Some new types of filter limit theorems for topological group-valued measures

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Abstract

Some new types of limit theorems for topological group-valued measures are proved in the context of filter convergence for suitable classes of filters. We investigate some fundamental properties of topological group-valued measures. We consider also Schur-type theorems, using the sliding hump technique, and prove some convergence theorems in the particular case of positive measures. We deal with the notion of uniform filter exhaustiveness, by means of which we prove some theorems on existence of the limit measure, some other kinds of limit theorems and their equivalence, using known results on existence of countably additive restrictions of strongly bounded measures.

We study some main properties of topological group-valued measures and some Drewnowski-type theorems on existence of countably additive restrictions of (s)-bounded measures. We investigate some different kinds of limit theorems. First, we consider some particular classes of filters and Schur-type theorems for measures defined on the class of all subsets of $\mathbb{N}$ and we deduce, as consequences, some Vitali-Hahn-Saks, Nikodým and Dieudonné-type theorems. We examine in particular positive measures, showing that in this case it is possible to prove some versions of these kinds of theorems under weaker assumptions on the filter involved. Moreover we deal with the powerful tool of filter exhaustiveness, which has allowed us to find a sub-sequence of the original sequence of measures, indexed by a suitable element of the filter involved, to which it is possible to apply some classical theorems to prove some equivalence results between limit theorems in this setting.

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Let Q be a countable set and $\mathcal{F}$ be a filter of Q. A subset of Q is *$\mathcal{F}$-stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of Q. If $I \in \mathcal{F}^*$, then the *trace* $\mathcal{F}(I)$ of $\mathcal{F}$ on I is the family $\{F \cap I : F \in \mathcal{F}\}$.

A free filter $\mathcal{F}$ of $\mathbb{N}$ is a *P-filter* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ there is a sequence $(B_n)_n$ in $\mathcal{F}$, such that the symmetric difference $A_n \triangle B_n$ is finite for all $n \in \mathbb{N}$ and $\bigcap\limits_{n=1}^{\infty} B_n \in \mathcal{F}$.

A filter $\mathcal{F}$ of Q is said to be *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$. Note that every P-filter $\mathcal{F}$ is diagonal.

From now on $\mathcal{F}$ denotes a free filter of $\mathbb{N}$, $R = (R, +)$ is a Hausdorff complete abelian topological group satisfying the first axiom of countability, with neutral element 0, and $\mathcal{J}(0)$ denotes a basis of closed and symmetric neighborhoods of 0. Moreover, given $k \in \mathbb{N}$ and U, $U_1, \ldots, U_k \subset R$, put $U_1 + \cdots + U_k := \{u_1 + \ldots + u_k \colon\ u_1 \in U_1, \ldots, u_k \in U_k\}$, and $k\,U := U + \cdots + U$ (k times).

A sequence $(x_n)_n$ in R *$\mathcal{F}$-converges to* $x_0 \in R$ iff for every $U \in \mathcal{J}(0)$, $\{n \in \mathbb{N} : x_n - x_0 \in U\} \in \mathcal{F}$, and we write $(\mathcal{F}) \lim\limits_n x_n = x_0$. Moreover, we say that a sequence $(B_n)_n$ of subsets of R *$\mathcal{F}$-converges to* 0 iff for each $U \in \mathcal{J}(0)$ the set $\{n \in \mathbb{N} : B_n \subset U\}$ belongs to $\mathcal{F}$, and we write $(\mathcal{F}) \lim\limits_n B_n = 0$.

Given an infinite set $I \subset Q$, a *blocking* of I is a countable partition $\{D_k : k \in \mathbb{N}\}$ of I into nonempty finite subsets.

A filter $\mathcal{F}$ of Q is said to be *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of I there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\sharp(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\sharp$ denotes the number of elements of the set into brackets.

Let Σ be a σ-algebra of parts of an abstract infinite set G. We say that a finitely additive measure $m : \Sigma \to R$ is *(s)-bounded* on Σ iff

$$\lim_k m(C_k) = 0 \quad \text{for each disjoint sequence } (C_k)_k \text{ in } \Sigma. \tag{1}$$

A finitely additive measure $m : \Sigma \to R$ is said to be *σ-additive* on Σ iff

$$m\Big(\bigcup_{k=1}^{\infty} C_k\Big) = \sum_{k=1}^{\infty} m(C_k) := \lim_i \Big(\sum_{k=1}^{i} m(C_k)\Big) \tag{2}$$

for every disjoint sequence $(C_k)_k$ in Σ.

A *submeasure* $\eta : \Sigma \to [0, +\infty]$ is a set function with $\eta(\emptyset) = 0$, $\eta(A) \leq \eta(B)$ whenever A, $B \in \Sigma$, $A \subset B$, and $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever A, $B \in \Sigma$ and $A \cap B = \emptyset$.

For every σ-algebra $\mathcal{L} \subset \Sigma$, set $m^{\mathcal{L}}(A) := \bigcup \{m(B) : B \in \mathcal{L}, B \subset A\}$, $A \in \mathcal{L}$. Moreover, put $m^+(A) := m^{\Sigma}(A) = \bigcup \{m(B) : B \in \Sigma, B \subset A\}$, $\quad A \in \Sigma$. Given two finitely additive measures $m : \Sigma \to R$, $\lambda : \Sigma \to [0, +\infty]$, we say that m is λ-*absolutely continuous* or shortly λ-*continuous* on Σ, iff $\lim_k m^+(H_k) = 0$ for every decreasing sequence $(H_k)_k$ in Σ such that $\lim_k \lambda(H_k) = 0$.

Let τ be a Fréchet-Nikodým topology on Σ. A finitely additive measure $m : \Sigma \to R$ is τ-*continuous* on Σ, iff $\lim_k m^+(H_k) = 0$ for each decreasing sequence $(H_k)_k$ in Σ, with τ-$\lim_k H_k = \emptyset$.

A finitely additive measure $m : \Sigma \to R$ is said to be *positive* iff

$$m^+(A) = \{m(A)\} \quad \text{for every } A \in \Sigma. \tag{3}$$

We now state a Drewnowski-type theorem on existence of σ-additive restrictions of (s)-bounded topological group-valued measures.

Theorem 0.1 *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of finitely additive measures. Then for any disjoint sequence $(C_k)_k$ in Σ there exists an infinite subset $P \subset \mathbb{N}$, with*

$$\lim_h \Big(\bigcup \Big\{ m_j \Big(\bigcup_{k \in Y, k \geq h} C_k \Big) : Y \subset P \Big\} \Big) = 0$$

for every $j \in \mathbb{N}$, and each m_j is σ-additive on the σ-algebra generated by the sets C_k, $k \in P$.

The finitely additive measures $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *uniformly* (s)-*bounded* on Σ iff $\lim_k \Big(\bigcup_{j=1}^{\infty} m_j^+(C_k) \Big) = 0$ for each disjoint sequence $(C_k)_k$ in Σ. The m_j's are *uniformly σ-additive* on Σ iff $\lim_k \Big(\bigcup_{j=1}^{\infty} m_j^+(H_k) \Big) = 0$ for each decreasing sequence $(H_k)_k$ in Σ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$. If λ is a finitely additive measure on Σ, then the m_j's are said to be *uniformly λ-absolutely continuous* or shortly *uniformly λ-continuous* on Σ iff $\lim_k \Big(\bigcup_{j=1}^{\infty} m_j^+(H_k) \Big) = 0$ for each decreasing sequence $(H_k)_k$ in Σ with $\lim_k \lambda(H_k) = 0$. If τ is a Fréchet-Nikodým topology on Σ, then the m_j's are *uniformly τ-continuous* on Σ iff $\lim_k \Big(\bigcup_{j=1}^{\infty} m_j^+(H_k) \Big) = 0$ for each decreasing sequence $(H_k)_k$ in Σ with τ-$\lim_k H_k = \emptyset$.

Let now $\mathcal{G}$, $\mathcal{H} \subset \Sigma$ be two lattices, such that $\mathcal{G}$ is closed with respect to countable disjoint unions, and the complement of every element of $\mathcal{H}$ (with respect to G) belongs to $\mathcal{G}$. We say that $m : \Sigma \to R$ is *regular* on Σ iff for every $A \in \Sigma$ there exist two sequences $(G_k)_k$ in $\mathcal{G}$, $(F_k)_k$ in $\mathcal{H}$, with $F_k \subset F_{k+1} \subset A \subset G_{k+1} \subset G_k$ for every k and $\lim_k m^+(G_k \setminus F_k) = 0$. Observe that, if $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are regular measures, then the sequences $(G_k)_k$, $(F_k)_k$ can be taken independently of j. The measures $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are said to be *uniformly regular* on Σ iff to every $A \in \Sigma$ there correspond two sequences $(G_k)_k$ in $\mathcal{G}$, $(F_k)_k$ in $\mathcal{H}$, with $F_k \subset F_{k+1} \subset A \subset G_{k+1} \subset G_k$ for every k and $\lim_k \Big(\bigcup_{j=1}^{\infty} m_j^+(G_k \setminus F_k) \Big) = 0$.

We now state the following Schur-type theorem, in which the hypothesis that the involved filter is block-respecting is essential, even when $R = \mathbb{R}$.

Theorem 0.2 *Let $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive measures, and assume that*

(i) $\lim_j m_j(\{n\}) = 0$ for any $n \in \mathbb{N}$, and

(ii) $(\mathcal{F}) \lim_j m_j(A) = 0$ for every $A \subset \mathbb{N}$.

Then we have:

β) $(\mathcal{F}) \lim_j m_j^+(\mathbb{N}) = 0$;

$\beta\beta$) if $\mathcal{F}$ is also diagonal, then the only condition (ii) is sufficient to get β).

We now state a Vitali-Hahn-Saks-type theorem.

Theorem 0.3 *Let $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$, τ be a Fréchet-Nikodým topology on Σ, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of τ-continuous measures, with*

$$(\mathcal{F}) \lim_j m_j(A) = 0 \quad \textit{for every } A \in \Sigma. \tag{4}$$

Then for each decreasing sequence $(H_k)_k$ in Σ with τ-$\lim_k H_k = \emptyset$ and for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there is an $\mathcal{F}$-stationary set $J \subset I$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^{\mathcal{L}}(H_k) \Big) = 0,$$

where $\mathcal{L}$ is the σ-algebra generated by the H_k's in H_1.

We have also the following Nikodým convergence-type theorem.

Theorem 0.4 *Let $\mathcal{F}$ be as above, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive measures, satisfying condition (4). Then for each decreasing sequence $(H_k)_k$ in Σ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$ and for every $I \in \mathcal{F}^*$ there exists $J \in \mathcal{F}^*$, $J \subset I$, with*

$$\lim_k \Big(\bigcup_{j \in J} m_j^{\mathcal{L}}(H_k) \Big) = 0.$$

In the next theorems, which are formulated for positive topological group-valued measures, the involved filter is required to be only diagonal, and not necessarily block-respecting. An example of such a filter is the class of all subsets of $\mathbb{N}$ having asymptotic density 1, which is also a P-filter, and is related to the statistical convergence.

We have a Vitali-Hahn-Saks-type theorem.

Theorem 0.5 *Let τ be a Fréchet-Nikodým topology on Σ, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of positive finitely additive (s)-bounded and τ-continuous measures. Assume that $m_0(E) := (\mathcal{F}) \lim_j m_j(E)$ exists in R for each $E \in \Sigma$, and that m_0 is σ-additive and positive on Σ.*

Then for every set $I \in \mathcal{F}^*$ and for each decreasing sequence $(H_k)_k$ in Σ with τ-$\lim_k H_k = \emptyset$ there exists a set $J \in \mathcal{F}^*$, $J \subset I$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^{+}(H_k) \Big) = \lim_k \Big(\bigcup_{j \in J} m_j(H_k) \Big) = 0.$$

We get also a Nikodým-type theorem.

Theorem 0.6 *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of positive σ-additive measures. If $m_0(A) := (\mathcal{F}) \lim_j m_j(A)$ exists in R for each $A \in \Sigma$, and m_0 is σ-additive and positive on Σ, then for each $I \in \mathcal{F}^*$ and for every decreasing sequence $(H_k)_k$ in Σ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$ there exists $J \in \mathcal{F}^*$, $J \subset I$, with*

$$\lim_k \Big(\bigcup_{j \in J} m_j^{+}(H_k) \Big) = \lim_k \Big(\bigcup_{j \in J} m_j(H_k) \Big) = 0.$$

Finally, we state a Dieudonné-type theorem.

Theorem 0.7 *Let G, Σ, $\mathcal{F}$ be as in Theorem 0.5, $\mathcal{G}$, $\mathcal{H} \subset \Sigma$ be as above, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of positive regular measures, such that $m_0(E) := (\mathcal{F}) \lim_j m_j(E)$ exists in R for every $E \in \Sigma$, and m_0 is σ-additive and positive.*

Furthermore, let $A \in \Sigma$ and $(G_k)_k$, $(F_k)_k$ be two sequences in $\mathcal{G}$, $\mathcal{H}$ respectively, with $F_k \subset F_{k+1} \subset A \subset G_{k+1} \subset G_k$ for every $k \in \mathbb{N}$, and

$$\lim_k m_j(G_k \setminus F_k) = 0 \quad \textit{for every } j \in \mathbb{N}. \tag{5}$$

Then for each $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^+(G_k \setminus F_k) \Big) = 0. \tag{6}$$

Let $\mathcal{F}$ be a free filter of $\mathbb{N}$, and $\lambda : \Sigma \to [0, +\infty]$ be a finitely additive measure, such that Σ is separable with respect to the Fréchet-Nikodým topology generated by λ (shortly, *λ-separable*). Assume that $m_j : \Sigma \to R$, $j \in \mathbb{N}$, is a sequence of finitely additive measures.

We say that the m_j's are *λ-uniformly $\mathcal{F}$-exhaustive on* Σ iff for every $U \in \mathcal{J}(0)$ there exist $\delta > 0$ and $V \in \mathcal{F}$ with $m_j(E) - m_j(F) \in U$ whenever $E, F \in \Sigma$ with $|\lambda(E) - \lambda(F)| \leq \delta$ and for any $j \in V$.

A sequence of finitely additive measures $m_j : \Sigma \to R$, $j \geq 0$, together with λ, *satisfies property $(*)$ with respect to R and $\mathcal{F}$* iff it is λ-uniformly $\mathcal{F}$-exhaustive on Σ and $(\mathcal{F}) \lim_j m_j(E) = m_0(E)$ for any $E \in \Sigma$.

We now give a result on equivalence of filter limit theorems. Assume that

H) $\lambda : \Sigma \to [0, +\infty]$ is a finitely additive measure, Σ is a λ-separable σ-algebra, $\mathcal{F}$ is a P-filter of $\mathbb{N}$, m_0, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are finitely additive measures, satisfying together with λ property $(*)$ with respect to R and $\mathcal{F}$ on Σ, and Σ_0 is a sub-σ-algebra on Σ. Then the following four theorems hold and are equivalent.

Brooks-Jewett (BJ): *If the m_j's are (s)-bounded on Σ_0, then there exists a set $M_0 \in \mathcal{F}$, such that the measures m_j, $j \in M_0$, are uniformly (s)-bounded on Σ_0.*

Vitali-Hahn-Saks (VHS): *If every m_j is (s)-bounded and τ-continuous on Σ_0, then there exists a set $M_0 \in \mathcal{F}$, such that the measures m_j, $j \in M_0$, are uniformly (s)-bounded and uniformly τ-continuous on Σ_0.*

Nikodým (N): *If each m_j is σ-additive on Σ_0, then there is $M_0 \in \mathcal{F}$, such that the measures m_j, $j \in M_0$, are uniformly σ-additive on Σ_0.*

Dieudonné (D): *If each m_j is (s)-bounded and regular on Σ_0, then there is $M_0 \in \mathcal{F}$ with the property that the measures m_j, $j \in M_0$, are uniformly (s)-bounded and uniformly regular on Σ_0.*

Observe that in general, even when $R = \mathbb{R}$, the hypothesis of λ-uniform $\mathcal{F}$-exhaustiveness in general cannot be dropped.