integers i, $1 \le i \le m$ be as in Theorem 3.1. If V is simply-connected, then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(B_m^T, x_0)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) | \ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0) \right\rangle^N},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ and $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ are homomorphisms induced by inclusion mappings.

§5. Fundamental Groups of Spaces

5.1 Determine fundamental groups of combinatorial spaces

Definition 5.1 *Let $\widetilde{M}$ be a combinatorial manifold underlying a graph $G[\widetilde{M}]$. An edge-induced graph $G^{\theta}[\widetilde{M}]$ is defined by*

$$V(G^{\theta}[\widetilde{M}]) = \{x_M, x_{M'}, x_1, x_2, \cdots, x_{\mu(M,M')} |\ for\ \forall (M, M') \in E(G[\widetilde{M}])\},$$

$$E(G^{\theta}[\widetilde{M}]) = \{(x_M, x_{M'}), (x_M, x_i), (x_{M'}, x_i) |\ 1 \leq i \leq \mu(M, M')\},$$

where $\mu(M, M')$ is called the edge-index of (M, M') with $\mu(M, M') + 1$ equal to the number of arcwise-connected components in $M \cap M'$.

By definition, $G^{\theta}[\widetilde{M}]$ of a combinatorial manifold $\widetilde{M}$ is gotten by replacing each edge (M, M') in $G[\widetilde{M}]$ by a subgraph $TB^{T}_{\mu(M,M')}$ shown in Fig.5.1 with $x_M = M$ and $x_{M'} = M'$.

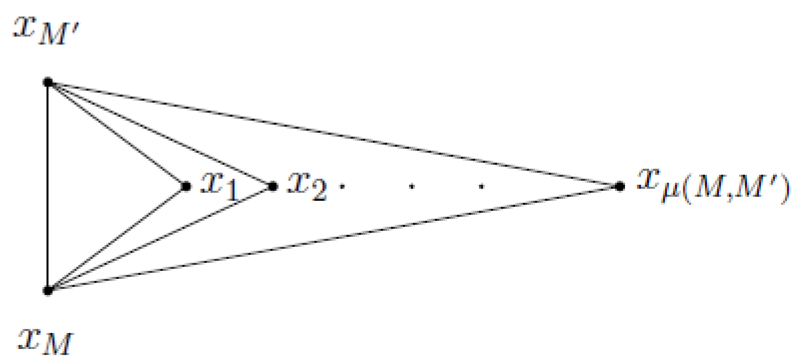
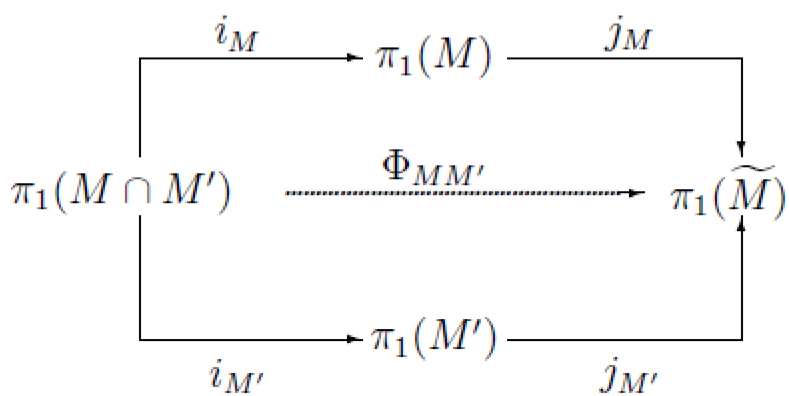


Fig.5.1

Theorem 5.1 *Let $\widetilde{M}$ be a finitely combinatorial manifold. Then*

$$\pi_1(\widetilde{M}) \cong \frac{\left(\prod\limits_{M\in V(G[\widetilde{M}])} \pi_1(M)\right) * \pi_1(G^\theta[\widetilde{M}])}{\left\langle (i_1^E)^{-1}(g)\cdot i_2^E(g) |\ g \in \prod\limits_{(M_1,M_2)\in E(G[\widetilde{M}])} \pi_1(M_1\bigcap M_2)\right\rangle^N},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_M : \pi_1(M\cap M') \to \pi_1(M)$, $i_{M'} : \pi_1(M\cap M') \to \pi_1(M')$ such as those shown in the following diagram:



for $\forall (M, M') \in E(G[\widetilde{M}])$.

Sketch of Proof This result is obvious for $|G[\widetilde{M}]| = 1$. Notice that $G^\theta[\widetilde{M}] = B^T_{\mu(M,M')+1}$ if $V(G[\widetilde{M}]) = \{M,\ M'\}$. Whence, it is an immediately conclusion of Theorem 4.4 for $|G[\widetilde{M}]| = 2$.

Let $U = \widetilde{M} \setminus (M \setminus \widetilde{M})$ and $V = M$. By definition, they are both opened. Applying Theorem 4.4, we get that

$$\pi_1(\widetilde{M}) \cong \frac{\pi_1(\widetilde{M} - M) * \pi_1(M) * \pi_1(B^T_m)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i) \right\rangle^N},$$

where C_i is an arcwise-connected component in $M \cap (\widetilde{M} - M)$ and

$$m = \sum_{(M,M') \in E(G[\widetilde{M}])} \mu(M, M').$$

Applying the induction assumption, we get that

$$\pi_1(\widetilde{M}) \cong \frac{\left(\prod_{M \in V(G[\widetilde{M}])} \pi_1(M)\right) * \pi_1(G^\theta[\widetilde{M}])}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{(M_1,M_2) \in E(G[\widetilde{M}])} \pi_1(M_1 \bigcap M_2) \right\rangle^N}. \qquad \square$$

Corollary 5.1([9],[10]) *Let $\widetilde{M}$ be a finitely combinatorial manifold. If for* $\forall(M_1, M_2) \in E(G^L[\widetilde{M}])$, $M_1 \cap M_2$ *is simply connected, then*

$$\pi_1(\widetilde{M}) \cong \left(\bigotimes_{M \in V(G[\widetilde{M}])} \pi_1(M)\right) \bigotimes \pi_1(G[\widetilde{M}]).$$

[9] L.F.Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[10] L.F.Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest, USA, 2009.

5.2 Determine fundamental groups of manifolds

If we choose $M \in V(G[\widetilde{M}])$ to be a chart $(U_\lambda, \varphi_\lambda)$ with $\varphi_\lambda : U_\lambda \to \mathbf{R}^n$ for $\lambda \in \Lambda$ in Theorem 5.1, i.e., an n-manifold, we get the fundamental group of n-manifold following.

Theorem 5.2 *Let M be a compact n-manifold with charts $\{(U_\lambda, \varphi_\lambda) | \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. Then*

$$\pi_1(M) \cong \frac{\pi_1(G^\theta[M])}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) | \ g \in \prod_{(U_\mu, U_\nu) \in E(G[M])} \pi_1(U_\mu \cap U_\nu) \right\rangle^N},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_{U_\mu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\mu)$, $i_{U_\nu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\nu)$, $\mu, \nu \in \Lambda$.

Corollary 5.2 *Let M be a simply connected manifold with charts $\{(U_\lambda, \varphi_\lambda)|\ \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. Then $G^\theta[M] = G[M]$ is a tree.*

Corollary 5.3 *Let M be a compact n-manifold with charts $\{(U_\lambda, \varphi_\lambda)|\ \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. If $U_\mu \cap U_\nu$ is simply connected for $\forall \mu, \nu \in \Lambda$, then*

$$\pi_1(M) \cong \pi_1(G[M]).$$

§6. Furthermore Discussions

If objects considered are smooth, then we can establish a differential theory on combinatorial manifold, i.e., combinatorially differential geometry (see [12] for detail), which can be used to characterizing the behavior of multi-spaces in Universe. More such applications can be found in references [11]-[13].

[11] L.F.Mao, Curvature equations on combinatorial manifolds with applications to theoretical physics, *International J.Math. Combin.*, Vol.1(2008), Vol.1, 1-25.

[12] L.F.Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest, USA, 2009.

[13] L.F.Mao, Relativity in combinatorial gravitational fields, *Progress in Physics*, Vol.3 (2010), 33-44.