Fractal Spacetime as Tentative Solution for the Cosmological and Coincidence Problems

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Abstract

Recent years have hinted that the concept of fractal spacetime may play a role in both the "would-be" physics beyond the Standard Model and the large-scale structure of the Universe. Here we explore a scenario where classical spacetime equipped with minimal fractality appears to provide a natural solution for two major challenges of relativistic cosmology, the cosmological constant and coincidence problems.

Key words: cosmological constant problem, coincidence problem, fractal space-time, effective refractive index, de Sitter metric, Friedman-Robertson-Walker metric, cosmic string network.

1. Introduction

As it is well known, the principle of *general* (or *diffeomorphism*) *covariance* lies at the core of General Relativity (GR) [1, 2]. This principle asserts that all physical laws must take the same mathematical form regardless of the coordinate system used by observers in arbitrary relative motion. Stated differently, general covariance means invariance of physical laws under all possible coordinate transformations and is typically formulated in terms of tensor fields. An implicit assumption of general covariance is that any coordinate transformation and its inverse are smooth functions that can be differentiated arbitrarily many times. However, as it is also known, there is a plethora of non-differentiable curves and surfaces in Nature, as repeatedly discovered since the introduction of fractal geometry in 1983 [3]. The inevitable conclusion is

GR assigns a preferential status to differentiable transformations and the smooth geometry of spacetime, which is at odds with the very spirit of general covariance.

A natural question arises on how to properly integrate fractal geometry in the classical framework of GR without spoiling its internal consistency. Echoing the points made in [4-13], we believe that a promising avenue along these lines is through the use of *fractional field theory* or *dimensional reduction*. As we explain below, although both approaches target primarily the realm of sub-nuclear scales (that is, physics beyond the Standard Model of elementary particles as well as the ultraviolet regime of Quantum Gravity), they may be also of great utility on macroscopic scales, where classical behavior and GR become relevant. A direct outcome of this scenario in the limit of low-level fractionality is that the description of classical fields on flat spacetime matches the description of the same fields coupled to gravitation. This viewpoint enables one to understand gravitation as an "*effective*" medium induced by the fractal attributes of spacetime [12-13] and makes it possible to equate classical trajectories of either massive or massless particles in GR with their paths on flat fractal manifolds.

In fact, the duality between motion in non-Euclidean metric and on fractal manifolds can be substantiated using a simple argument from classical optics: light bending in gravitational fields is analogous to light bending in an optical medium having a varying refractive index [15-16]. Since a flat spacetime manifold having an underlying fractal structure can be shown to act like a medium with a locally defined effective refractive index (ERI) [6, 9-10, 12], the analogy may be used to gain insight into some of the open puzzles of relativistic cosmology. In particular, as we argue below, exploiting this duality leads to a natural resolution of both *cosmological constant* and *coincidence problems*. Moreover, the proposed solution steers clear of any anthropic reasoning and fine-tuning arguments, does not require the presence of new fields or symmetries in the theory and does not make any prior assumptions on the dynamic nature of the cosmological constant.

To gain credibility, our findings must be backed up by concurrent theoretical work and further input from astrophysical observations. Being aware of the unsettled nature of some of the topics discussed here, we caution that our sole intent is sketching an unexplored viewpoint on two open problems of contemporary cosmology.

The paper is organized as follows: the formal analogy between classical field theory coupled to gravitation and the same theory defined on flat fractal spacetime is introduced in the second section. The idea of ERI stemming from the fractal geometry of spacetime forms the topic of the third section. Next section establishes the connection between ERI on fractal spacetime and ERI defined in optical analogues of classical gravity. Building on these findings, the last section shows how this analogy enables a straightforward resolution of the cosmological constant and of the coincidence problems.

2. Fractal space-time as analog of classical gravity

Consider a classical scalar field theory defined on Minkowski spacetime [4-5]

$$S = \int d^4x[-\frac{1}{2}\partial_\mu \varphi \, \partial^\mu \varphi - V(\varphi)] \qquad (1)$$

Straightforward generalization of (1) on fractal spacetime can be carried out by defining a non-trivial measure of the form

$$d\rho_\alpha(x) = d^4x \, \mathrm{v}_\alpha(x) \qquad (2)$$

$$\mathrm{v}_{\alpha}(x) = \prod_{\mu} \frac{\left|x^{\mu}\right|^{\alpha-1}}{\Gamma(\alpha)} \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function and $0 < \alpha \leq 1$ is the fractal dimension, assumed equal for all coordinates. Ordinary flat spacetime surfaces in the limit $\alpha = 1$. Introducing the operator

$$\mathrm{D}_{\mu} = \frac{1}{\sqrt{\mathrm{v}(x)}} \partial_{\mu}(\sqrt{\mathrm{v}(x)}\,\cdot) \tag{4}$$

turns (1) into

$$S = \int d\rho(x)[-\frac{1}{2}\mathrm{D}_{\mu}\varphi \mathrm{D}^{\mu}\varphi - V(\varphi)] \tag{5}$$

Furthermore, assuming a polynomial potential $V(\varphi) \sim \varphi^{n}$ yields

$$S = \int d^{4}x[\mathrm{v}_{\alpha}(x)(-\frac{1}{2})\partial_{\mu}\varphi\, \partial^{\mu}\varphi - \mathrm{v}_{\alpha}(x)^{1-\frac{n}{2}} V(\varphi)] \tag{6}$$

Compare now (6) with classical scalar field theory in curved spacetime induced by gravitational fields [1, 18]

$$S = \int d^{4}x\sqrt{-g}[-\, g^{\mu\nu}(x)(\frac{1}{2}\partial_{\mu}\varphi \partial^{\nu}\varphi) - V(\varphi)] \tag{7}$$

It is apparent that the contribution of fractal spacetime carried by the nontrivial measure (2) and (3) is formally identical with the contribution of the metric potential $g^{\mu\nu}$ as in

$$\boxed{\sqrt{-g}\, g^{\mu\nu}(x) \Rightarrow \mathrm{v}_{\alpha}(x)} \tag{8}$$

$$\boxed{\sqrt{-g} \Rightarrow \mathrm{v}_\alpha(x)^{1-\frac{n}{2}}} \tag{9}$$

It is also apparent that the passage to flat space-time ($\det(\sqrt{-g}) \to 1$, $g^{\mu\nu} \to \eta^{\mu\nu}$ with signature $\eta^{\mu\nu} = (-1,+1,+1,+1)$) occurs in the limit $\alpha = 1$.We are led to conclude from this brief analysis that, at least in principle, *scalar field theory on fractal space time is formally reducible to scalar field theory coupled to classical gravitation* [7, 9-10].

3. The effective refractive index of fractal space-time

A remarkable property of fractal spacetime is that it turns the classical vacuum into an "effective" medium equipped with polarization properties [12]. In particular, the departure from integer dimensionality changes the magnitude of electrical and magnetic fields, as well as of currents and charges in empty space. Three coefficients contribute to these deviations, namely:

$$c_1(\gamma, \mathbf{r}_0) = \frac{2^{1-\gamma}\Gamma(1/2)}{\Gamma(\gamma/2)} |\mathbf{r}_0|^{\gamma-1} \tag{10}$$

$$c_2(d, \mathbf{r}_0) = \frac{2^{2-d}}{\Gamma(d/2)} |\mathbf{r}_0|^{d-2} \tag{11}$$

$$c_3(D, \mathbf{r}_0) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} |\mathbf{r}_0|^{D-3} \tag{12}$$

Here, $\mathbf{r}_0$ represents the position vector in a spherical coordinates, γ, d and D are non-integers numbers that generalize the familiar one, two and three dimensions of ordinary spaces and $\Gamma(\cdot)$ stands for the Gamma function. It follows that the ERI for the fractal spacetime is given by

$$n_{eff} = \sqrt{\varepsilon_{eff}\mu_{eff}} = \frac{c_2(d,\mathbf{r}_0)}{c_1(\gamma,\mathbf{r}_0)} \tag{13}$$

Consider now that the space-time has low-level fractality in one, two and three spatial dimensions, which amounts to

$$\gamma = d = D = O(\varepsilon), \quad \varepsilon << 1 \tag{14}$$

In particular, let us take $\gamma = O(\varepsilon)$ and $d = O(\varepsilon')$, with ε and ε' arbitrarily close to each other. In this case the ERI (13) becomes

$$n_{eff} = 1 + O(\varepsilon - \varepsilon') = 1 + O(\varepsilon^2) \tag{15}$$

The above relation tells us that the departure from unity of the ERI scales quadratically with the dimensional parameter ε, when $\varepsilon << 1$. We shall use (15) in the next section.

4. Optical analogues of General Relativity

Modeling gravitation as an optical medium is an old idea, first formulated by Eddington and developed by others [15-16]. Here we follow [15] and briefly review the basis for this analogy. If the space is considered isotropic and the metric time-independent, the line element may be written as

$$ds^2 = \Omega^2(\mathbf{r})c_0^2 dt^2 - \Phi^{-2}(\mathbf{r})\left|d\mathbf{r}\right|^2 \tag{16}$$

in which Ω and Φ are functions of the isotropic coordinates $\mathbf{r} = (r,\theta,\varphi)$ and c_0 is the speed of light in vacuum. The speed of light at any point in the gravitational field results from setting $ds = 0$, which yields

$$c(\mathbf{r}) = \left| {d\mathbf{r}}/{dt} \right| = c_0 \Phi(\mathbf{r}) \Omega(\mathbf{r}) \tag{17}$$

The ERI induced by gravitation is thus

$$n(\mathbf{r}) = \frac{c_0}{c(\mathbf{r})} = \Phi^{-1}(\mathbf{r}) \Omega^{-1}(\mathbf{r}) \tag{18}$$

Using (18) the ERI in the de Sitter (*dS*) model is given by

$$n_{dS}(\mathbf{r}) = \frac{1}{1 - \frac{\Lambda_n(\mathbf{r})}{12}} \tag{19}$$

where Λ represents the dimensionful cosmological constant and $\Lambda_n(\mathbf{r}) = \Lambda \mathbf{r}^2$ is the same constant expressed in dimensionless form. Due to the exceedingly small magnitude of $\Lambda_n(\mathbf{r})$, (19) is well approximated by

$$n_{dS}(\mathbf{r}) = 1 + O(\Lambda_n(\mathbf{r})) \tag{20}$$

Likewise, the ERI in the Friedman-Robertson-Walker (*FRW*) model is

$$n_{FRW}(\mathbf{r}, t) = \frac{a(t)}{1 + \frac{k\mathbf{r}^2}{4}} = \frac{a(t)}{1 + \frac{K}{4}} \tag{21}$$

where $a(t)$ is the expansion parameter, k represents the Gaussian curvature with units [length] ^ (-2) and K is the dimensionless curvature [15]. Consider now an asymptotically flat FRW model corresponding to an open Universe ($K_U < 0$, $K_U \to 0$) of maximal observable radius

$$R_U = \sqrt{1 \big/ \left|K_U\right|} \tag{22}$$

It follows that (21) is well approximated in this limit by

$$n_{FRW}(\mathbf{r},t) = 1 + O(R_U^{-2}) \tag{23}$$

We note that both (20) and (23) become consistent with (15), if

$$\varepsilon = O(R_U^{-1}) = O(\Lambda_n^{1/2}) \tag{24}$$

5. Solving the cosmological constant and coincidence problems

5.1) Current estimates place the diameter of the observable Universe at $2R_U \approx 9.3\times10^{10}$ light-years $\approx 8.8\times10^{26}$ m [21]. Since the Planck length is 1.6162×10^{-35} m, the radius of the observable Universe expressed in Planck units (PU) amounts to

$$R_U \approx 2.7\times10^{61}\,\text{(PU)} \tag{25}$$

As seen from (20), (23) and (24), both effective refractive indices for the *dS* and *FRW* models deviate slightly from unity, under the assumption that the radial coordinate approaches its maximal observed value ($\left|\mathbf{r}\right| \to R_U,\ K^{-2} \to R_U$). Combining (20), (23) and (25) yields

$$\boxed{\Lambda_n = O(R_U^{-2}) = O(10^{-123})\ \text{(PU)}} \tag{26}$$

The magnitude of the cosmological constant given by (26) is in close agreement with current astrophysical observations [17].

5.2) The concept of fractal texture underlying the large scale geometry of spacetime resonates well with the hypothesis of the so-called *cosmic string network* [19-20]. In line with this hypothesis, consider that the overall mass M of matter and radiation in the observable Universe is uniformly distributed along a random web of nearly one-dimensional strings (that, in turn, may be interpreted as *topological defects* of the space-time continuum). It is then natural to assume that M scales almost linearly with R_U as in

$$M \sim R_U^{1-\varepsilon}\,,\ \varepsilon << 1 \text{ (PU)} \tag{27}$$

The density of matter and radiation of the present epoch takes the form

$$\rho_M \sim \frac{M}{R_U^3} \sim R_U^{-(2+\varepsilon)} \text{ (PU)} \tag{28}$$

It follows that (28) falls close to the numerical value of the cosmological constant (26), that is,

$$\boxed{\rho_M = O(\Lambda_n) \text{ (PU)}} \tag{29}$$

6. Conclusions

Our brief analysis reveals that classical spacetime equipped with minimal fractality may provide a natural explanation for both the cosmological constant and coincidence problems. The proposed solution is not based on any anthropic reasoning and fine-tuning arguments, does not invoke the presence of new fields or symmetries in the theory and does not make any prior assumptions on the dynamic nature of the cosmological constant.

References

[1] P. J. E. Peebles, “Principles of physical cosmology”, Princeton University Press, Princeton, New Jersey, 1993.

[2] C. Misner, K. Thorne and J. Wheeler, “Gravitation”, Freeman, San Francisco, 1973.

[3] B. Mandelbrot, “The fractal geometry of nature”, Freeman, San Francisco, 1982.

[4] G. Calcagni, “Multiscale Gravity and Cosmology”, http://arxiv.org/abs/1307.6382.

[5] G. Calcagni and G. Nardelli, “Symmetries and propagator in multi-fractional scalar field theories”, Phys. Rev. D, vol. 87, issue 8, 085008 (2013).

[6] E. Goldfain, “Fractional dynamics and the TeV regime of field theory”, http://www.sciencedirect.com/science/article/pii/S1007570406001183.

[7] E. Goldfain, “Fractal Space-Time as Underlying Structure of the Standard Model”, Quantum Matter, 2013 (in press).

[8] S. Vacaru, “Fractional Dynamics from Einstein Gravity, General Solutions and Black Holes”, Intl. J. Theor. Phys., vol. 51, issue 5, 1338 (2011).

[9] E. Goldfain, “Non-equilibrium Theory, Fractional Dynamics and Physics of the Terascale Sector” in *New Developments in the Standard Model*, Nova Science Publishers, 41-74 (2012).

[10] E. Goldfain, “Fractional Field Theory and High-Energy Physics: New Developments” in *Horizons in World Physics*, 279, Nova Science Publishers, 69-92 (2013).

[11] L. Pietronero, "The Fractal Structure of the Universe: Correlations of Galaxies and Clusters", Physica A (144), 257, 1987.

[12] V. Tarasov, “Electromagnetic fields on fractals”, Mod. Phys. Lett. A, 21, 1587 (2006).

[13] V. Tarasov, “Gravitational field of fractal distribution of particles”, http://theory.sinp.msu.ru/~tarasov/PDF/CMDA2006.pdf

[15] J. Evans *et al.*, “The Optical-Mechanical Analogy in General Relativity: Exact Newtonian Forms for the Equations of Motion of Particles and Photons”, Gen. Relativity and Gravitation, 28, 4, 413 (1996).

[16] C. Barcelo, S. Liberati and M. Visser, “Analogue Gravity”, http://arxiv.org/abs/gr-qc/0505065

[17] J. D. Barrow and D. J. Shaw, “The Value of the Cosmological Constant”, Intl. J. Mod. Phys. D, 20, 14, 2875 (2011).

[18] L. Parker and D. Toms, “Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity”, Cambridge University Press, 2009.

[19] T. Hiramatsu *et al.*, “Type I cosmic string network”, http://arxiv.org/pdf/1307.0308v1.pdf

[20] Planck Collaboration, “Planck 2013 results. XXV. Searches for cosmic strings and other topological defects”, http://arxiv.org/pdf/1303.5085v1.pdf

[21] Vaudrevange P. M. *et al.*, “Constraints on the Topology of the Universe: Extension to General Geometries”, http://arxiv.org/pdf/1206.2939v1.pdf