E. Hitzer, R. Bujack, G. Scheuermann, Vector field computations in Clifford's geometric algebra, Proc. of the Third SICE Symposium on Computational Intelligence, August 30, 2013, Osaka University, Osaka, pp. 91-95.

# Vector field computations in Clifford's geometric algebra

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**Abstract**– Exactly 125 years ago G. Peano introduced the modern concept of vectors in his 1888 book "Geometric Calculus - According to the Ausdehnungslehre (Theory of Extension) of H. Grassmann". Unknown to Peano, the young British mathematician W. K. Clifford (1846-1879) in his 1878 work "Applications of Grassmann's Extensive Algebra" had already 10 years earlier perfected Grassmann's algebra to the modern concept of geometric algebras, including the measurement of lengths (areas and volumes) and angles (between arbitrary subspaces). This leads currently to new ideal methods for vector field computations in geometric algebra, of which several recent exemplary results will be introduced.

**Keywords:** Vector field, Clifford's geometric algebra, Geometric calculus

## 1 Introduction

The descriptions of L. Kannenberg's first complete translation [2] of G. Peano's *Calcolo Geometrico* says:

> *Calcolo Geometrico*, G. Peano's first publication in mathematical logic, is a model of expository writing, with a significant impact on 20th century mathematics. . . . In Chapter IX, with the innocent-sounding title "Transformations of a linear system," one finds the crown jewel of the book: Peano's axiom system for a vector space, the first-ever presentation of a set of such axioms. The very wording of the axioms (which Peano calls "definitions") has a remarkably modern ring, almost like a modern introduction to linear algebra. Peano also presents the basic calculus of set operation, introducing the notation for 'intersection,' 'union,' and 'element of,' many years before it was accepted.
> Despite its uniqueness, *Calcolo Geometrico* has been strangely neglected by historians of mathematics, and even by scholars of Peano.

The mathematical biography of G. Peano at the University of St. Andrews [3] writes about *Calcolo Geometrico*:

> In 1888 Peano published the book *Geometrical Calculus* which begins with a chapter on mathematical logic. . . . A more significant feature of the book is that in it Peano sets out with great clarity the ideas of Grassmann which certainly were set out in a rather obscure way by Grassmann himself. This book contains the first definition of a vector space given with a remarkably modern notation and style and, although it was not appreciated by many at the time, this is surely a quite remarkable achievement by Peano.

In the preface G. Peano himself writes in February 1888 about his expectations for his work [2]:

> . . . I will be satisfied with my work in writing this book (which would be the only recompense I could expect), if it serves to disclose among mathematicians some of the ideas of Grassmann [1]. It is however my opinion that, before long, this geometric calculus, or something analogous, will be substituted for the methods actually in use in higher education. It is indeed true that the study of this calculus, as with that of every science, requires time; but I do not believe that it exceeds that necessary for the study of, e.g., the fundamentals of analytic geometry; and then the student will find himself in possession of a method which comprehends that of analytic geometry as a particular case, but which is much more powerful, and which lends itself in a marvelous way to the study of geometric applications of infinitesimal calculus, of mechanics, and of graphic statics; indeed, some part of such sciences are already observed to have taken possessions of that calculus. . . .

Grassmann's work was the source for W.K. Clifford [4, 5] in England to introduce the modern concept of geometric algebras, which includes the measurement of lengths (areas and volumes) and angles (between arbitrary subspaces). He wrote [4]:

> . . . I propose to communicate in a brief form

> some applications of Grassmann's theory ... I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science. ...

A recent review (covering works until early 2012) of modern applications of Clifford's geometric algebra can be found in [7]. We will therefore concentrate on more recent advances.

Regarding recent progress, we want to report in Section 2 about the detection of outer and total rotations in two-dimensional and three-dimensional vector fields using iterative geometric correlation [9, 10, 11]. Another area of major progress is based on the in depth study of square roots of $-1$ in Clifford's geometric algebras [12]. This has lead to new research (see Section 3) in quaternion and Clifford Fourier transformations [16, 17, 18, 19, 20, 22, 23, 24]. Next we explain (Section 4) about the establishment of the algebraic foundations of split hypercomplex nonlinear adaptive filtering [26]. And finally (Section 5) we show how even in material science, Clifford's geometric algebra allows to find new geometric descriptions for fundamental symmetry properties [27].

## 2 Progress in vector field detection

Correlation is a common technique for the detection of shifts. Its generalization to the multidimensional geometric correlation in Clifford algebras additionally contains information with respect to rotational misalignment. It has proven a useful tool for the registration of vector fields that differ by an *outer rotation*. In [11] we have recently proved that applying the geometric correlation iteratively has the potential to detect the total rotational misalignment for linear two-dimensional vector fields. We have further analyzed its effect on general analytic vector fields and showed how the rotation can be calculated from their power series expansions.

So far the exact correction of a three-dimensional outer rotation could only be achieved in certain special cases. In [10] we further prove that applying the geometric correlation iteratively can detect the outer rotational misalignment even for *arbitrary three-dimensional* vector fields. Thus, we developed a foundation applicable for image registration, color image processing and pattern matching. Based on the theoretical work we have established a new algorithm and tested it on several principle examples.

In [9] we further present the explicit iterative algorithm, and analyze its efficiency for detecting the rotational misalignment in the color space of a color image. The experiments suggest a method for the acceleration of the algorithm, which has now been practically tested with great success.

## 3 Progress in quaternion and Clifford Fourier transforms

Vector fields can be embedded with great advantage in a Clifford algebra, which contains the corresponding vector space as a subset. Due to the availability of new types of Fourier transformations in the embedding Clifford algebra, new methods can be established for vector field processing. A major new type of Clifford Fourier transformation relies on the detailed study [12] of square roots of $-1$ in Clifford algebras $Cl(p,q)$, $n = p+q$.

It is well known that real Clifford (geometric) algebra offers a real geometric interpretation for square roots of $-1$ in the form of blades that square to minus one. This extends to a geometric interpretation of quaternions as the side face bivectors of a unit cube. Systematic research has been done [14] on the biquaternion roots of $-1$, abandoning the restriction to blades. Biquaternions are isomorphic to the Clifford (geometric) algebra $Cl(3,0)$ of $\mathbb{R}^3$. Further research on general algebras $Cl(p,q)$ has explicitly derived the geometric roots of $-1$ for $p+q \leq 4$ [13]. The new research [12] abandons this dimension limit and uses the Clifford algebra to matrix algebra isomorphisms in order to algebraically characterize the continuous manifolds of square roots of $-1$ found in the different types of Clifford algebras, depending on the type of associated ring ($\mathbb{R}$, $\mathbb{H}$, $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$). This allows to establish explicit computer generated tables of representative square roots of $-1$ for all Clifford algebras with $n = 5, 7$, and $s = 3 \,(\mathrm{mod}\ 4)$ with the associated ring $\mathbb{C}$. This includes, e.g., $Cl(0,5)$ important in Clifford analysis, and $Cl(4,1)$ which in applications [7] is at the foundation of conformal geometric algebra. All these roots of $-1$ are immediately useful in the construction of new types of geometric Clifford Fourier transformations (CFT).

Basically in the kernel of the complex Fourier transform the imaginary unit $j$ in $\mathbb{C}$ (complex numbers) is replaced by a real geometric multivector square root of $-1$ in $Cl(p,q)$. The recent (one-sided) CFT [18] thus obtained generalizes previously known and applied CFTs [15], which replaced $j$ in $\mathbb{C}$ only by blades (usually pseudoscalars) squaring to $-1$. A major advantage of real Clifford algebra CFTs is their completely real geometric interpretation. Established have been so far (left and right) linearity of the CFT for constant multivector coefficients in $Cl(p,q)$, translation ($\boldsymbol{x}$-shift) and modulation ($\boldsymbol{\omega}$-shift) properties, and signal dilations. The new CFTs have an inversion theorem. Moreover, they have been applied to vector differentials, partial derivatives, vector derivatives and spatial moments of signals. Plancherel and Parseval identities as well as a general convolution theorem have been derived.

This research has subsequently been extended to general *two-sided* CFTs [19], and their properties (from linearity to convolution) have been studied too. Two general *multivector square roots* $\in Cl(p,q)$ *of* $-1$ are used both to split multivector signals, and to construct the left and right CFT kernel factors.

The classical Fourier Mellin transform [21], which transforms functions $f$ representing, e.g., a gray level image defined over a compact set of $\mathbb{R}^2$ has recently been generalized [20] to Hamilton's quaternions. Note that quaternions are isomorphic to $Cl(0,2)$ and to the even subalgebra $Cl^+(3,0)$. The quaternionic Fourier Mellin transform (QFMT) applies to functions $f : \mathbb{R}^2 \to \mathbb{H}$, for which $|f|$ is summable over $\mathbb{R}_+^* \times \mathbb{S}^1$ under the measure $d\theta \frac{dr}{r}$. $\mathbb{R}_+^*$ is the multiplicative group of positive and nonzero real numbers. The properties of the QFMT have been investigated, similar to the investigation of the quaternionic Fourier Transform (QFT) in [8]. The next step of generalization achieved in [22] for the complex Fourier-Mellin transform is to Clifford algebra valued signal functions over the domain $\mathbb{R}^{p,q}$ taking values in $Cl(p,q)$, $n = p + q = 2$.

The two-sided quaternionic Fourier transformation (QFT) was introduced and applied in [25] for the analysis of 2D linear time-invariant partial-differential systems. In further theoretical investigations [8] a special split of quaternions was introduced, then called $\pm$split. In the most recent research [23] this split has been analyzed further, and interpreted geometrically as an *orthogonal 2D planes split* (OPS), and generalized to a freely steerable split of $\mathbb{H}$ into two orthogonal 2D analysis planes. The new general form of the OPS split allows to find new geometric interpretations for the action of the QFT on the signal. The second major new result is a variety of *new steerable forms* of the QFT, their geometric interpretation, and for each form, OPS split theorems, which allow fast and efficient numerical implementation with standard FFT software.

The increasing demand for Fourier transforms on geometric algebras (CFTs) has resulted in an increasing variety of new transforms. In [16] we therefore introduced one single straight forward definition of a general geometric Fourier transform covering most versions in the literature (up to 2011/2012). We showed which constraints are additionally necessary to obtain certain features like linearity or a shift theorem. As a result, we can provide guidelines for the target-oriented design of yet unconsidered transforms that fulfill requirements in a specific application context. Furthermore, the standard theorems do not need to be shown in a slightly different form every time a new geometric Fourier transform (CFT) is developed since they are proved here once and for all. In further research [17] we extended these results by a general CFT convolution theorem.

## 4 Progress in hypercomplex nonlinear adaptive filtering

A split hypercomplex learning algorithm for the training of nonlinear finite impulse response adaptive filters for the processing of hypercomplex signals of any dimension has recently been proposed in [26]. This includes possible applications to vector signals of any dimension. The derivation of the algorithm strictly took into account the laws of hypercomplex algebra and hypercomplex calculus, some of which have previously been neglected in existing learning approaches (e.g. for quaternions). Already in the case of quaternions it became possible to predict improvements in performance of hypercomplex processes. The convergence of the proposed algorithms has been rigorously analyzed.

## 5 Progress in geometric symmetry description

In the field of material science, recent research work [27] explains how, following the representation of 3D crystallographic space groups in Clifford's geometric algebra [28], has made it possible to similarly represent all 162 so called *subperiodic groups* of crystallography in Clifford's geometric algebra. A new compact geometric algebra group representation symbol has thus been constructed, which allows to read off the complete set of geometric algebra generators. For clarity moreover the chosen generators have been stated explicitly. The subperiodic group symbols are based on the representation of point groups in geometric algebra by versors (Clifford products of invertible vectors).

This is yet another indication, that in many fields the proper way of multiplying vectors is Clifford's associative and invertible geometric product. This way of handling vectors brings many simplifications, new geometric understanding and opens up new avenues of research and development. The current paper mainly reviews our own recent work, which is but a small part of the recent new developments in the field. Further researches and new information can be found in [29] and [30], etc.

## Acknowledgment

E.H. wants to acknowledge God [31]:

> In the beginning God created the heavens and the earth. ... And God said, "Let there be light," and there was light.
> In the beginning was the Word, and the Word was with God, and the Word was God. He was with God in the beginning. Through him all things were made; without him nothing was made that has been made.

E.H. further wants to thank his beloved family, as well as T. Nitta and Y. Kuroe.

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