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Derivation of the Fine Structure Constant from Fractional Electrodynamics

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Abstract

Both classical and quantum electrodynamics assume that random fluctuations are absent from the steady-state evolution of the underlying physical system. Our work goes beyond this approximation and accounts for the continuous exposure to stochastic fluctuations. It is known that the asymptotic limit of quantum field dynamics, dominated by large and persistent perturbations, may be described as an anomalous diffusion process. We use fractional calculus as an appropriate tool to handle this highly non-trivial regime. It is shown that the fine structure constant can be recovered from the fractional evolution equation of the density matrix under standard normalization conditions.

2. Transition from conventional to fractional dynamics

The effect of generic fluctuations may be conveniently modeled using the concept of density matrix. As known from quantum statistical mechanics, the density matrix measures the probability distribution for transitions between various locations in either coordinate or momentum space. Assuming throughout natural units ($\hbar = c = 1$), the density matrix in coordinate space obeys the following evolution equation [7]

$$\frac{\partial \rho(\mathbf{r}, \mathbf{r}'; t)}{\partial t} = -\mathbf{H}\rho(\mathbf{r}, \mathbf{r}'; t) \tag{1}$$

where $\mathbf{H}$ represents the Hamiltonian operator. If fluctuations generate a "white noise" distribution at $t = 0$, (1) becomes subject to the initial condition

$$\rho(\mathbf{r}, \mathbf{r}'; 0) = \rho_0 \tag{2}$$

and satisfies normalization to a positive constant C

$$\int_R \rho(\mathbf{r}, \mathbf{r}'; t)\, \mathrm{d}\mathbf{r} = C \tag{3}$$

over the spatial domain $\mathbf{r} \in \{\mathbf{R}\}$.

By analogy with the standard treatment of Brownian motion [8], Hamiltonian $\mathbf{H}$ is a linear addition of a deterministic component ($\mathbf{H}_\mathrm{d}$), created by applied external potentials, and a stochastic component produced by steady-state fluctuations ($\mathbf{H}_\mathrm{s}$).

Let us consider the interaction of a static electron with an isotropic Coulomb potential $V(r) = e(k)^2/4\pi r$ created by a neighboring point charge. The deterministic Hamiltonian is given by

$$\mathbf{H}_\mathrm{d} = -\frac{1}{2m}\nabla^2 - \frac{e(k)^2}{4\pi r} \tag{4}$$

in which m is the electron mass and the fine structure constant $\alpha(k) = e(k)^2/4\pi$ determines the coupling strength at the sliding momentum scale $k = 1/r$ [9].

The stochastic energy may be modeled as stationary noise which vanishes upon ensemble average

$$\langle H_\mathrm{s}(r, t)\rangle_\mathrm{ens} = 0 \tag{5}$$

and whose autocorrelation has the general factorized form [10]

$$\langle H_\mathrm{s}(r, t) H_\mathrm{s}(r', t')\rangle = \varphi(t - t')w(r - r') \tag{6}$$

$$w(r - r') = \Delta^2 \exp\left[-\frac{1}{2}\left(\frac{r - r'}{\Delta}\right)^2\right] \tag{7}$$

In (6) the time correlation function $\varphi(t - t')$ is arbitrary and parameter Δ determines the linear scale of fluctuations.

To further simplify the approach and without loss of generality, we proceed with the following assumptions:

(a) the potential part of the energy is asymptotically higher than its kinetic counterpart. This is a legitimate hypothesis in the strong coupling limit of short distances.
(b) $\mathbf{r}'$ is kept fixed and coincident with the origin of the reference frame ($\mathbf{r}' = 0$).
(c) if L denotes the linear extent of $\{\mathbf{R}\}$, i.e. $[\mathbf{R}] = L^3$, the limits of the spatial domain are set by Δ and L. The strong fluctuation regime of short distances is characterized by $\Delta \gg L$, therefore L represents the lower limit of the spatial domain and Δ its upper limit.
(d) the electron space–time path driven by stochastic fluctuations is considered a CTRW. The closest analog of position fluctuations may be represented by the so-called Feynman's Chessboard Model (FCM), in which deviations from

the nominal location are restricted to "forward" and "backward" moves of random amplitude along the radial position vector [27]. Unlike FCM, the CTRW model assumes non-uniform time intervals between consecutive spatial jumps.

The evolution equation may be brought to a non-dimensional form with the following substitutions

$$r^0 = \frac{r}{L}, \quad t^0 = \frac{t}{L}, \quad \rho^0 = \rho L, \quad k^0 = \frac{1}{r^0}$$

It is known that, to properly account for the short-distance behavior of QED, a vacuum polarization correction needs to be added to the classical Coulomb potential [11,12]. This contribution is known as the Uehling term. The complete potential energy, including the Uehling term and expressed in non-dimensional form, reads [13]

$$H_{\mathrm{d}}^0 = \frac{-e_0^2}{4\pi r^0}[(1 - b\ln r^0) - \mathrm{O}(e_0^4)] \tag{8}$$

where

$$b = \frac{e_0^2}{6\pi^2} \tag{9}$$

Here e_0 represents the electric charge at the reference momentum scale, related to the "running" electric charge $e(k^0)$ via [14]

$$e(k^0)^2 = \frac{e_0^2}{1 - b\ln\frac{1}{r^0}} \tag{10}$$

After taking the ensemble average of the overall energy and accounting for (5) we arrive at [1]

$$\frac{\partial\rho^0(r^0,t^0)}{\partial t^0} = \left[\frac{e_0^2}{4\pi r^0}(1 - b\ln r^0)\right]\rho^0(r^0,t^0) \tag{11}$$

which is solved by

$$\rho^0(r^0,t^0) = \rho_0 L\exp\left\{\frac{e_0^2}{4\pi r^0}[1 - b\ln r^0]t^0\right\} \tag{12}$$

For finite time intervals satisfying the constraint

$$t^0 \ll \frac{4\pi r^0}{e_0^2(1 - b\ln r^0)} \tag{13}$$

the argument of the exponential function may be dropped and (12) reduces to

$$\rho^0(r^0,t^0) = \rho_0 L \tag{14}$$

The normalization required by (3) yields

$$\rho_0 = \frac{C}{\Delta - L} \tag{15}$$

and the first two moments for the ensemble distribution of r^0, computed using (14), amount to

$$\langle r^0\rangle = \int_1^{\Delta/L} r^0\rho_0 L\,\mathrm{d}r^0 = \frac{C(\Delta + L)}{2L} \tag{16}$$

$$\langle (r^0)^2\rangle = \int_1^{\Delta/L} (r^0)^2\rho_0 L\,\mathrm{d}r^0 = \frac{C}{3L^2}\frac{\Delta^3 - L^3}{\Delta - L} \tag{17}$$

It is seen that, according to assumption (c), both moments diverge in the strong fluctuation limit of short distances. As known from the stochastic theory of CTRW's, the divergence of either one of the two moments signals the transition to

[1] To simplify notation, we omit for the remainder of the paper the index referring to ensemble averaging.

fractional dynamics [15]. This regime is defined by a typical loss of characteristic scale (as $\Delta \gg L$) and requires fractional calculus to properly describe its time evolution [16,17].

The object of the next section is to analyze the impact of fractional dynamics on density matrix and its temporal behavior. [2]

3. Fractional evolution equation

The natural generalization of (11) in the framework of fractional calculus is

$$D_t^v \rho^0(r^0, t^0) = \left[\frac{e_0^2}{4\pi r^0}(1 - b\ln r^0)\right]\rho^0(r^0, t^0) \tag{18}$$

in which D_t^v stands for the fractional differential operator with respect to time and $v \in [0, 1]$ is the differentiation order [18]. Eq. (18) subject to (2) models a fractional Cauchy problem [19] whose closed-form solution is

$$\rho^0(r^0, t^0) = (\rho_0 L)E_{v,1}\left[\frac{e_0^2}{4\pi r^0}(1 - b\ln r^0)(t^0)^v\right] \tag{19}$$

where $E_{v,1}(z)$ represents the Mittag–Leffler function [15,19]

$$E_{v,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(vk + 1)} \tag{20}$$

Repeating the argument previously made, we further simplify expansion (19) by restricting the analysis to time intervals satisfying (13). This condition enables one to write (19) in a linearized form, that is:

$$\rho^0(r^0, t^0) \approx (\rho_0 L)\left[1 + \frac{e_0^2(1 - b\ln r^0)(t^0)^v}{4\pi r^0 \Gamma(1 + v)}\right] \tag{21}$$

The normalization constraint (3) with the natural choice

$$\rho_0 L = C \tag{22}$$

yields explicitly

$$\left(\frac{\Delta}{L} - 1\right) + \frac{(t^0)^v \alpha}{\Gamma(1 + v)} \ln\left(\frac{\Delta}{L}\right) - \frac{(t^0)^v \alpha}{2\Gamma(1 + v)} b\left[\ln\left(\frac{\Delta}{L}\right)\right]^2 = 1 \tag{23}$$

in which α is the low-energy fine structure constant ($\alpha = e_0^2/4\pi$).

Although, in principle, one can use (23) as a basis for retrieving α, this proves to be ultimately unfeasible since Δ/L is a large unknown variable and there are no realistic grounds for a suitable choice of parameters t^0 and v. As the next two sections show, invoking the asymptotic behavior of QED and linking the onset of fractional dynamics to critical phenomena, provides a way to overcome this difficulty.

4. The high energy region of QED

The short-distance behavior of QED is characterized by small valued denominators in (10). Since Δ/L is an unknown parameter, it makes sense to formulate (10) in its original form [14], that is

$$e(k)^2 = \frac{e_0^2}{1 - b\ln\left(\frac{k}{k_{\text{ref}}}\right)} \tag{24a}$$

[2] As known, Eq. (1) is valid only in the context of equilibrium statistical mechanics. The screening effect of vacuum polarization [14], accounted for by the Uehling correction, is to transform the classical Coulomb field into a short-range interaction. This, in turn, allows use of equilibrium statistical mechanics as a legitimate framework for analysis [26].

Here k is the sliding momentum scale, k_{ref} the reference momentum scale and $e_0^2 = e^2(k_{\text{ref}})$. Let k_c represent the momentum cutoff assigned to the high energy region. We have

$$k_{\text{ref}} \leqslant k \leqslant k_c \tag{24b}$$

It is now convenient to redefine the spatial limits of the region in terms of k_{ref} and k_c. Instead of considering 1 and Δ/L as the lower and upper boundaries, as it was previously done, we are going to substitute them with $r_c^0 = (r_c/r_{\text{ref}}) = (k_{\text{ref}}/k_c)$ and 1, respectively. Stated differently, r_c^0 is set to play the role of the distance cutoff. Consequently, for any sliding distance $r^0 = r/r_{\text{ref}}$

$$r_c^0 \leqslant r^0 \leqslant r_{\text{ref}}^0 \tag{24c}$$

where $r_{\text{ref}}^0 = r_{\text{ref}}/r_{\text{ref}} = 1$.

In what follows we posit that (a) the reference distance r_{ref} of the high energy region corresponds to the standard length scale of 10^{-15} m [2,14] and (b) the distance cutoff r_c of the high energy region is set by the upper limit of electron radius. The second hypothesis is based on recent studies performed at TEVATRON on Drell-Yan processes and reported at 95% confidence level [20]. Thus

$$r_{\text{ref}} = 10^{-15} \text{ m}, \quad r_c = 5.69 \times 10^{-19} \text{ m} \tag{25}$$

In the high energy region of short-time intervals, t^0 can be thought of as having the same order of magnitude as r_c^0. Hence we take

$$t^0 \sim r_c^0 \tag{26}$$

Using these new definitions, normalization (3) along with the choice (22) leads to

$$(1 - r_c^0) - \frac{(r_c^0)^v \alpha}{\Gamma(1+v)} \ln(r_c^0) \left[1 - \frac{b \ln(r_c^0)}{2} \right] = 1 \tag{27}$$

This is the main result of the paper. The next section deals with the relationship between the fractional order v and the theory of phase transitions from statistical physics.

5. Connection to critical behavior

As known from the theory of second order phase transitions, the loss of characteristic length scale is a universal feature of systems exhibiting critical behavior [21,22]. This property enables us to draw a natural analogy between the onset of fractional dynamics outlined in Section 2 and the approach to criticality in statistical physics.

Consider the traditional Ising model consisting of a three-dimensional lattice of spins σ_i having random orientations and undergoing a continuous change in temperature [9,22]. Away from the transition point (T_c), the correlation function between the ith and the jth spins displays an exponential decay at large distances according to

$$g_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \sim \exp\left(-\frac{|i-j|}{\xi} \right) \tag{28}$$

where $|i-j| \to \infty$. Here ξ, stands for the correlation length. At the transition point the correlation length diverges as

$$\xi \sim \left(\frac{T}{T_c} - 1 \right)^{-v} \tag{29}$$

In (29) v is the critical exponent of the correlation length whose numerical value for the Ising model is $v = 0.626$ [9].

Universality of the approach to critical behavior is outlined in the Appendix A. From this standpoint, the long range order induced by fluctuations and embodied by (16) and(17) is equivalent to the long range order expressed by (29). Based on this argument [3] we take

$$v = 0.626 \tag{30}$$

[3] It is instructive to highlight here the formal equivalence of FCM with the Ising model, as discussed in [27].

6. Computation of the fine structure constant

We are now in the position to compute α. From (25)–(27) and (30) we derive

$$\alpha = \frac{3\pi}{2\ln r_c^0}\left(1 - \sqrt{1 + \frac{4(r_c^0)^{1-v}\Gamma(1+v)}{3\pi}}\right) \tag{31}$$

where

$$r_c^0 = \frac{r_c}{r_{ref}} = \frac{5.69 \times 10^{-19}}{10^{-15}} = 5.69 \times 10^{-4} \tag{32}$$

We find

$$\alpha = \frac{1}{137.043} \tag{33}$$

which is in close agreement with the experimental value of $1/137.036$ [3,9].

8. Conclusions

We have outlined a derivation of the fine structure constant based upon the fractional time evolution of stochastic electrodynamics. The work has focused on the asymptotic limit of QED which is defined by large fluctuations in all relevant variables. The loss of characteristic scale, typical for anomalous diffusion and "complex behavior", was linked to the theory of critical phenomena. The fine structure constant was found to emerge from the fractional evolution equation of the density matrix, subject to standard normalization conditions.

Appendix A

Under the most general circumstances, space and time are treated as independent variables of the CTRW induced by fluctuations. In this case the reference length scale (r_{ref}) and the reference time scale (t_{ref}) are different and we may define $t^0 = (t/t_{ref})$.

Constraints (13) and (26) require

$$t_{ref} \gg \frac{\alpha t(1 - b\ln r^0)}{r^0} \tag{A.1}$$

$$t_{ref} = \frac{t r_{ref}}{r_c} \tag{A.2}$$

$$r_c \ll \frac{r}{\alpha\left[1 - b\ln\left(\frac{r}{r_{ref}}\right)\right]} \tag{A.3}$$

This condition is satisfied by sampling a spatial region well outside the cutoff limit, i.e. if $r_c \ll r \approx r_{ref}$. Based on this assumption and using (19), the position ensemble average is given by

$$\langle r^0 \rangle \sim F(\alpha, r_c^0, v, t^v) t_{ref}^{-v} \tag{A.4}$$

where

$$F(\alpha, r_c^0, v, t) = \frac{C\alpha[(1 - r_c^0)(1 + b) + b r_c^0 \ln r_c^0]t^v}{\Gamma(1+v)} \tag{A.5}$$

It is seen that (A.4) is formally equivalent with (29) upon identifying fluctuations in radial position of the electron with the Ising spin field:

$$\langle r^0 \rangle \Longleftrightarrow \xi \tag{A.6a}$$

$$t_{ref} \to 0 \Longleftrightarrow T \to T_c \tag{A.6b}$$

(A.6b) corresponds to probing the deep ultraviolet domain of quantum field theory, i.e. the Planck length region ($t_{ref} \sim (L_p/c)$).