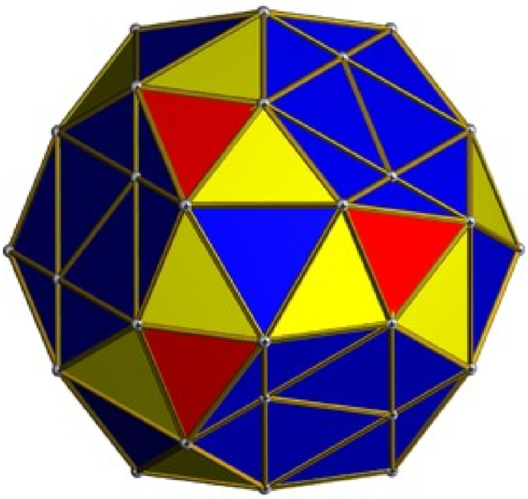
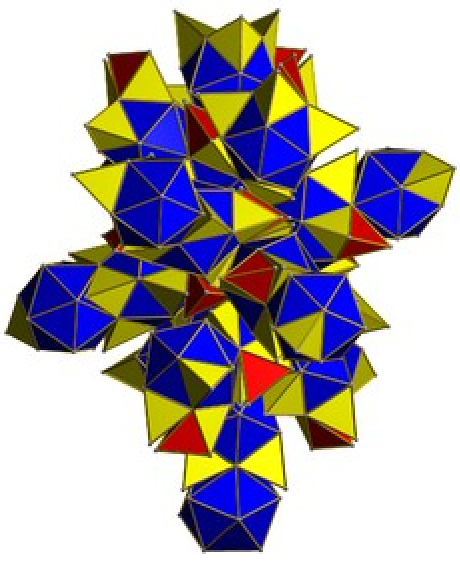
OP2 and the
G2 to B3 to D4 to B4 to F4
Magic Triangle





By John Frederick Sweeney

Abstract

Mathematicians and physicists have long wondered why the Octionic Projective Plane (OP2), the Freudenthal – Tits Magic Square, or Magic Triangle and certain functions of the Octonions and Sedenions abruptly end. This paper lays out the various elements included in this conundra, with the assumption that irregularities and undiscovered relationships between these structures account for the anomalies. In addition to the above, this paper investigates the G2 to B3 to D4 to B4 to F4 Magic Triangle, the twisted product of S7 x S7 x G2, which leads to the Sedenions, the exceptional singularities, Kleinian singularities, Coxeter Groups H3 and H4, Polytope (3,3,5) , the 600 – cell and the binary icosahedral group.

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Introduction

In order to understand OP2, it is best to understand that projective planes exist for Reals, Complex, Quarternions and Octonions, but not for Sedenions or Trigintaduonions. OP2 ends the series. Many writers never make clear whether they refer to a real, complex, or quarternionic Hopf Fibration, which implies that they themselves remain unaware of any differences.

Why write about OP2? In the course of building the Qi Men Dun Jia Model, the author has discovered various anomalies with regard to G2, the Sedenions, the Magic Square and the Magic Triangle, Triality, and the language used to describe the Hopf Fibration in terms of the BC – Helix.

Most writers who describe the Hopf Fibration in terms of the BC – Helix fail to offer any sense of scale, or to differentiate about which type of Hopf Fibration they refer, while others prefer loose terminology such as Hopf Map or Hopf Bundle, or S03, all of which lead to confusion. Since the mathematical mind was not designed for language, humanity shall forever suffer this problem.

This paper is an effort, in part, to straighten out the language problems. Yet in the large part, this paper is an attempt to lay out the various elements of what appear to be related conundrums: the loss of functionality of Cayley Numbers, the higher one travels, the odd dimensions of the Hopf Fibration and the even dimension or degree of the lattices, the Magic Square, E8, the Exceptional Lie Algebras, and the Octonion Projective Plane. Why 3 Hopf Fibrations and then a final OP2?

In the spirit of adventure we set out to lay out these elements and to draw connections, primarily to H3 and to H4 and their related Platonic Solids, for this seems to be the thrust of the Qi Men Dun Jia Model, as we follow the Golden Ratio higher into the Golden Field.

Real Projective Plane

Mridul Aanjaneya introduces the Real Projective Plane and its qualities.

What is the Real Projective Plane?

Definition

The real projective plane $\mathbb{P}^2$ is a set of points in *one-to-one correspondence* with the lines of a vector space $\mathbb{V}^3$ in $\mathbb{R}^3$, with the points in $\mathbb{P}^2$ linearly dependent iff the corresponding lines of $\mathbb{V}^3$ are linearly dependent.

Triangulating the Real Projective Plane, by MRIDUL AANJANEYA

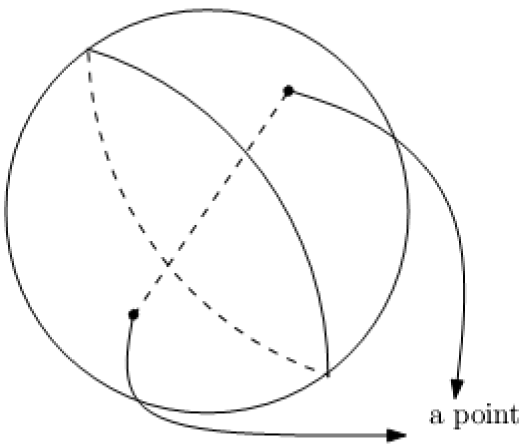
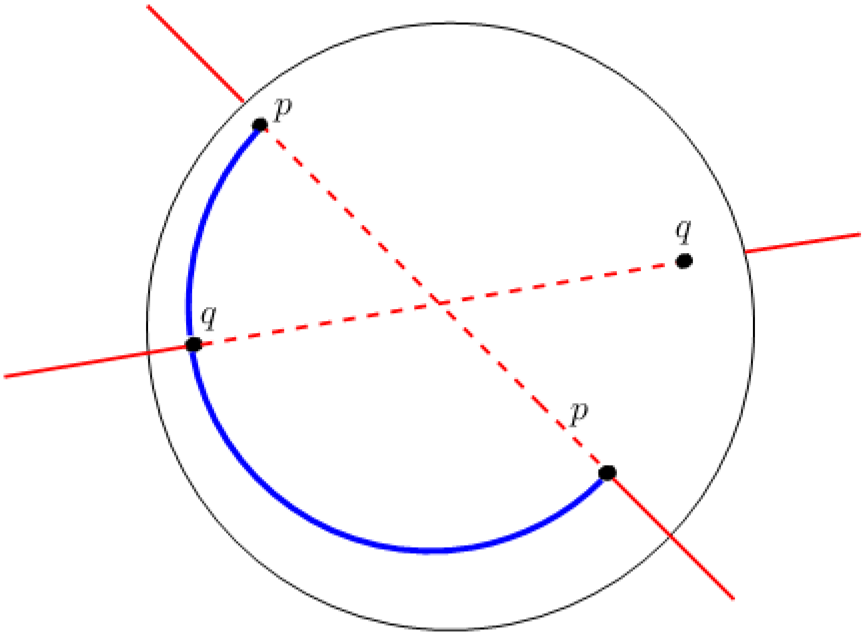


Figure: The sphere model of $\mathbb{P}^2$

- $p = (x, y, z) = \lambda(x, y, z)$.
- There is no notion of “distance" on $\mathbb{P}^2$!



Definition

A *triangulation* of $\mathbb{P}^2$ is a simplicial complex such that each face is bounded by a 3-cycle.

Suppose $\mathcal{M}$ is a map on $\mathbb{P}^2$ and e is an edge of $\mathcal{M}$.

- *Contraction of e* in $\mathcal{M}$ is to remove e and identify its two endpoints.
- Contraction is allowed only if the resulting graph $\mathcal{H}$ is a simplicial complex.
- If $\mathcal{M}$ has no contractible edge, then $\mathcal{M}$ is called *irreducible*.

Theorem (Barnette, 1982)

The real projective plane $\mathbb{P}^2$ admits exactly two irreducible triangulations.

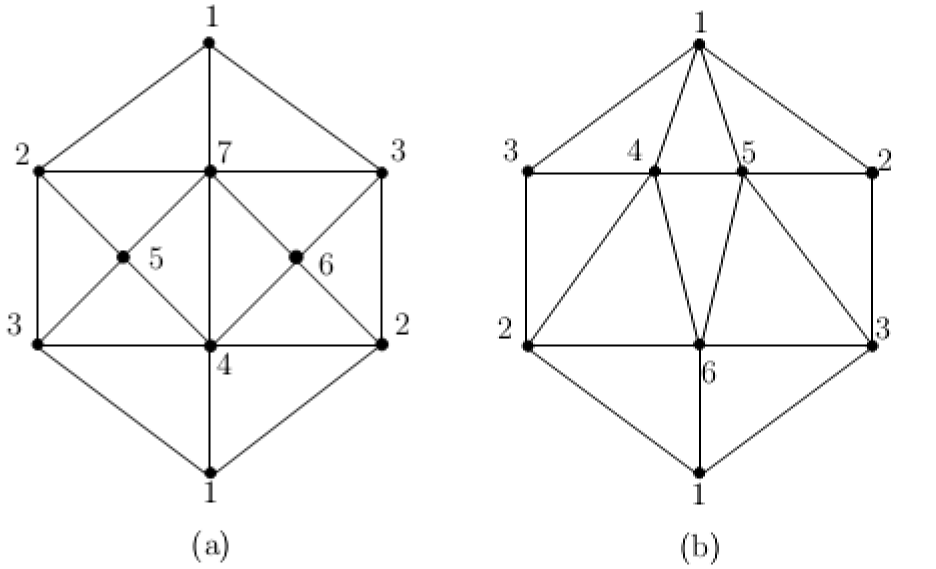
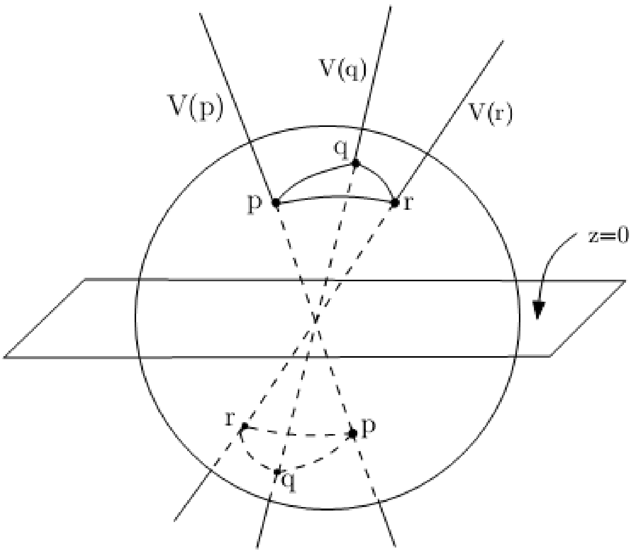


Figure: The two irreducible triangulations of $\mathbb{P}^2$

- The real projective plane with a *disk* cut out is a *Möbius band*.
- The interior of a triangle can be unambiguously defined if we associate a *distinguishing plane* with it.



Lemma

If among every set of four points in $\mathcal{P}$ at least three are collinear, then at least $(n-1)$ points in $\mathcal{P}$ are collinear.

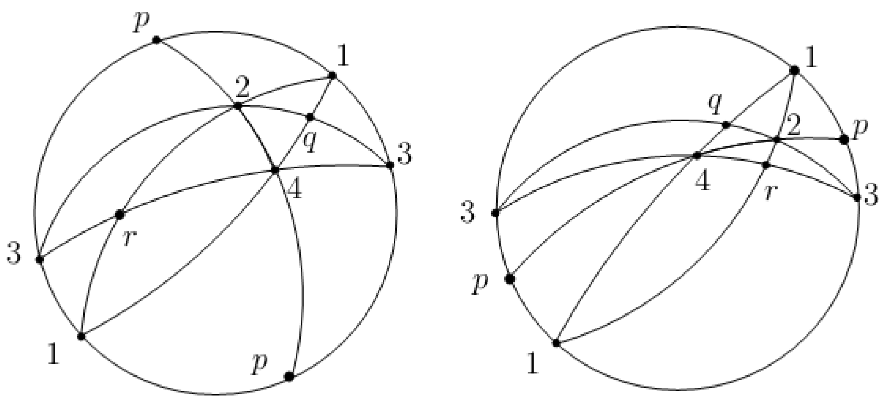
Corollary

If no set of $(n-1)$ points in $\mathcal{P}$ are collinear, then there exists a set of four points no three of which are collinear.

Such a set of four points is called a K_4-*quadrangulation*.

Lemma

A K_4-quadrangulation can be used to construct a triangulation of $\mathbb{P}^2$.



(a) (b)

Lemma

If there exists a K_4-quadrangulation $\mathcal{A}$ such that at least two points in $\mathcal{P}$ lie in different regions of $\mathcal{A}$, then it is possible to triangulate $\mathbb{P}^2$.

Such a K_4-quadrangulation is called a *canonical set*.

A set of points $\mathcal{S}$ is in *general position* if no three points in $\mathcal{S}$ are collinear.

Lemma

If at least six points in $\mathcal{P}$ are in general position, then there always exists a canonical set.

Algorithm

1: *Find a set $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ of six points such that no three points in $\mathcal{S}$ are collinear.*
2: *Construct a projective triangulation with the set $\mathcal{S}$. Associate distinguishing planes with every triangle of the triangulation.*
3: ***for** all points $p \in \mathcal{P} \setminus \mathcal{S}$ **do***
4:     *Identify the triangle $\triangle abc$ in which p lies.*
5:     *Make p adjacent to the vertices a, b and c. Make the distinguishing plane of $\triangle apb, \triangle bpc$, and $\triangle cpa$ the same as that for $\triangle abc$.*
6: ***end for***
7: **return**(*triangulation of $\mathbb{P}^2$*).

The Complex Projective Plane

complex projective plane, usually denoted $P^2(C)$, is the two-dimensional complex projective space. It is a complex manifold described by three complex coordinates

$$(z_1, z_2, z_3) \in \mathbf{C}^3, \qquad (z_1, z_2, z_3) \neq (0,0,0)$$

where, however, the triples differing by an overall rescaling are identified:

$$(z_1, z_2, z_3) \equiv (\lambda z_1, \lambda z_2, \lambda z_3); \quad \lambda \in \mathbf{C}, \qquad \lambda \neq 0.$$

That is, these are homogeneous coordinates in the traditional sense of projective geometry.

Contents

Topology

The Betti numbers of the complex projective plane are

1, 0, 1, 0, 1, 0, 0, ...

The middle dimension 2 is accounted for by the homology class of the complex projective line, or Riemann sphere, lying in the plane. The nontrivial homotopy groups of the complex projective plane are $\pi_2 = \pi_5 = \mathbb{Z}$. The fundamental group is trivial and all other higher homotopy groups are those of the 5-sphere, i.e. torsion.

Algebraic geometry

In birational geometry, a complex rational surface is any algebraic surface birationally equivalent to the complex projective plane. It is known that any non-singular rational variety is obtained from the plane by a sequence of blowing up transformations and their inverses ('blowing down') of curves, which must be of a very particular type. As a special case, a non-singular complex quadric in $\mathbf{P}^3$ is obtained from the plane by blowing up two points to curves, and then blowing down the line through these two points; the inverse of this transformation can be seen by taking a point P on the quadric Q, blowing it up, and projecting onto a general plane in $\mathbf{P}^3$ by drawing lines through P.

The group of birational automorphisms of the complex projective plane is the Cremona group.

Differential geometry

As a Riemannian manifold, the complex projective plane is a 4-dimensional manifold whose sectional curvature is quarter-pinched. The rival normalisations are for the curvature to be pinched between 1/4 and 1; alternatively, between 1 and 4. With respect to the former normalisation, the imbedded surface defined by the complex projective line has Gaussian curvature 1. With respect to the latter normalisation, the imbedded real projective plane has Gaussian curvature 1.

Cremona group, introduced by Cremona (1863, 1865), is the group of birational automorphisms of the n-dimensional projective space over a field k. It is denoted by $\mathrm{Cr}(\mathbf{P}^n(k))$ or $\mathrm{Bir}(\mathbf{P}^n(k))$ or $\mathrm{Cr}_n(k)$.

The Cremona group is naturally identified with the automorphism group $\mathrm{Aut}_k(k(x_1, \ldots, x_n))$ of the field of the rational functions in n indeterminates over k, or in other words a pure transcendental extension of k, with transcendence degree n.

The projective general linear group of order n+1, of projective transformations, is contained in the Cremona group of order n. The two are equal only when $n=0$ or $n=1$, in which case both the numerator and the denominator of a transformation must be linear.

The Cremona group in 2 dimensions

In two dimensions, Max Noether and Castelnuovo showed that the complex Cremona group is generated by the standard quadratic transformation, along with PGL(3, k), though there was some controversy about whether their proofs were correct, and Gizatullin (1983) gave a complete set of relations for these generators. The structure of this group is still not well understood, though there has been a lot of work on finding elements or subgroups of it.

- Cantat & Lamy (2010) showed that the Cremona group is not simple as an abstract group;
- Blanc showed that it has no normal subgroups other than the trivial group and itself that are also closed in a natural topology.
- For the finite subgroups of the Cremona group see Dolgachev & Iskovskikh (2009).

The Cremona group in higher dimensions

There is little known about the structure of the Cremona group in three dimensions and higher though many elements of it have been described. Blanc (2010) showed that it is (linearly) connected, answering a question of Serre (2010). There is no easy analogue of the Noether–Castelnuovo theorem as Hudson (1927) showed that the Cremona group in dimension at least 3 is not generated by its elements of degree bounded by any fixed integer.

De Jonquières groups

A De Jonquières group is a subgroup of a Cremona group of the following form. Pick a transcendence basis $x_1, \ldots, x_n$ for a field extension of k. Then a De Jonquières group is the subgroup of automorphisms of $k(x_1, \ldots, x_n)$ mapping the subfield $k(x_1, \ldots, x_r)$ into itself for some $r \leq n$. It has a normal subgroup given by the Cremona group of automorphisms of $k(x_1, \ldots, x_n)$ over the field $k(x_1, \ldots, x_r)$, and the quotient group is the Cremona group of $k(x_1, \ldots, x_r)$ over the field k. It can also be regarded as the group of birational automorphisms of the fiber bundle $P^r \times P^{n-r} \to P^r$.

When n=2 and r=1 the De Jonquières group is the group of Cremona transformations fixing a pencil of lines through a given point, and is the semidirect product of $PGL_2(k)$ and $PGL_2(k(t))$.

Quaternionic Projective Space

quaternionic projective space is an extension of the ideas of real projective space and complex projective space, to the case where coordinates lie in the ring of quaternions **H**. Quaternionic projective space of dimension *n* is usually denoted by

$$\mathbb{HP}^n$$

and is a closed manifold of (real) dimension *4n*. It is a homogeneous space for a Lie group action, in more than one way.

Its direct construction is as a special case of the projective space over a division algebra. The homogeneous coordinates of a point can be written

$$[q_0, q_1, \ldots, q_n]$$

where the q_i are quaternions, not all zero. Two sets of coordinates represent the same point if they are 'proportional' by a left multiplication by a non-zero quaternion *c*; that is, we identify all the

$$[cq_0, cq_1 \ldots, cq_n].$$

In the language of group actions, $\mathbb{HP}^n$ is the orbit space of $\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$ by the action of $\mathbb{H}^{\times}$, the multiplicative group of non-zero quaternions. By first projecting onto the unit sphere inside $\mathbb{H}^{n+1}$ one may also regard $\mathbb{HP}^n$ as the orbit space of $\mathbb{S}^{4n+3}$ by the action of $\mathrm{Sp}(1)$, the group of unit quaternions.[1] The sphere $\mathbb{S}^{4n+3}$ then becomes a principal Sp(1)-bundle over $\mathbb{HP}^n$:

$$\mathrm{Sp}(1) \to \mathbb{S}^{4n+3} \to \mathbb{HP}^n.$$

There is also a construction of $\mathbb{HP}^n$by means of two-dimensional complex subspaces of $\mathbb{H}^{2n}$, meaning that $\mathbb{HP}^n$lies inside a complex Grassmannian.

Projective line

The one-dimensional projective space over **H** is called the "projective line" in generalization of the complex projective line. For example, it was used (implicitly) in 1947 by P. G. Gormley to extend the Möbius group to the quaternion context with "linear fractional transformations". For the linear fractional transformations of an associative ring with 1, see projective line over a ring and the homography group GL(2,*A*).

From the topological point of view the quaternionic projective line is the 4-sphere, and in fact these are diffeomorphic manifolds. The fibration mentioned previously is from the 7-sphere, and is an example of a Hopf fibration.

Infinite-dimensional quaternionic projective space

The space $\mathbb{HP}^\infty$is the classifying space BS^3. The homotopy groups of $\mathbb{HP}^\infty$are given by $\pi_i(\mathbb{HP}^\infty) = \pi_i(BS^3) \cong \pi_{i-1}(S^3)$. These groups are known to be very complex and in particular they are non-zero for infinitely many values of i. However, we do have that $\pi_i(\mathbb{HP}^\infty) \otimes \mathbb{Q} \cong \mathbb{Q}$if $i = 4$and $\pi_i(\mathbb{HP}^\infty) \otimes \mathbb{Q} = 0$if $i \neq 4$. It follows that rationally, i.e. after localisation of a space, $\mathbb{HP}^\infty$is an Eilenberg-Maclane space $K(\mathbb{Q}, 4)$. That is $\mathbb{HP}^\infty_{\mathbb{Q}} \simeq K(\mathbb{Z}, 4)_{\mathbb{Q}}$. (cf. the example K(Z,2)). See rational homotopy theory.

Quaternionic projective plane

The 8-dimensional $\mathbb{HP}^2$ has a circle action, by the group of complex scalars of absolute value 1 acting on the other side (so on the right, as the convention for the action of *c* above is on the left). Therefore the quotient manifold

$$\mathbb{HP}^2/\mathrm{U}(1)$$

may be taken, writing U(1) for the circle group. It has been shown that this quotient is the 7-sphere, a result of Vladimir Arnold from 1996, later rediscovered by Edward Witten and Michael Atiyah.

Tony Smith Excerpts

In What is a Lie Algebra, Frank "Tony" Smith writes,

The A series contains the complex rotations in the unit circle, S1, and S1 is a Lie group.

The B and C series both contain the quaternion rotations on the unit sphere, S3, and S3 is a Lie group.

The D series contains the Lorentz group in 4-dim space, consisting of two copies of S3 (3 rotations and 3 boosts). Note that in some sense all nonablelian Lie groups can be constructed from nonabelian S3. Roughly, you can take as many copies of S3 as the rank of the Lie group, and then add additional root vectors according to the symmetry of the Weyl group.

HOWEVER, the exceptional Lie groups do NOT include S7, because octonion non-associativity forces S7 to expand, so that S7 is the only unit sphere in a division algebra that is not a Lie group.

To what Lie group does S7 expand?
S7 expands to the twisted product of S7 x S7 x G2,

which is the D-series Lie group D4 = Spin(0,8).

Spin(8) is the spin covering of the rotations in 8-dimensional space, the space of the octonions.

The D4 Lie group Spin(0,8) lives in BOTH:
the standard series Lie groups, as D4;
and
the exceptional octonion Lie groups.

Therefore,
you would expect Spin(0,8) to be a very special Lie group,
and it is.
So much so, that it is the basis of my D4-D5-E6-E7 physics model.

Then somewhat later,

NOW WE CAN LOOK AT THE COMMUTATOR ALGEBRAS
OF THE SPHERES S1, S3, and S7:

Complex S1 is a Lie algebra.	[S1,S1] = 0	S1 COLLAPSES!
Quaternion S3 is a Lie algebra.	[S3,S3] = S3	S3 IS STABLE!
Octonion S7 is NOT a Lie algebra because it does NOT satisfy the Jacobi identity.	[S7,S7] = S7xS7xG2 = Spin(0,8) (x=twisted fibration product)	S7 EXPANDS!

Another way to describe Spin(8) is based on Clifford Algebras:

Spin(8) is the Lie Group whose Lie Algebra is the commutator algebra of bivectors of the real Clifford Algebra Cl(8)
with basis elements:
G0 G1 G2 G3 G4 G5 G6 G7
These 8 real basis elements form an 8-real-dimensional

representation space for Spin(8).

Denote the basis of the complex numbers {1, i}.
The 8-real-dimensional basis of Cl(8) can be rewritten
as a 4-complex dimensional basis:

G0-iG1   G2-iG3   G4-iG5   G6-iG7

These 4 basis elements form a 4-complex-dimensional
representation space for the SU(4) subgroup of Spin(8).

Denote the basis of the quaternions {1, i, j, k}.
The 8-real-dimensional basis of Cl(8) can be rewritten
as a 2-quaternionic dimensional basis:

G0-iG1-jG2-kG3   G4-iG5-jG6-kG7

These 2 basis elements form a 2-quaternionic-dimensional
representation space for the Sp(2) subgroup of Spin(8).

Denote the basis of the octonions {1, i, j, k, E, I, J, K}.
The 8-real-dimensional basis of Cl(8) can be rewritten
as a 1-octonionic dimensional basis:

G0-iG1-jG2-kG3-EG4-IG5-JG6-KG7

This 1 basis element forms a 1-octonionic-dimensional
representation space for an S7 subset of Spin(8).
The S7 subset of Spin(8) is acted upon
by the G2 subgroup of Spin(8).
Notice that the 7-sphere S7 is not a Lie algebra,
but if you extend it to make a Lie algebra, you get Spin(8),
which has an 8-real-dimensional representation space,
that corresponds to the 1-octonionic-dimensional space.

The late Guillermo Moreno wrote

that the set of zero divisors (with entries of norm one)
in the sedenions is homeomorphic to the Lie group G2.
If the sedenions are regarded as the Cayley-Dickson
product of two octonion spaces, then:
if you take one 7-sphere S7 in each octonion space,
and
if you take G2 as the space of zero divisors,
then
YOU CAN CONSTRUCT FROM THE SEDENIONS the Lie group Spin(0,8)
as the twisted fibration product S7 x S7 x G2.
Such a structure is represented
by the design of the Temple of Luxor.

We may assume here that this construction of G2 has something to do with

By a theorem of Frobenius (1877), there are three and only three associative finite division algebras:
1-dimensional real numbers R;
2-dimensional complex numbers C; and
4-dimensional quaternions Q.

In 1958, Kervaire and Milnor proved independently of each other that the finite-dimensional real division algebras have dimensions 1, 2, 4, or 8.
To do so, they used the periodicity theorem of Bott on the homotopy groups of unitary and orthogonal groups.

In 1960, Adams proved that a continuous multiplication in R(n+1) with two-sided unit and with norm product exists only for n+1 = 1, 2, 4, or 8.

In 1962, Adams proved that the maximum number of linearly independent tangent vector fields on a sphere Sn is equal to n for (n+1) = 2, 4, or 8,
and is less than n for all other n.
(It had been proven earlier that the only parallelizable spheres are S1, S3, and S7.)

So we get (n-1) for the odd dimensional spheres. This seems to be the opposite of combinatorial, in fact it appears to be a reduction in size.

real division algebras must have dimension 1,2,4,8
"...has been derived on the basis of topological reasoning on a seven-dimensional sphere. A pure algebraic proof of the theorem is still unknown."

Topological study of such manifolds has produced the only presently known proofs that the dimension of a division algebra must be 1, 2, 4, or 8.
No algebraic proof is now known (Okubo (1995)).

Hopf fibrations can only be done for spheres of dimension 1,3,7,15:

S0	- S1	- S1	(S0 = point)	based on real numbers
S1	- S3	- S2		based on complex numbers
S3	- S7	- S4		based on quaternions
S7	- S15	- S8		based on octonions

```
The 0-grade 1-dimensional scalar space of Cl(0,15)
represents the sedenion real axis.

There is an 8 to 1 correspondence between the
1x128 minimal ideal SSPINOR on which SL acts by Clifford action,
and
the 1x16 sedenion column vectors on which
SL acts by matrix-vector action.

This leads
to failure of the division algebra property of sedenions,
because the map SL from SSPINOR to S is 8 to 1
and invertible.
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the derivation algebra of ALL Cayley-Dickson algebra

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For sedenions, you lose the following properties:

the division algebra (over R) property
     xy = 0 only if x =/= 0 and y =/= 0 ;
(A concrete example of zero divisors in terms of that basis
is given by Guillermo Moreno in q-alg/9710013:
(e1 + e10)(e15 - e4)  =  -e14 - e5 + e5 + e14 = 0.)

linear alternativity
     (x,y,z) = (xy)z - x(yz) = (-1)P(Px,Py,Pz)
     where P is a permutation of sign (-1)P ;

and the Moufang identities
     (xy)(zx) = x(yz)x
     (xyx)z = x(y(xz))
     z(xyx) = ((zx)y)x .
```

the derivation algebra of ALL Cayley-Dickson algebra

```
at the level of octonions or larger,
that is, of dimension 2^N  where N = 3 or greater,
is the exceptional Lie algebra G2,
the Lie algebra of the automorphism group of the octonions.

The exceptional Lie algebra G2 is 14-dimensional,
larger than the 8-dimensional octonions,
but smaller than the 16-dimensional sedenions.

Each 8-dimensional half-spinor space of Cl(0,8)
has a 7-dimensional subspace of "pure" spinors
(see Penrose and Rindler, Spinors and space-time,
                          vol. 2, Cambridge 1986)
that correspond by triality to the
7-dimensional null light-cone of the 8-dim vector space.

The 7-dimensional spaces are 7-dimensional representations
of the 14-dimensional Lie algebra G2.
```

For sedenions, you retain the following properties:

anticommutativity of basic units $xy = -yx$;

and nonlinear alternativity of basic units
$(xx)y = x(xy)$ and $(xy)y = x(yy)$.

The 28 new associative triple cycles of the sedenions are related to the 28-dimensional Lie algebra Spin(0,8), and to the 28 different differentiable structures on the 7-sphere S7 that are used to construct exotic structures on differentiable manifolds.

Freudenthal – Tits Magic Square

In mathematics, the **Freudenthal magic square** (or **Freudenthal–Tits magic square**) is a construction relating several Lie algebras (and their associated Lie groups). It is named after Hans Freudenthal and Jacques Tits, who developed the idea independently. It associates a Lie algebra to a pair of division algebras *A*, *B*. The resulting Lie algebras have Dynkin diagrams according to the table at right. The "magic" of the Freudenthal magic square is that the constructed Lie algebra is symmetric in *A* and *B*, despite the original construction not being symmetric, though Vinberg's symmetric method gives a symmetric construction; it is not a magic square as in recreational mathematics.

The Freudenthal magic square includes all of the exceptional Lie groups apart from G_2, and it provides one possible approach to justify the assertion that "the exceptional Lie groups all exist because of the octonions": G_2 itself is the automorphism group of the octonions (also, it is in many ways like a classical Lie group because it is the stabilizer of a generic 3-form on a 7-dimensional vector space - see prehomogeneous vector space).

A \ B	R	C	H	O
R	A_1	A_2	C_3	F_4
C	A_2	$A_2 \times A_2$	A_5	E_6
H	C_3	A_5	D_6	E_7
O	F_4	E_6	E_7	E8

Magic Triangle

Bruce Westbury 2005

In this paper we introduce the sextonions as a six dimensional real alternative algebra intermediate between the split quaternions and the split octonions. Then we argue that the above magic square, magic rectangle and magic triangle should all be extended to include an extra row and column. If the rows or columns are indexed by division algebras then this extra row or column is indexed by the sextonions. In the following extended magic square we give the derived algebras of the intermediate algebras.

(1)

A_1	A_2	C_3	$C_3.H_{14}$	F_4
A_2	$2A_2$	A_5	$A_5.H_{20}$	E_6
C_3	A_5	D_6	$D_6.H_{32}$	E_7
$C_3.H_{14}$	$A_5.H_{20}$	$D_6.H_{32}$	$D_6.H_{32}.H_{44}$	$E_7.H_{56}$
F_4	E_6	E_7	$E_7.H_{56}$	E_8

The notation in this table is that $G.H_n$ means that G has a representation V of dimension n with an invariant symplectic form, ω. Then H_n means the Heisenberg algebra of (V, ω) and $G.H_n$ means the semidirect product of G and H_n. The entry at the intersection of the additional row and column is the bigraded algebra (20).

5. The magic square

There are three constructions of the magic square. All three constructions take a pair of composition algebras $(\mathbb{A}, \mathbb{B})$ and produce a semisimple Lie algebra $L(\mathbb{A}, \mathbb{B})$. The original construction is due to Freudenthal-Tits. Other constructions are Vinberg construction and the triality construction. These constructions are shown to give isomorphic Lie algebras in [BS03]. In all these cases we can extend the construction to include the sextonions and all constructions give isomorphic Lie algebras. Again we find that the intermediate subalgebra of $L(\mathbb{A}, \widetilde{\mathbb{O}})$ is $L(\mathbb{A}, \widetilde{\mathbb{S}})$. This Lie algebra is non-negatively graded; the sum of the components with positive degree is the nilpotent radical; the degree zero component is a complement and is the reductive subalgebra $L(\mathbb{A}, \widetilde{\mathbb{H}})$.

SEXTONIONS AND THE MAGIC SQUARE, BRUCE W. WESTBURY

5.2. **Magic triangle.** There is another approach to the magic square based on dual reductive pairs. This constructs a magic triangle. This magic triangle is given in [Cvi], [Rum97] and [DG02]. This is also implicit in [CJLP99].

The involution which sends $\mathfrak{g}$ to the centraliser in E_8 corresponds to the involution

$$m \mapsto \frac{-2m}{m+2}$$

If we include the Lie algebra $E_7.H_{56}$ with $m = 6$ then this suggests that we should also include a Lie algebra for $m = -3/2$. This Lie algebra is given as the Lie superalgebra $\mathfrak{osp}(1|2)$. Taken literally this suggests that $\mathfrak{osp}(1|2)$ and $E_7.H_{56}$ are a dual reductive pair in E_8. However $\mathfrak{osp}(1|2)$ is not a subalgebra and $E_7.H_{56}$ is not reductive.

In his conclusion, Bruce Westbury indicates the proper path that should have been taken through the Lie Algebras to reach the Magic Triangle:

These two constructions show that, for $0 \leq p \leq 8$, $\mathfrak{a}(\mathbb{A}, W_p)$ and $\mathfrak{so}(Z_{9-p})$ are a dual reductive pair in $L(\mathbb{A}, \mathbb{O})$. Some of these dual reductive pairs are constructed in [Rub94]. Since these Lie algebras also form a sequence of subalgebras this gives a second magic triangle. We will not consider this second magic triangle. Instead we note that two of these can be inserted in the sequence of subalgebras giving the original magic triangle as follows

$$G_2 \to B_3 \to D_4 \to B_4 \to F_4$$

In particular this suggests that from the point of view of the magic triangle the exceptional series should be further extended to include B_3 and B_4. These two cases are not consistent with the numerology of the exceptional series.

From this point of view the magic triangle should be extended to include the inclusions

$$\mathfrak{der}(\mathbb{A}) \to \mathfrak{int}(\mathbb{A}) \to \mathfrak{tri}(\mathbb{A}) \to \mathfrak{a}(\mathbb{A}, \mathbb{R}^2) \to L(\mathbb{A}, \mathbb{R})$$

This sequence makes for $\mathbb{A} = \widetilde{\mathbb{S}}$ and gives the Lie algebras intermediate between the sequence for $\mathbb{A} = \widetilde{\mathbb{S}}$ and the sequence for $\mathbb{A} = \widetilde{\mathbb{H}}$.

Thus our primary interest lies in the path from G2 to B3 to D4 to B4 to F4. Herein lies the keys to the conundrum in the author's view.

Magic Triangle

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						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $3\mathbf{A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $2\mathbf{A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $3\mathbf{A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Kleinian Singularities

In this section we provide excerpts from Andrei Gabrielov to illustrate the key role of Kleinian Singularities in this problem:

Coxeter-Dynkin diagrams and singularities

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Consider, for example, classification of critical points of germs of analytic functions $(\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$. Each simple object in this classification appears, after a change of variables in $(\mathbf{C}^3, 0)$, in the following (A, D, E) list [1, 5, 8]:

$A_k,\ k \geq 1$	$D_k,\ k \geq 4$	E_6	E_7	E_8
$x^{k+1} + y^2 + z^2$	$x^{k-1} + xy^2 + z^2$	$x^4 + y^3 + z^2$	$x^3y + y^3 + z^2$	$x^5 + y^3 + z^2$
cyclic	*dihedral*	*tetrahedral*	*octahedral*	*icosahedral*

These are exactly Kleinian singularities [5, 8] associated with finite subgroups of $SL_2(\mathbf{C})$. The algebra of invariants of the natural action of such a group on $\mathbf{C}^2$ was computed by Klein. It is generated by three polynomials x, y, z, with a single relation. These relations appear in the (A, D, E) list above, for cyclic groups and for binary dihedral, tetrahedral, octahedral, icosahedral groups, respectively. Direct connection between finite subgroups of $SL_2(\mathbf{C})$ and (extended) Dynkin diagrams is provided by McKay correspondence (see [8, 9]).

Several other classification problems produce the same A, D, E list: Lagrangian and Legendrian singularities that appear in optics as the singularities of caustics and wave fronts [2,4]. These singularities are closely related to the singularities of critical points of the corresponding hamiltonians.

Binary Icosahedral Group . Wikipedia

TABLE I: Multiplication table of the binary octahedral group.

	V_0	V_+	V_-	V_1	V_2	V_3
V_0	V_0	V_+	V_-	V_1	V_2	V_3
V_+	V_+	V_-	V_0	V_3	V_1	V_2
V_-	V_-	V_0	V_+	V_2	V_3	V_1
V_1	V_1	V_2	V_3	V_0	V_+	V_-
V_2	V_2	V_3	V_1	V_-	V_0	V_+
V_3	V_3	V_1	V_2	V_+	V_-	V_0

Elements A^* and B^* are defined in (3). One can use Table I to show that the set of quaternions V_0 and $V_+ \oplus V_-$ are separately left invariant under $W(SO(9))$. In other words, they are the $W(SO(9))$ orbits of sizes 8 and 16, respectively. The orbits V_0 and $V_+ \oplus V_-$ are called the 16-cell and 8-cell

This group plays a pivotal role in the QMDJ Model, providing the linkage to Polytope (3,3,5)

Relation to 4-dimensional symmetry groups

The 4-dimensional analog of the icosahedral symmetry group I_h is the symmetry group of the 600-cell (also that of its dual, the 120-cell).

Just as the former is the Coxeter group of type H_3, the latter is the Coxeter group of type H_4, also denoted [3,3,5]. Its rotational subgroup, denoted $[3,3,5]^+$ is a group of order 7200 living in SO(4).

Coxeter group	**symmetry group**			**Double Cover**	**Order**
H_3	600-cell		SO(3		
H_4	120-cell	Polytope [3,3,5]	SO(4)	Spin(4)	14400

The coset space Spin(3) / $2I = S^3$ / $2I$ is a spherical 3-manifold called the Poincaré homology sphere. It is an example of a homology sphere, i.e. a 3-manifold whose homology groups are identical to those of a 3-sphere. The fundamental group of the Poincaré sphere is isomorphic to the binary icosahedral group, as the Poincaré sphere is the quotient of a 3-sphere by the binary icosahedral group.

The author notes here that Andrei observed the presence of singularities with H3 and with H4.

SO(4) has a double cover called Spin(4) in much the same way that Spin(3) is the double cover of SO(3). Similar to the isomorphism Spin(3) = Sp(1), the group Spin(4) is isomorphic to Sp(1) × Sp(1).

The preimage of $[3,3,5]^+$ in Spin(4) (a four-dimensional analogue of $2I$) is precisely the product group $2I \times 2I$ of order 14400. The rotational symmetry group of the 600-cell is then

$$[3,3,5]^+ = (2I \times 2I) / \{ \pm 1 \}.$$

Various other 4-dimensional symmetry groups can be constructed from $2I$. For details, see (Conway and Smith, 2003).

The 600 – Cell

The 600 – Cell ties directly to the Hopf Fibration, according to one writer.

Of primary importance is that the 600 – Cell leads directly to the number 24:

The snub 24-cell may be obtained from the 600-cell by removing the vertices of an inscribed 24-cell and taking the convex hull of the remaining vertices. This process is a *diminishing* of the 600-cell. In geometry, the **snub 24-cell** is a convex uniform polychoron composed of 120 regular tetrahedral and 24 icosahedral cells. Five tetrahedra and three icosahedra meet at each vertex. In total it has 480 triangular faces, 432 edges, and 96 vertices.

The snub 24-cell is related to the truncated 24-cell by an alternation operation. Half the vertices are deleted, the 24 truncated octahedron cells become 24 icosahedron cells, the 24 cubes become 24 tetrahedron cells, and the 96 deleted vertex voids create 96 new tetrahedron cells.

The snub 24-cell may also be constructed by a particular *diminishing* of the 600-cell: by removing 24 vertices from the 600-cell corresponding to those of an inscribed 24-cell, and then taking the convex hull of the remaining vertices. This is equivalent to removing 24 icosahedral pyramids from the 600-cell.

Conversely, the 600-cell may be constructed from the snub 24-cell by augmenting it with 24 icosahedral pyramids.

Coordinates

The vertices of a snub 24-cell centered at the origin of 4-space, with edges of length 2, are obtained by taking even permutations of

$$(0, \pm 1, \pm \varphi, \pm \varphi^2)$$

(where $\phi = (1+\sqrt{5})/2$ is the golden ratio).

These 96 vertices can be found by partitioning each of the 96 edges of a 24-cell into the golden ratio in a consistent manner, in much the same way that the 12 vertices of an icosahedron or "snub octahedron" can be produced by partitioning the 12 edges of an octahedron in the golden ratio. This is done by first placing vectors along the 24-cell's edges such that each two-dimensional face is bounded by a cycle, then similarly partitioning each edge into the golden ratio along the direction of its vector.[4] The 96 vertices of the snub 24-cell, together with the 24 vertices of a 24-cell, form the 120 vertices of the 600-cell.

Structure

Each icosahedral cell is joined to 8 other icosahedral cells at 8 triangular faces in the positions corresponding to an inscribing octahedron. The remaining triangular faces are joined to tetrahedral cells, which occur in pairs that share an edge on the icosahedral cell.

The tetrahedral cells may be divided into two groups, of 96 cells and 24 cells respectively. Each tetrahedral cell in the first group is joined via its triangular faces to 3 icosahedral cells and one tetrahedral cell in the second group, while each tetrahedral cell in the second group is joined to 4 tetrahedra in the first group.

Symmetry

The snub 24-cell has three vertex-transitive colorings based on a Wythoff construction on a Coxeter group from which it is alternated: F_4 defines 24 interchangeable icosahedra, while the BC_4 group defines two groups of icosahedra in a 8:16 counts, and finally the D_4 group has 3 groups of icosahedra with 8:8:8 counts.

This last section bears repeating and amplification since it provides essential information:

F_4 defines 24 interchangeable icosahedra, while the BC_4 group defines two groups of icosahedra in a 8:16 counts, and finally the D_4 group has 3 groups of icosahedra with 8:8:8 counts.

Here we see links between the BC4 group, then the D4 Group and finally the F4 Group, which reflects the general G2 to B3 to D4 to B4 to F4. In other words, the geometry in the seventh dimension or degree which links S7 x S7 X G2 to the fourteenth dimension and the Sedenions, can be found right here in the linkages of these icosahedra. Tony Smith understands the importance of this sequence, which he mentions above, and as may be seen in this chart:

Dim	**Number**	**Lie Alge bra**	**Hopf Fibration**	**Lattice**	**Division Algebra**	**SPIN**
1	Real		S1		Div Alg	
2			Z2	Square Lattice	Div Alg	
3	Complex		S3 Hopf			
4		D4			Div Alg	
5						
6						
7			S7 Hopf			
8	Quarternion			Root Lattice	Div Alg	
9						
10						
11						
12						
13						
14		G2	S7 x S7 x G2			
15			S7 – S15 – S8			
16	Sedenion		Octionic Projective Plane	Laminat ed Lattice L16		
17						
18						
19						

It is one of three semiregular polychora made of two or more cells which are platonic solids, discovered by Thorold Gosset in his 1900 paper. He called it a **tetricosahedric** for being made of tetrahedron and icosahedron cells. (The other two are the rectified 5-cell and rectified 600-cell.)

Wikipedia on the 600 – Cell

In geometry, the **600-cell** (or **hexacosichoron**) is the convex regular 4-polytope, or polychoron, with Schläfli symbol {3,3,5}. Its boundary is composed of 600 tetrahedral cells with 20 meeting at each vertex. Together they form 1200 triangular faces, 720 edges, and 120 vertices. The edges form 72 flat regular decagons. Each vertex of the 600-cell is a vertex of six such decagons.

The mutual distances of the vertices, measured in degrees of arc on the circumscribed hypersphere, only have the values 36° = $\pi/5$, 60° = $\pi/3$, 72° = $2\pi/5$, 90° = $\pi/2$, 108° = $3\pi/5$, 120° = $2\pi/3$, 144° = $4\pi/5$, and 180° = π.

Departing from an arbitrary vertex V one has at 36° and 144° the 12 vertices of an icosahedron, at 60° and 120° the 20 vertices of a dodecahedron, at 72° and 108° again the 12 vertices of an icosahedron, at 90° the 30 vertices of an icosadodecahedron, and finally at 180° the antipodal vertex of V. *References:* S.L. van Oss (1899); F. Buekenhout and M. Parker (1998).

The 600-cell is regarded as the 4-dimensional analog of the icosahedron, since it has five tetrahedra meeting at every edge, just as the icosahedron has five triangles meeting at every vertex. It is also called a **tetraplex** (abbreviated from "tetrahedral complex") and **polytetrahedron**, being bounded by tetrahedral cells.

Its vertex figure is an icosahedron, and its dual polytope is the 120-cell.

Each cell touches, in some manner, 56 other cells. One cell contacts each of the four faces; two cells contact each of the six edges, but not a face; and ten cells contact each of the four vertices, but not a face or edge.

The snub 24-cell is related to the truncated 24-cell by an alternation operation. Half the vertices are deleted, the 24 truncated octahedron cells become 24 icosahedron cells, the 24 cubes become 24 tetrahedron cells, and the 96 deleted vertex voids create 96 new tetrahedron cells.

Since the elements of the symmetric group S_4 permute the quaternionic units $1, e_1, e_2, e_3$ [2] we obtain 24 sets of elements of (19) in the following form

$$\begin{aligned}
&\pm a \pm be_1 \pm ce_2 \pm de_3, \quad \pm a \pm ce_1 \pm de_2 \pm be_3,\\
&\pm a \pm de_1 \pm be_2 \pm ce_3, \quad \pm a \pm be_1 \pm de_2 \pm ce_3,\\
&\pm a \pm de_1 \pm ce_2 \pm be_3, \quad \pm a \pm ce_1 \pm be_2 \pm de_3,\\
&\pm b \pm ce_1 \pm de_2 \pm ae_3, \quad \pm b \pm de_1 \pm ae_2 \pm ce_3,\\
&\pm b \pm ae_1 \pm ce_2 \pm de_3, \quad \pm b \pm ce_1 \pm ae_2 \pm de_3,\\
&\pm b \pm ae_1 \pm de_2 \pm ce_3, \quad \pm b \pm de_1 \pm ce_2 \pm ae_3,\\
&\pm c \pm de_1 \pm ae_2 \pm be_3, \quad \pm c \pm ae_1 \pm be_2 \pm de_3,\\
&\pm c \pm be_1 \pm de_2 \pm ae_3, \quad \pm c \pm de_1 \pm be_2 \pm ae_3,\\
&\pm c \pm be_1 \pm ae_2 \pm de_3, \quad \pm c \pm ae_1 \pm de_2 \pm be_3,\\
&\pm d \pm ae_1 \pm be_2 \pm ce_3, \quad \pm d \pm be_1 \pm ce_2 \pm ae_3,\\
&\pm d \pm ce_1 \pm ae_2 \pm be_3, \quad \pm d \pm ae_1 \pm ce_2 \pm be_3,\\
&\pm d \pm be_1 \pm ce_2 \pm ae_3, \quad \pm d \pm be_1 \pm ae_2 \pm ce_3.
\end{aligned} \tag{21}$$

Why 24?

Geoffrey Dixon seems obsessed with this question, in one of his thought provoking essays on his website.

Above the reader may find partial answers

3 x 8 = 24, and 8 is the primary unit in the stable 8 x 8 Satva state of matter.

2 x 12 = 24, and 12 is an essential number in astrology, houses or Earth Branches.

6 x 4 = 24

1 x 24 = 24, and the traditional Chinese calendar has 24 seasons.

28 – 4 = 24, and 28 is a critical number for Sedenion triplets.

John Baez on H3 and H4

7) Andreas Fring and Christian Korff, Non-crystallographic reduction of Calogero-Moser models, Jour. Phys. A 39 (2006), 1115-1131. Also available as hep-th/0509152.

We apply a recently introduced reduction procedure based on the embedding of non-crystallographic Coxeter groups into crystallographic ones to Calogero–Moser systems. For rational potentials the familiar generalized Calogero Hamiltonian is recovered. For the Hamiltonians of trigonometric, hyperbolic and elliptic types, we obtain novel integrable dynamical systems with a second potential term which is rescaled by the golden ratio. We explicitly show for the simplest of these non-crystallographic models, how the corresponding classical equations of motion can be derived from a Lie algebraic Lax pair based on the larger, crystallographic Coxeter group.

They set up a nice correspondence between some non-crystallographic Coxeter groups and some crystallographic ones:

the H2 Coxeter group and the A4 Coxeter group,
the H3 Coxeter group and the D6 Coxeter group,
the H4 Coxeter group and the E8 Coxeter group.

A Coxeter group is a finite group of linear transformations of R^n that's generated by reflections. We say such a group is "non-crystallographic" if it's not the symmetries of any lattice. The ones listed above are closely tied to the number 5:

H2 is the symmetry group of a regular pentagon.
H3 is the symmetry group of a regular dodecahedron.
H4 is the symmetry group of a regular 120-cell.

Note these live in 2d, 3d and 4d space. Only in these dimensions are there regular polytopes with 5-fold rotational symmetry! Their symmetry groups are non-crystallographic, because no lattice can have 5-fold rotational symmetry.

A Coxeter group is "crystallographic", or a "Weyl group", if it *is* symmetries of a lattice. In particular:

A4 is the symmetry group of a 4-dimensional lattice also called A4.
D6 is the symmetry group of a 6-dimensional lattice also called D6.
E8 is the symmetry group of an 8-dimensional lattice also called E8.

H_3 is the group of symmetries of the dodecahedron or icosahedron. H_4 is the group of symmetries of a regular solid in 4 dimensions which I talked about in "week20". This regular solid is also called the "unit icosians" - it has 120 vertices, and is a close relative of the icosahedron and dodecahedron. One amazing thing is that it itself *is* a group in a very natural way. There are no "hypericosahedra" or "hyperdodecahedra" in dimensions greater than 4, which is related to the fact that the H series quits at this point.

- end

The author of this paper notes here that this is the critical part of Baez's paper:

H2 is the symmetry group of a regular pentagon.
H3 is the symmetry group of a regular dodecahedron.
H4 is the symmetry group of a regular 120-cell.

Which sets up the relationship between H2 and A5, which proves key to the Five Elements and to the construction of Time.

The Dodecahedron and the 120-cell play important roles in the formation of matter at higher stages, and here Baez sets up the relationship. That singularities should occur in these regions indicates the special importance of these groups in the process of the formation of matter.

Conclusion

This paper has laid out, autopsy – style, the various parts and pieces the author believes fit into the puzzle before us. Given the isolated nature of mathematicians and physicists, separated by time, distance, discipline, specialties, language and jargon, it becomes difficult to see the common areas of a problem. For this reason the author has found it necessary to deliberately excise parts from one paper and another, in order to lay out all the parts together on the table as it were, to begin analysis.

This is the method preferred by my mentor, Chalmers Johnson, whose autopsy of People’s War helped make his scholarly reputation. From a heuristic perspective, laying out the parts helps to see all of the relationships clearly, so that one may begin to analyze how the whole functions. While constructing the Qi Men Dun Jia Model the author noticed many anomalies – as well as that few seemed to have any answers for them.

The author notices, as Tony Smith wrote, that the three types of Hopf Fibrations occur in odd dimensions, while lattices appear only in even dimensions, and the same with division algebras. This has to do with the formation of matter into two distinct types: the stable 8 x 8 Satva type of matter, which develops in even dimensions and eventually forms into stable structures, such as the DNA helix, which enjoys an isomorphic relationship with the 64 hexagrams of the I Ching, as was noted in 1970 by Jungian analyst Marie – Louise von Franz.

The other type of matter is that of 9 x 9: a more dynamic, Raja type of structure associated with the Tai Xuan Jing, the Dao De Jing, a section of the Huang Di Nei Jing, and with the Pearce Cluster, as the author will soon describe in a forthcoming paper.

This division of visible matter into different types explains the chain of anomalies which mark the path of the emergence of matter into solid forms – crystals or otherwise. Thus H3 and H4 and their associated quirks, the strangeness in degree or dimension seven of S7 squared added to G2 to jump to degree fourteen. This indicates a relationship with the Sedenions and higher Octonions, perhaps even the Sextonions as noted by Bruce Westbury, and the Twisted Octonions, since the process of matter creation requires a sudden twist, which may as well explain the twisted fibration product noted by Tony Smith.

The next paper in this series moves into strange and rare territory of the 600 – cell, which is introduced here and mentioned briefly in the paper on the BC – Helix. One motivation for this paper was the observation that the Hopf Fibration seemed many things to many people, with no sense of scale.

The Hopf Fibration both appears to comprise the BC – Helix and to float and rotate around it, while few tried to account for the associated Clifford Tori associated with the Hopf Fibration. It may well prove the case that specific types of Hopf Fibrations appear in different sizes and have associations with the Clifford Tori, but this needs clarification.

Putting the Hopf Fibration into perspective will help manage questions about the 600 – cell and the relations between the Platonic Solids and the Golden Ratio. At this point the author theorizes that the Golden Ratio functions to separate the two states of visible matter, one from the other, forming the border between them, thus the appearance of the Golden Ratio and the Platonic Solids in the process of the formation of visible matter.

Old Plato knew his stuff and of what he wrote, it is the shame of previous generations to have lacked the imagination to have taken him seriously. The universe is indeed made of tiny triangles – the Fano Plane, the multiplication table of the Octonions. And in passing, *en passant*, we note the case of Roger Penrose as one of the "lost causes" of physics, for lacking the imagination to have seen the octonions for what they are – Plato's tiny triangles, and equilateral ones at that.

From here the author ventures into Polytope (3,3,5), briefly mentioned herein, but in need of complete analysis, with the prospect of macroscopic and microscopic Polytope (3,3,5)'s, the macro forming the structure of Time itself.

Bibliography

Much of the bibliography has been directly noted in passing, for ease of reference, and we thank Frank “Tony” Smith for the excerpts from his extensive website, which one ought frequently read and re-read until true understanding emerges.

Thanks to John Baez for his contributions.

Wikipedia and Wolfram provide many definitions.

Other authors as noted.

Appendix I John Baez on OP2

John Baez has written extensively on the Octontions as well as the Octonionic Projective Plane, which he describes in detail here, with its relation to the exceptional Lie Algebras E6 and F4.

The second smallest of the exceptional Lie groups is the 52-dimensional group $\mathbf{F}_4$. The geometric meaning of this group became clear in a number of nearly simultaneous papers by various mathematicians. In 1949, Jordan constructed the octonionic projective plane using projections in $\mathfrak{h}_3(\mathbb{O})$. One year later, Armand Borel [8] noted that $\mathbf{F}_4$ is the isometry group of a 16-dimensional projective plane. In fact, this plane is none other than than $\mathbb{OP}^2$. Also in 1950, Claude Chevalley and Richard Schafer [18] showed that $\mathbf{F}_4$ is the automorphism group of $\mathfrak{h}_3(\mathbb{O})$. In 1951, Freudenthal [35] embarked upon a long series of papers in which he described not only $\mathbf{F}_4$ but also the other exceptional Lie groups using octonionic projective geometry. To survey these developments, one still cannot do better than to read his classic 1964 paper on Lie groups and the foundations of geometry [38].

Let us take Chevalley and Schafer's result as the definition of $\mathbf{F}_4$:

$$\mathbf{F}_4 = \mathrm{Aut}(\mathfrak{h}_3(\mathbb{O})).$$

Its Lie algebra is thus

$$\mathfrak{f}_4 = \mathfrak{der}(\mathfrak{h}_3(\mathbb{O})).$$

As we saw in Section 3.4, points of $\mathbb{OP}^2$ correspond to trace-1 projections in the exceptional Jordan algebra. It follows that F_4 acts as transformations of $\mathbb{OP}^2$. In fact, we can equip $\mathbb{OP}^2$ with a Riemannian metric for which F_4 is the isometry group. To get a sense of how this works, let us describe $\mathbb{OP}^2$ as a quotient space of F_4.

In Section 3.4 we saw that the exceptional Jordan algebra can be built using natural operations on the scalar, vector and spinor representations of $\mathrm{Spin}(9)$. This implies that $\mathrm{Spin}(9)$ is a subgroup of F_4. Equation (3.4) makes it clear that $\mathrm{Spin}(9)$ is precisely the subgroup fixing the element

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since this element is a trace-one projection, it corresponds to a point of $\mathbb{OP}^2$. We have already seen that F_4 acts transitively on $\mathbb{OP}^2$. It follows that

$$\mathbb{OP}^2 \cong \mathrm{F}_4/\mathrm{Spin}(9).$$

This fact has various nice spinoffs. First, it gives an easy way to compute the dimension of F_4:

$$\dim(\mathrm{F}_4) = \dim(\mathrm{Spin}(9)) + \dim(\mathbb{OP}^2) = 36 + 16 = 52.$$

Second, since $\mathbf{F}_4$ is compact, we can take any Riemannian metric on $\mathbb{O}\mathbb{P}^2$ and average it with respect to the action of this group. The isometry group of the resulting metric will automatically include $\mathbf{F}_4$ as a subgroup. With more work [5], one can show that actually

$$\mathbf{F}_4 = \mathrm{Isom}(\mathbb{O}\mathbb{P}^2)$$

and thus

$$\mathfrak{f}_4 = \mathfrak{isom}(\mathbb{O}\mathbb{P}^2).$$

Equation (4.2) also implies that the tangent space of our chosen point in $\mathbb{O}\mathbb{P}^2$ is isomorphic to $\mathfrak{f}_4/\mathfrak{so}(9)$. But we already know that this tangent space is just $\mathbb{O}^2$, or in other words, the spinor representation of $\mathfrak{so}(9)$. We thus have

$$\mathfrak{f}_4 \cong \mathfrak{so}(9) \oplus S_9$$

as vector spaces, where $\mathfrak{so}(9)$ is a Lie subalgebra. The bracket in $\mathfrak{f}_4$ is built from the bracket in $\mathfrak{so}(9)$, the action $\mathfrak{so}(9) \otimes S_9 \to S_9$, and the map $S_9 \otimes S_9 \to \mathfrak{so}(9)$ obtained by dualizing this action. We can also rewrite this description of $\mathfrak{f}_4$ in terms of the octonions, as follows:

$$\mathfrak{f}_4 \cong \mathfrak{so}(\mathbb{O} \oplus \mathbb{R}) \oplus \mathbb{O}^2$$

This last formula suggests that we decompose $\mathfrak{f}_4$ further using the splitting of $\mathbb{O} \oplus \mathbb{R}$ into $\mathbb{O}$ and $\mathbb{R}$. It is easily seen by looking at matrices that for all n, m we have

$$\mathrm{so(n+m)} \cong \mathfrak{so}(n) \oplus \mathfrak{so}(m) \oplus V_n \otimes V_m.$$

Moreover, when we restrict the representation S_9 to $\mathfrak{so}(8)$, it splits as a direct sum $S_8^+ \oplus S_8^-$. Using these facts and equation (4.2), we see

$$\mathfrak{f}_4 \cong \mathfrak{so}(8) \oplus V_8 \oplus S_8^+ \oplus S_8^-$$

This formula emphasizes the close relation between $\mathfrak{f}_4$ and triality: the Lie bracket in $\mathfrak{f}_4$ is completely built out of maps involving $\mathfrak{so}(8)$ and its three 8-dimensional irreducible representations! We can rewrite this in a way that brings out the role of the octonions:

$$\mathfrak{f}_4 \cong \mathfrak{so}(\mathbb{O}) \oplus \mathbb{O}^3$$

While elegant, none of these descriptions of $\mathfrak{f}_4$ gives a convenient picture of all the derivations of the exceptional Jordan algebra. In fact, there is a nice picture of this sort for $\mathfrak{h}_3(\mathbb{K})$ whenever $\mathbb{K}$ is a normed division algebra. One way to get a derivation of the Jordan algebra $\mathfrak{h}_3(\mathbb{K})$ is to take a derivation of $\mathbb{K}$ and let it act on each entry of the matrices in $\mathfrak{h}_3(\mathbb{K})$. Another way uses elements of

$$\mathfrak{sa}_3(\mathbb{K}) = \{x \in \mathbb{K}[3]\colon\; x^* = -x,\; \mathrm{tr}(x) = 0\}.$$

Given $x \in \mathfrak{sa}_3(\mathbb{K})$, there is a derivation ad_x of $\mathfrak{h}_3(\mathbb{K})$ given by

$$\mathrm{ad}_x(a) = [x, a].$$

In fact [4], every derivation of $\mathfrak{h}_3(\mathbb{K})$ can be uniquely expressed as a linear combination of derivations of these two sorts, so we have

$$\mathfrak{der}(\mathfrak{h}_3(\mathbb{K})) \cong \mathfrak{der}(\mathbb{K}) \oplus \mathfrak{sa}_3(\mathbb{K})$$

as vector spaces. In the case of the octonions, this decomposition says that

$$\mathfrak{f}_4 \cong \mathfrak{g}_2 \oplus \mathfrak{sa}_3(\mathbb{O}).$$

In equation (4.2), the subspace $\mathfrak{der}(\mathbb{K})$ is always a Lie subalgebra, but $\mathfrak{sa}_3(\mathbb{K})$ is not unless $\mathbb{K}$ is commutative and associative -- in which case $\mathfrak{der}(\mathbb{K})$ vanishes. Nonetheless, there is a formula for the brackets in $\mathfrak{der}(\mathfrak{h}_3(\mathbb{K}))$ which applies in every case [70]. Given $D, D' \in \mathfrak{der}(\mathbb{K})$ and $x, y \in \mathfrak{sa}_3(\mathbb{K})$, we have

$$\begin{array}{rcl}
[D, D'] &=& DD' - D'D \\
{[D, \mathrm{ad}_x]} &=& \mathrm{ad}_{Dx} \\
{[\mathrm{ad}_x, \mathrm{ad}_y]} &=& \mathrm{ad}_{[x,y]_0} + \dfrac{1}{3} \displaystyle\sum_{i,j=1}^{3} D_{x_{ij}, y_{ij}}
\end{array}$$

where D acts on x componentwise, $[x,y]_0$ is the trace-free part of the commutator $[x,y]$, and $D_{x_{ij},y_{ij}}$ is the derivation of $\mathbb{K}$ defined using equation (4.1).

Summarizing these different descriptions of $\mathfrak{f}_4$, we have:

Theorem 5. The compact real form of $\mathfrak{f}_4$ is given by

$$\begin{aligned}
\mathfrak{f}_4 &\cong \mathfrak{isom}(\mathbb{OP}^2) \\
&\cong \mathfrak{der}(\mathfrak{h}_3(\mathbb{O})) \\
&\cong \mathfrak{der}(\mathbb{O}) \oplus \mathfrak{sa}_3(\mathbb{O}) \\
&\cong \mathfrak{so}(\mathbb{O} \oplus \mathbb{R}) \oplus \mathbb{O}^2 \\
&\cong \mathfrak{so}(\mathbb{O}) \oplus \mathbb{O}^3
\end{aligned}$$

where in each case the Lie bracket is built from natural bilinear operations on the summands.

The smallest nontrivial representations of $\mathbf{E}_6$ are 27-dimensional: in fact it has two inequivalent representations of this dimension, which are dual to one another. Now, the exceptional Jordan algebra is also 27-dimensional, and in 1950 this clue led Chevalley and Schafer [18] to give a nice description of $\mathbf{E}_6$ as symmetries of this algebra. These symmetries do not preserve the product, but only the determinant.

More precisely, the group of determinant-preserving linear transformations of $\mathfrak{h}_3(\mathbb{O})$ turns out to be a noncompact real form of $\mathbf{E}_6$. This real form is sometimes called $\mathbf{E}_{6(-26)}$, because its Killing form has signature -26. To see this, note that any automorphism of $\mathfrak{h}_3(\mathbb{O})$ preserves the determinant, so we get an inclusion

$$\mathbf{F}_4 \hookrightarrow \mathbf{E}_{6(-26)}.$$

This means that $\mathbf{F}_4$ is a compact subgroup of $\mathbf{E}_{6(-26)}$. In fact it is a maximal compact subgroup, since if there were a larger one, we could average a Riemannian metric group on $\mathbb{OP}^2$ with respect to this group and get a metric with an isometry group larger than $\mathbf{F}_4$, but no such metric exists. It follows that the Killing form on the Lie algebra $\mathfrak{e}_{6(-26)}$ is negative definite on its 52-dimensional maximal compact Lie algebra, $\mathfrak{f}_4$ and positive definite on the complementary 26-dimensional subspace, giving a signature of $26-52=-26$.

We saw in Section 3.4 that the projective plane structure of $\mathbb{OP}^2$ can be constructed starting only with the determinant function on the vector space $\mathfrak{h}_3(\mathbb{O})$. It follows that $\mathbf{E}_{6(-26)}$ acts as **collineations** on $\mathbb{OP}^2$, that is, line-preserving transformations. In fact, the group of collineations of $\mathbb{OP}^2$ is precisely $\mathbf{E}_{6(-26)}$:

$$\mathbf{E}_{6(-26)} \cong \mathrm{Coll}(\mathbb{OP}^2).$$

Moreover, just as the group of isometries of $\mathbb{OP}^2$ fixing a specific point is a copy of $\mathrm{Spin}(9)$, the group of collineations fixing a specific point is $\mathrm{Spin}(9,1)$. This fact follows with some work starting from equation (3.4), and it gives us a commutative square of inclusions:

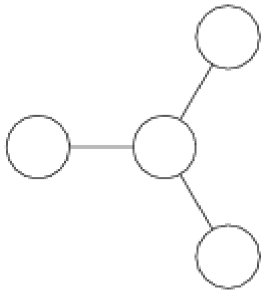
$$\begin{array}{ccc}
\mathrm{Spin}(9) & \longrightarrow & \mathrm{Isom}(\mathbb{OP}^2) \cong \mathbf{F}_4 \\
\downarrow & & \downarrow \\
\mathrm{Spin}(9,1) & \longrightarrow & \mathrm{Coll}(\mathbb{OP}^2) \cong \mathbf{E}_{6(-26)}
\end{array}$$

where the groups on the top are maximal compact subgroups of those on the bottom. Thus in a very real sense, $\mathbf{F}_4$ is to 9-dimensional Euclidean geometry as $\mathbf{E}_{6(-26)}$ is to 10-dimensional Lorentzian geometry.

Appendix II Triality by Wikipedia

Triality

From Wikipedia, the free encyclopedia



The automorphisms of the Dynkin diagram D_4 give rise to triality in Spin(8).

In mathematics, **triality** is a relationship among three vector spaces, analogous to the duality relation between dual vector spaces. Most commonly, it describes those special features of the Dynkin diagram D_4 and the associated Lie group Spin(8), the double cover of 8-dimensional rotation group SO(8), arising because the group has an outer automorphism of order three. There is a geometrical version of triality, analogous to duality in projective geometry.

Of all simple Lie groups, Spin(8) has the most symmetrical Dynkin diagram, D_4. The diagram has four nodes with one node located at the center, and the other three attached symmetrically. The symmetry group of the diagram is the symmetric group S_3 which acts by permuting the three legs. This gives rise to an S_3 group of outer automorphisms of Spin(8). This automorphism group permutes the three 8-dimensional irreducible representations of Spin(8); these being the vector representation and two chiral spin representations. These automorphisms do not project to automorphisms of SO(8). The vector representation - the natural action of SO(8) (hence Spin(8)) on K^8 - is also known as the "defining module", while the chiral spin representations are also known as "half-spin representations", and all three of these are fundamental representations.

No other Dynkin diagram has an automorphism group of order greater than 2; for other D_n (corresponding to other even Spin groups, Spin($2n$)), there is still the automorphism corresponding to switching the two half-spin representations, but these are not isomorphic to the vector representation.

Roughly speaking, symmetries of the Dynkin diagram lead to automorphisms of the Bruhat-Tits building associated with the group. For special linear groups, one obtains projective duality. For Spin(8), one finds a curious phenomenon involving 1, 2, and 4 dimensional subspaces of 8-dimensional space, historically known as "geometric triality".

The exceptional 3-fold symmetry of the D_4diagram also gives rise to the Steinberg group $^3D_4$.

General formulation

A duality between two vector spaces over a field **F** is a nondegenerate bilinear map

$$V_1 \times V_2 \to \mathbb{F},$$

i.e., for each nonzero vector v in one of the two vector spaces, the pairing with v is a nonzero linear functional on the other.

Similarly, a triality between three vector spaces over a field **F** is a nondegenerate trilinear map

$$V_1 \times V_2 \times V_3 \to \mathbb{F},$$

i.e., each nonzero vector in one of the three vector spaces induces a duality between the other two.

By choosing vectors e_i in each V_i on which the trilinear map evaluates to 1, we find that the three vector spaces are all isomorphic to each other, and to their duals. Denoting this common vector space by V, the triality may be reexpressed as a bilinear multiplication

$$V \times V \to V$$

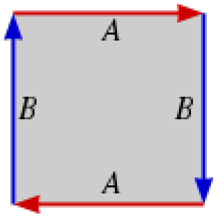
where each e_i corresponds to the identity element in V. The nondegeneracy condition now implies that V is a division algebra. It follows that V has dimension 1, 2, 4 or 8. If further **F** = **R** and the identification of V with its dual is given by positive definite inner product, V is a normed division algebra, and is therefore isomorphic to **R**, **C**, **H** or **O**.

Conversely, the normed division algebras immediately give rise to trialities by taking each V_i equal to the division algebra, and using the inner product on the algebra to dualize the multiplication into a trilinear form.

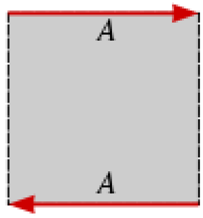
An alternative construction of trialities uses spinors in dimensions 1, 2, 4 and 8. The eight dimensional case corresponds to the triality property of Spin(8).

Appendix III

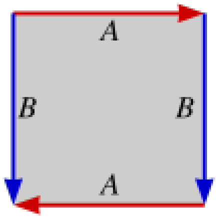
The Real Projective Plane / Wikipedia



The fundamental polygon of the projective plane.



The Möbius strip with a single edge, can be closed into a projective plane by gluing opposite open edges together.



In comparison the Klein bottle is a Möbius strip closed into a cylinder.

In mathematics, the **real projective plane** is an example of a compact non-orientable two-dimensional manifold, that is, a one-sided surface. It cannot be embedded in our usual three-dimensional space without intersecting itself. It has basic applications to geometry, since the common construction of the real projective plane is as the space of lines in $\mathbf{R}^3$ passing through the origin.

The plane is also often described topologically, in terms of a construction based on the Möbius strip: if one could glue the (single) edge of the Möbius strip to itself in the correct direction, one would obtain the projective plane. (This cannot be done in our three-dimensional space.) Equivalently, gluing a disk along the boundary of the Möbius strip gives the projective plane. Topologically, it has Euler characteristic 1, hence a demigenus (non-orientable genus, Euler genus) of 1.

Since the Möbius strip, in turn, can be constructed from a square by gluing two of its sides together, the real projective plane can thus be represented as a unit square (that is, $[0,1] \times [0,1]$) with its sides identified by the following equivalence relations:

$$(0, y) \sim (1, 1 - y) \quad \text{for } 0 \leqslant y \leqslant 1$$

and

$$(x, 0) \sim (1 - x, 1) \quad \text{for } 0 \leqslant x \leqslant 1,$$

as in the leftmost diagram on the right.

Projective geometry is not necessarily concerned with curvature and the real projective plane may be twisted up and placed in the Euclidean plane or 3-space in many different ways.[1] Some of the more important examples are described below.

The projective plane cannot be embedded (that is without intersection) in three-dimensional Euclidean space. The proof that the projective plane does not embed in three-dimensional Euclidean space goes like this: Assuming that it does embed, it would bound a compact region in three-dimensional Euclidean space by the generalized Jordan curve theorem. The outward-pointing unit normal vector field would then give an orientation of the boundary manifold, but the boundary manifold would be projective space, which is not orientable. This is a contradiction, and so our assumption that it does embed must have been false.

The projective sphere

Consider a sphere, and let the great circles of the sphere be "lines", and let pairs of antipodal points be "points". It is easy to check that this system obeys the axioms required of a projective plane:

- any pair of distinct great circles meet at a pair of antipodal points; and
- any two distinct pairs of antipodal points lie on a single great circle.

If we identify each point on the sphere with its antipodal point, then we get a representation of the real projective plane in which the "points" of the projective plane really are points. This means that the projective plane is the quotient space of the sphere obtained by partitioning the sphere into equivalence classes under the equivalence relation ~, where x ~ y if y = −x. This quotient space of the sphere is homeomorphic with the collection of all lines passing through the origin in $\mathbf{R}^3$.

The quotient map from the sphere onto the real projective plane is in fact a two sheeted (i.e. two-to-one) covering map. It follows that the fundamental group of the real projective plane is the cyclic group of order 2, i.e. integers modulo 2. One can take the loop *AB* from the figure above to be the generator.

The projective hemisphere

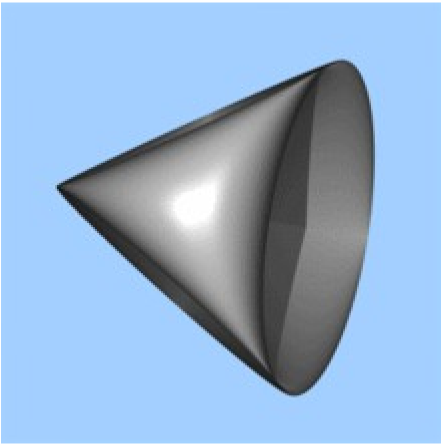
Because the real projective plane covers the sphere twice, it may be represented as a hemisphere around whose rim opposite points are similarly identified.[2]

Boy's surface – an immersion

The projective plane can be immersed (local neighbourhoods of the source space do not have self-intersections) in 3-space. Boy's surface is an example of an immersion.

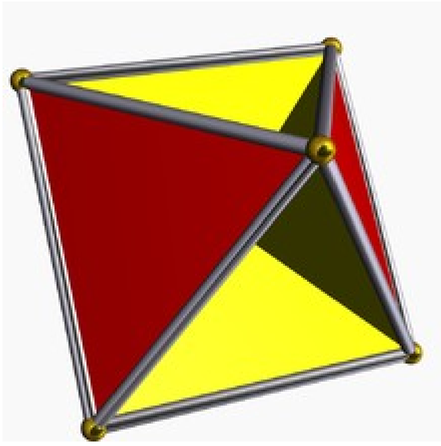
Polyhedral examples must have at least nine faces.[3]

Roman surface



An animation of the Roman Surface

Steiner's Roman surface is a more degenerate map of the projective plane into 3-space, containing a cross-cap.



The tetrahemihexahedron is a polyhedral representation of the real projective plane.

A polyhedral representation is the tetrahemihexahedron,[4] which has the same general form as Steiner's Roman Surface, shown to the right.

Hemi polyhedra

Looking in the opposite direction, certain abstract regular polytopes — hemi-cube, hemi-dodecahedron, and hemi-icosahedron — can be constructed as regular figures in the *projective plane;* see also projective polyhedra.

Planar projections

Various planar (flat) projections or mappings of the projective plane have been described. In 1874 Klein described the mapping

$$k(x,y) = (1 + x^2 + y^2)^{1/2}.(x,y)$$[1]

Central projection of the projective hemisphere onto a plane yields the usual infinite projective plane, described below.

Cross-capped disk

A closed surface is obtained by gluing a disk to a cross-cap. This surface can be represented parametrically by the following equations:

$$X(u,v) = r\,(1+\cos v)\,\cos u,$$

$$Y(u,v) = r\,(1+\cos v)\,\sin u,$$

$$Z(u,v) = -\tanh(u-\pi)\;r\,\sin v,$$

where both u and v range from 0 to 2π. These equations are similar to those of a torus. Figure 1 shows a closed cross-capped disk.

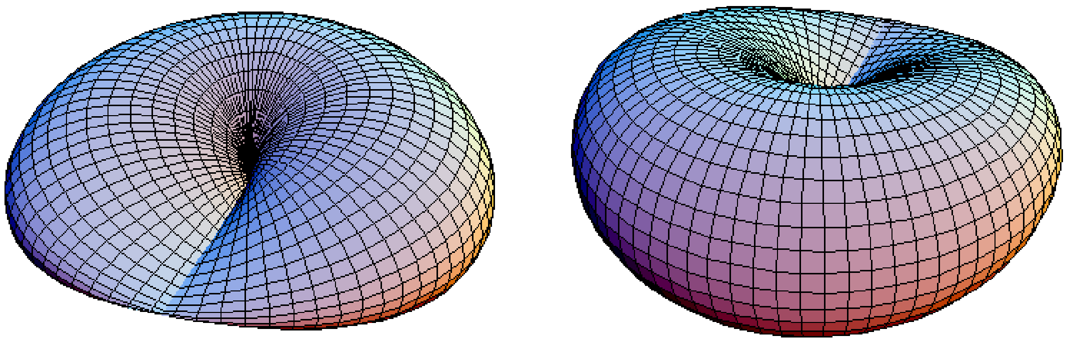


Figure 1. Two views of a cross-capped disk.

A cross-capped disk has a plane of symmetry which passes through its line segment of double points. In Figure 1 the cross-capped disk is seen from above its plane of symmetry $z = 0$, but it would look the same if seen from below.

A cross-capped disk can be sliced open along its plane of symmetry, while making sure not to cut along any of its double points. The result is shown in Figure 2.

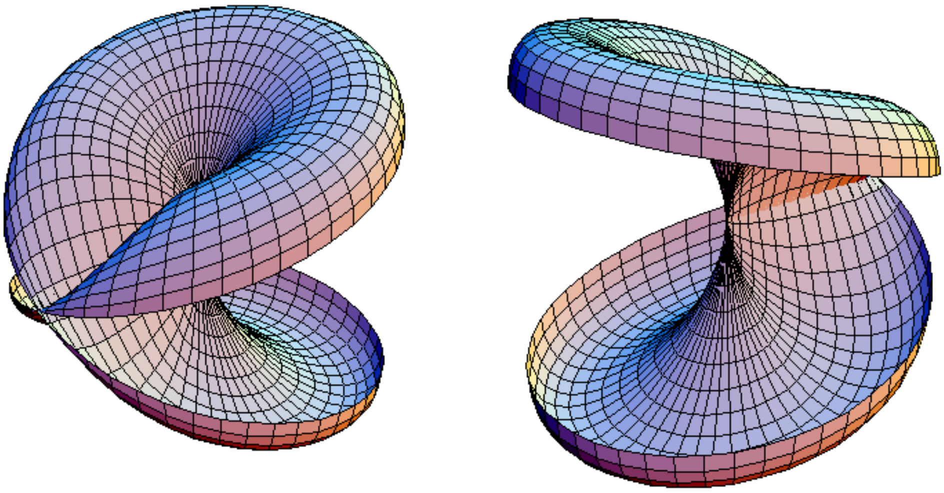


Figure 2. Two views of a cross-capped disk which has been sliced open.

Once this exception is made, it will be seen that the sliced cross-capped disk is homeomorphic to a self-intersecting disk, as shown in Figure 3.

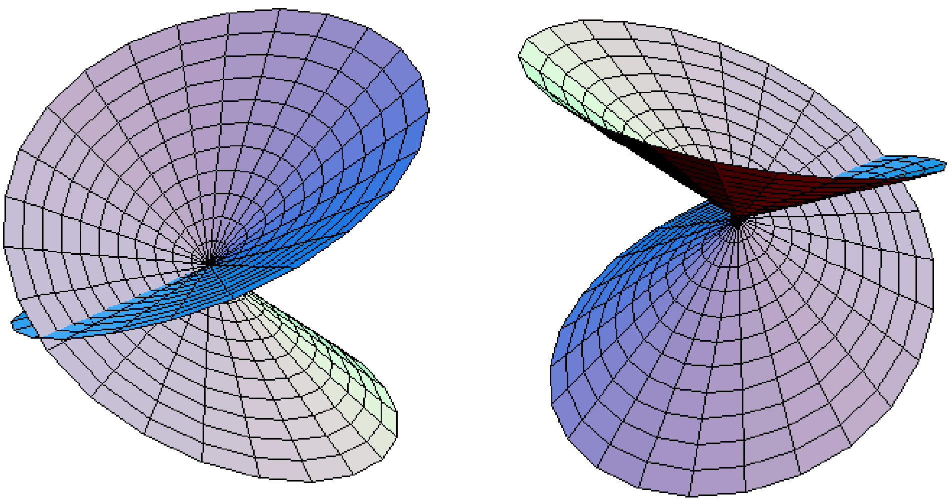


Figure 3. Two alternate views of a self-intersecting disk.

The self-intersecting disk is homeomorphic to an ordinary disk. The parametric equations of the self-intersecting disk are:

$$X(u,v) = r\,v\,\cos 2u,$$

$$Y(u,v) = r\,v\,\sin 2u,$$

$$Z(u,v) = r\,v\,\cos u,$$

where u ranges from 0 to 2π and v ranges from 0 to 1.

Projecting the self-intersecting disk onto the plane of symmetry (z = 0 in the parametrization given earlier) which passes only through the double points, the result is an ordinary disk which repeats itself (doubles up on itself).

The plane z = 0 cuts the self-intersecting disk into a pair of disks which are mirror reflections of each other. The disks have centers at the origin.

Now consider the rims of the disks (with v = 1). The points on the rim of the self-intersecting disk come in pairs which are reflections of each other with respect to the plane z = 0.

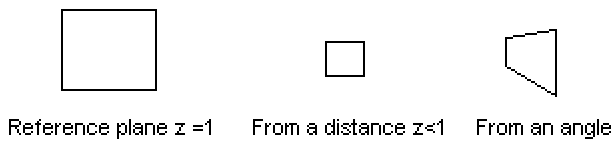
A cross-capped disk is formed by identifying these pairs of points, making them equivalent to each other. This means that a point with parameters $(u, 1)$ and coordinates $(r\,\cos 2u, r\,\sin 2u, r\,\cos u)$ is identified with the point $(u + \pi, 1)$ whose coordinates are $(r\,\cos 2u, r\,\sin 2u, -r\,\cos u)$. But this means that pairs of opposite points on the rim of the (equivalent) ordinary disk are identified with each other; this is how a real projective plane is formed out of a disk. Therefore the surface shown in Figure 1 (cross-cap with disk) is topologically equivalent to the real projective plane RP^2.

Homogeneous coordinates

The points in the plane can be represented by homogeneous coordinates. A point has homogeneous coordinates $[x : y : z]$, where the coordinates $[x : y : z]$ and $[tx : ty : tz]$ are considered to represent the same point, for all nonzero values of t. The points with coordinates $[x : y : 1]$ are the usual real plane, called the **finite part** of the projective plane, and points with coordinates $[x : y : 0]$, called **points at infinity** or **ideal points**, constitute a line called the **line at infinity**. (The homogeneous coordinates $[0 : 0 : 0]$ do not represent any point.)

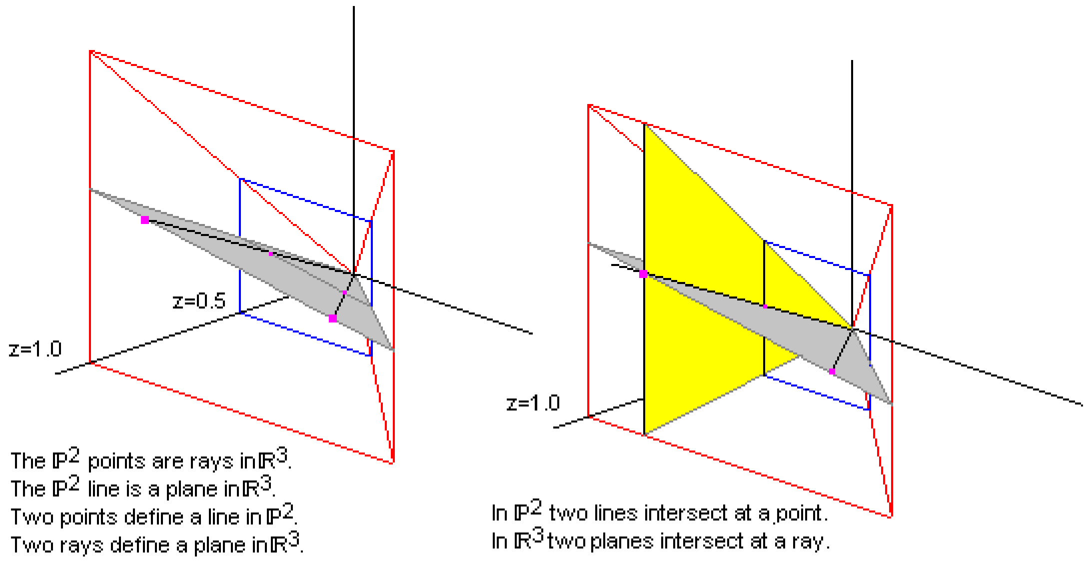
The lines in the plane can also be represented by homogeneous coordinates. A projective line corresponding to the plane $ax + by + cz = 0$ in $\mathbf{R}^3$ has the homogeneous coordinates $(a : b : c)$. Thus, these coordinates have the equivalence relation $(a : b : c) = (da : db : dc)$ for all nonzero values of d. Hence a different equation of the same line $dax + dby + dcz = 0$ gives the same homogeneous coordinates. A point $[x : y : z]$ lies on a line $(a : b : c)$ if $ax + by + cz = 0$. Therefore, lines with coordinates $(a : b : c)$ where a, b are not both 0 correspond to the lines in the usual real plane, because they contain points that are not at infinity. The line with coordinates $(0 : 0 : 1)$ is the line at infinity, since the only points on it are those with $z = 0$.

The flat projective plane



In the projective plane $\mathbf{P}^2$, a point x is represented by the homogeneous coordinate (x_1, x_2, x_3). If we think of (x_1, x_2, x_3) as a point in real space $\mathbf{R}^3$ with the third value of the homogeneous coordinate as a value in the z direction, then $\mathbf{P}^2$ can be visualized as $\mathbf{R}^3$.

Points, rays, lines, and planes



A line in $\mathbf{P}^2$ can be represented by the equation $ax + by + c = 0$. If we treat a, b, and c as the column vector $\boldsymbol{\ell}$ and x, y, 1 as the column vector $\mathbf{x}$ then the equation above can be written in matrix form as:

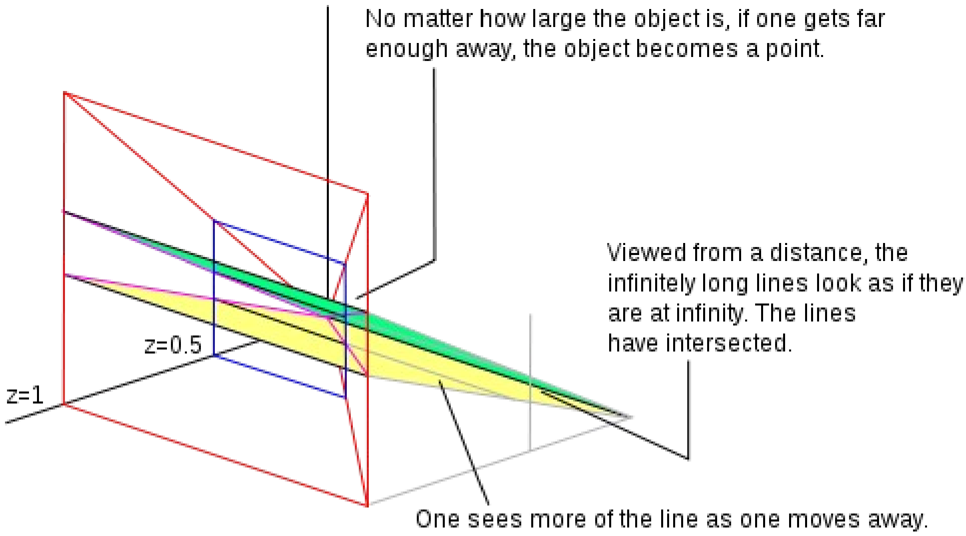
$$\mathbf{x}^T\boldsymbol{\ell} = 0 \text{ or } \boldsymbol{\ell}^T\mathbf{x} = 0.$$

Using vector notation we may instead write

$$\mathbf{x} \cdot \boldsymbol{\ell} = 0 \text{ or } \boldsymbol{\ell} \cdot \mathbf{x} = 0.$$

The equation $k(\mathbf{x}^T\boldsymbol{\ell}) = 0$ (which k is a non-zero scalar) sweeps out a plane that goes through zero in $\mathbf{R}^3$ and $k(x)$ sweeps out a ray, again going through zero. The plane and ray are linear subspaces in $\mathbf{R}^3$, which always go through zero.

Ideal points



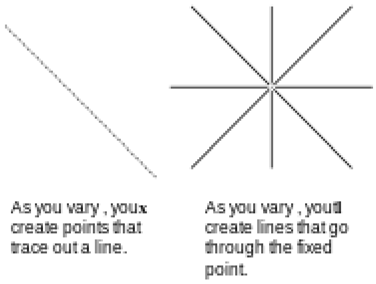
In $\mathbf{P}^2$ the equation of a line is $ax + by + c = 0$ and this equation can represent a line on any plane parallel to the x, y plane by multiplying the equation by k.

If $z = 1$ we have a normalized homogeneous coordinate. All points that have $z = 1$ create a plane. Let's pretend we are looking at that plane (from a position further out along the z axis and looking back towards the origin) and there are two parallel lines drawn on the plane. From where we are standing (given our visual capabilities) we can see only so much of the plane, which we represent as the area outlined in red in the diagram. If we walk away from the plane along the z axis, (still looking backwards towards the origin), we can see more of the plane. In our field of view original points have moved. We can reflect this movement by dividing the homogeneous coordinate by a constant. In the image to the right we have divided by 2 so the z value now becomes 0.5. If we walk far enough away what we are looking at becomes a point in the distance. As we walk away we see more and more of the parallel lines. The lines will meet at a line at infinity (a line that goes through zero on the plane at $z = 0$). Lines on the plane when $z = 0$ are ideal points. The plane at $z = 0$ is the line at infinity.

The homogeneous point (0, 0, 0) is where all the real points go when you're looking at the plane from an infinite distance, a line on the

z = 0 plane is where parallel lines intersect.

Duality



In the equation $\mathbf{x}^T\boldsymbol{\ell}$ = 0 there are two column vectors. You can keep either constant and vary the other. If we keep the point constant $\mathbf{x}$ and vary the coefficients $\boldsymbol{\ell}$ we create new lines that go through the point. If we keep the coefficients constant and vary the points that satisfy the equation we create a line. We look upon x as a point because the axes we are using are x, y, and z. If we instead plotted the coefficients using axis marked a, b, c points would become lines and lines would become points. If you prove something with the data plotted on axis marked x, y, and z the same argument can be used for the data plotted on axis marked a, b, and c. That is duality.

Lines joining points and intersection of lines (using duality)

The equation $\mathbf{x}^T\boldsymbol{\ell}$ = 0 calculates the inner product of two column vectors. The inner product of two vectors is zero if the vectors are orthogonal. To find the line between the points $\mathbf{x}_1$ and $\mathbf{x}_2$ you must find the column vector $\boldsymbol{\ell}$ that satisfies the equations $\mathbf{x}_1{}^T\boldsymbol{\ell} = 0$ and $\mathbf{x}_2{}^T\boldsymbol{\ell} = 0$, that is we must find a column vector $\boldsymbol{\ell}$ that is orthogonal to $\mathbf{x}_1$ and $\mathbf{x}_2$. In the case of $\mathbf{P}^2$, the cross product will find such a vector. The line joining two points is given by the equation $\mathbf{x}_1 \times \mathbf{x}_2$. To find the intersection of two lines you look to duality. If you plot $\boldsymbol{\ell}$ in the coefficient space you get rays. To find the point $\mathbf{x}$ that is orthogonal to the two rays you find the cross product. That is $\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2$.

While the cross product works in $\mathbf{P}^2$, it is not well-defined in arbitrary dimensions. However, this pair of equations is satisfied by

$$\mathbf{x}_1^{\mathrm{T}}\ell - \lambda\mathbf{x}_2^{\mathrm{T}}\ell = 0$$ [citation needed]

Embedding into 4-dimensional space

The projective plane embeds into 4-dimensional Euclidean space. The real projective plane $\mathbf{P}^2(\mathbf{R})$ is the quotient of the two-sphere

$$\mathbf{S}^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2+y^2+z^2 = 1\}$$

by the antipodal relation $(x, y, z) \sim (-x, -y, -z)$. Consider the function $\mathbf{R}^3 \rightarrow \mathbf{R}^4$ given by $(x, y, z) \mapsto (xy, xz, y^2-z^2, 2yz)$. This map restricts to a map whose domain is $\mathbf{S}^2$ and, since each component is a homogeneous polynomial of even degree, it takes the same values in $\mathbf{R}^4$ on each of any two antipodal points on $\mathbf{S}^2$. This yields a map $\mathbf{P}^2(\mathbf{R}) \rightarrow \mathbf{R}^4$. Moreover, this map is an embedding. Notice that this embedding admits a projection into $\mathbf{R}^3$ which is the Roman surface.

Higher non-orientable surfaces

By gluing together projective planes successively we get non-orientable surfaces of higher demigenus. The gluing process consists of cutting out a little disk from each surface and identifying (*gluing*) their boundary circles. Gluing two projective planes creates the Klein bottle.

The article on the fundamental polygon describes the higher non-orientable surfaces.

Appendix IV

Rosenfeld projective planes

Following the discovery of the Cayley projective plane or "octonionic projective plane" $\mathbf{P}^2(\mathbf{O})$ in 1933, whose symmetry group is the exceptional Lie group F_4, and with the knowledge that G_2 is the automorphism group of the octonions, it was proposed by Rozenfeld (1956) that the remaining exceptional Lie groups E_6, E_7, and E_8 are isomorphism groups of projective planes over certain algebras over the octonions:[1]

- the **bioctonions, $\mathbf{C} \otimes \mathbf{O}$**,
- the **quateroctonions, $\mathbf{H} \otimes \mathbf{O}$**,
- the **octooctonions, $\mathbf{O} \otimes \mathbf{O}$**.

This proposal is appealing, as there are certain exceptional compact Riemannian symmetric spaces with the desired symmetry groups and whose dimension agree with that of the putative projective planes ($\dim(\mathbf{P}^2(\mathbf{K} \otimes \mathbf{K}')) = 2\dim(\mathbf{K})\dim(\mathbf{K}')$), and this would give a uniform construction of the exceptional Lie groups as symmetries of naturally occurring objects (i.e., without an a priori knowledge of the exceptional Lie groups). The Riemannian symmetric spaces were classified by Cartan in 1926 (Cartan's labels are used in sequel); see classification for details, and the relevant spaces are:

- the octonionic projective plane – FII, dimension $16 = 2 \times 8$, F_4 symmetry, Cayley projective plane $\mathbf{P}^2(\mathbf{O})$,
- the bioctonionic projective plane – EIII, dimension $32 = 2 \times 2 \times 8$, E_6 symmetry, complexified Cayley projective plane, $\mathbf{P}^2(\mathbf{C} \otimes \mathbf{O})$,
- the "**quateroctonionic projective plane**"[2] – EVI, dimension $64 = 2 \times 4 \times 8$, E_7 symmetry, $\mathbf{P}^2(\mathbf{H} \otimes \mathbf{O})$,
- the "**octooctonionic projective plane**"[3] – EVIII, dimension $128 = 2 \times 8 \times 8$, E_8 symmetry, $\mathbf{P}^2(\mathbf{O} \otimes \mathbf{O})$.

The difficulty with this proposal is that while the octonions are a division algebra, and thus a projective plane is defined over them, the bioctonions, quateroctonions and octooctonions are not division algebras, and thus the usual definition of a projective plane does not work. This can be resolved for the bioctonions, with the resulting projective plane being the complexified Cayley plane, but the constructions do not work for the quateroctonions and octooctonions, and the spaces in question do not obey the usual axioms of projective planes,[1] hence the quotes on "(putative) projective plane". However, the tangent space at each point of these spaces can be identified with the plane $(\mathbf{H} \otimes \mathbf{O})^2$, or $(\mathbf{O} \otimes \mathbf{O})^2$ further justifying the intuition that these are a form of generalized projective plane.[2][3] Accordingly, the resulting spaces are sometimes called **Rosenfeld projective planes** and notated as if they were projective planes. More broadly, these compact forms are the **Rosenfeld elliptic projective planes**, while the dual non-compact forms are the **Rosenfeld hyperbolic projective planes**. A more modern presentation of Rosenfeld's ideas is in (Rosenfeld 1997), while a brief note on these "planes" is in (Besse 1987, pp. 313–316).[4]

The spaces can be constructed using Tit's theory of buildings, which allows one to construct a geometry with any given algebraic group as symmetries, but this requires starting with the Lie groups and constructing a geometry from them, rather than constructing a geometry independently of a knowledge of the Lie groups.[1]

Contact

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Jaq 2013 at out look dot com all of this connected with no spaces

“Some men see things and ask, why? I dream of things and I ask, why not?”

Robert Francis Kennedy (RFK), after George Bernard Shaw