ZEROS DISTRIBUTION OF THE RIEMANN ZETA - FUNCTION

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Abstract

Horizontal and vertical distributions of complex zeros of the Riemann zeta-function in the critical region are being found in general form in the paper on the basis of standard methods of function theory of complex variable.
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1 Introduction

Let's state in short, to make the picture complete, some elementary properties of zeta-function and point out its role in analytical number theory.

The Riemann zeta-function is defined by the Dirichlet series [1]:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1.1}$$

where $s = \sigma + i\lambda$, and a series converges to analytical function when $\sigma > 1$. To be exact [2], the series $\sum_{n=1}^{\infty} n^{-s}$ evenly converges when $\mathrm{Re}\, s \geqslant 1 + \varepsilon$, and the function, defined by this series, is regular in the semi - plane $\mathrm{Re}\, s > 1$. In this connection n^{-s} for complex s is given by the equality $n^{-s} = e^{-s \ln n}$, where $\ln n$ means a real logarithm of the positive number n.

The link between prime numbers and zeta-function is established in the following way [1,2]: when $\sigma > 1$ the next equality takes place

$$\prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n} \frac{1}{n^s} = \zeta(s), \tag{1.2}$$

where n runs through numbers $1, 2, 3, \dots$, and p runs through all prime numbers $2, 3, 5, 7, 11, \dots$.

Let's remind, that the main problem of the distribution of prime numbers theory is in investigation of $\pi(x)$, that is, of a quantitiys of prime number, which are less than or equal to x. If we consider sufficiently large number of terms in the sequence of prime numbers it is obvious that there is probably no elementary function, whith the help of which it will be possible to represent $\pi(x)$ for all integers $x > 0$, since the increase $\pi(x)$ occurs very uneven [2].

In 1896, Hadamard and (independently of him) Vallee Poussin proved a theorem on prime numbers with the help of the theory of whole functions of the finite order [2]. Its content consists of the description of asymptotical law of prime numbers distribution [2]:

$$\lim_{x \to \infty} \frac{\pi(x) \ln x}{x} = 1. \tag{1.3}$$

In this connection we had to prove, that there are no zeros $\zeta(s)$ on the straight line $\mathrm{Re}\, s = \sigma = 1$. Vallee Poussin proved more precise ratio [2]:

$$\pi(x) = \int_{2}^{x} \frac{d\xi}{\ln \xi} + R(x), \tag{1.4}$$

where for sufficiently large x

$$|R(x)| < c_1 x e^{-c_2\sqrt{\ln x}},$$

i.e. the error is less than $x/\ln^A x$ when the constant number A is as large as possible. Hence, it was shown, that the function

$$\operatorname{li} x = \int_2^x \frac{d\,\xi}{\ln \xi} + C \qquad (C = \operatorname{li} 2 = 1,04\ldots)$$

only gives a good approximation for the $\pi(x)$ function.

Earlier in his famous memoirs of 1860, Riemann showed [2], that the key for deep investigation of the prime numbers distribution is in study of $\zeta(s)$ as the function of complex variable s.

Two main results, proved by Riemann, are [3]:

1) $\zeta(s)$ function can be analytically extended all over the whole plane $\mathbb{C}$; it is meromorphic there and has a single simple pole with a residue 1 in the point $s = 1$. Otherwise, the function $\zeta(s) - (s-1)^{-1}$ is whole.

2) $\zeta(s)$ satisfies the functional equation:

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left(\frac{1}{2}(1-s)\right)\zeta(1-s), \tag{1.5}$$

which is equivalent to the statement that the function at the left, is an even function from $\left(s - \frac{1}{2}\right)$.

It is convenient to rewrite the equation (1.5) in the most concise form [4, 5, 10]

$$\varphi(s) = \varphi(1-s),$$

where function $\varphi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Functional equation permits to derive properties of $\zeta(s)$ when $\sigma < 0$ from its properties when $\sigma > 1$. In particular, the only zeros of $\zeta(s)$ when $\sigma < 0$ are the poles $\Gamma(s/2)$, i.e. the points $s = -2, -4, -6, \ldots$. They are called trivial zeros. The rest part of a plane, where $0 \leqslant \sigma \leqslant 1$, is called a critical zone.

Besides, Riemann made a series of remarkable assumptions [3]:

1′) $\zeta(s)$ has indefinitely many zeros in the critical zone. They are located symmetrically along the real axis, and also the central line $\sigma = \frac{1}{2}$ (the last by virtue of the functional equation).

2′) The number $N(T)$ of zeros $\zeta(s)$ in the critical zone with $0 < \lambda \leqslant T$ satisfies the asymptotical ratio [1,2]

$$N(T) = \frac{T}{2\pi}\ln\frac{T}{2\pi} - \frac{T}{2\pi} + O(\ln T).$$

At first it was proved with rather weak residual term by Mangoldt in 1895, and it was fully proved in 1905 [3].

3′) The whole function $\xi(s)$ determined by the equality

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s),$$

assumes representation in the form of the product

$$\xi(s) = e^{A+Bs}\prod_\rho\left(1 - \frac{s}{\rho}\right)e^{s/\rho}, \tag{1.6}$$

where A and B are constants, an ρ runs through zeros $\zeta(s)$ in the critical zone. This equality and the assumption $1'$) were proved by Hadamard in 1893, [2]. It played an important role when Hadamard and Vallee Poussin were proving the distribution of prime numbers law.

$4'$)) There exists the exact formula for $\pi(x) - \mathrm{li}\, x$, which is true when $x > 1$, the most essential part of which is the sum distributed on complex zeros ρ of the function $\zeta(s)$. Since it has a rather complicated form, let's replace it by the simplest, but closely related to it, formula for $\psi(x) - x$ (definition of the related to $\pi(x)$ function $\psi(x)$ see, for example, in [2,3]). It has the following form under some deals [3]:

$$\psi(x) - x = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}). \tag{1.7}$$

This fact was proved by Mangoldt in 1895, [3] (in the initial statement, belonging to Riemann).

Therefore, it turned out that zeros of $\zeta(s)$ in the zone $0 \leqslant \sigma \leqslant 1$, $\sigma = \mathrm{Re}\, s$ play an exceptional role in many questions of prime number theory.

$5'$) The remarkable Riemann hypothesis which has not been proved yet: all zeros of $\zeta(s)$ in the critical zone are located on the straight line $\sigma = 1/2$. In 1914, Hardy proved, that an indefinite number of zeros were located on that line [1], and in 1942 Selberg proved, that they had a positive density in the set of all zeros [3].

The given report is devoted to the Riemann hypothesis proof $5'$) with the help of complex integration methods, and also to the determination of the general expression describing distribution of all complex zeroes of zeta - function in the critical zone.

2 Necessary and sufficient condition

There exists a great number of conditions for the Riemann hypothesis to be true. For instance, Titchmarsh concretely gave two such conditions [1]. The first of them states that a series $\sum\limits_{1}^{\infty} \mu(n) n^{-s}$ converges, and its sum is equal to $\frac{1}{\zeta(s)}$ for all values of $\sigma > \frac{1}{2}$, and $\mu(n)$ being the Mobius function [2] ($\mu(1) = 1, \mu(n) = 0$, if n is divided by the square of a number other than one, and $\mu(n) = (-1)^k$ otherwise, where k is a number of prime divisors of the number n).

The next necessary and sufficient condition [1] is that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)! \zeta(2n)} = O\left(x^{\frac{1}{4}+\varepsilon}\right), \quad \text{when} \quad x \to \infty.$$

Franel [1] gave the condition of an absolutely another type. It's easy to see that these conditions are rather complicated by themselves and as a rule do not facilitate the problem of the Riemann hypothesis proof. For example, according to Titchmarsh, "problems dealing with $\frac{1}{\zeta(s)}$, are, apparently, extremely difficult" [1].

In this connection let's use (it seems to us more easy) "functional" condition given in details below. In fact, the given condition is an overformulating of the Riemann hypothesis from the function $\zeta(s)$ into the related to it function $F_{2\omega}(s)$ in the sense of zeros.

Consider the class of functions giving by the expression:

$$F_f(z) = \frac{1}{z^2 - \frac{1}{4}} + \Phi_f(z), \tag{2.1}$$

$$\Phi_f(z) = \frac{1}{2}\int\limits_1^{\infty} f(x)x^{-3/4}(x^{z/2}+x^{-z/2})\,dx, \quad z=\sigma+i\lambda, \tag{2.2}$$

where continuous functions $f(x)\colon [1,+\infty)\to\mathbb{R}$ may belong to diffinent functional spaces (the detailed definitions of which we omit).

Class $\{F_f(z)\}$ in (2.1) is initially considered in the zone $\Pi=\{z\in\mathbb{C}\ :\ |\operatorname{Re} z|<1/2\}$ or in the closed zone $\overline{\Pi}=\{z\in\mathbb{C}\ :\ |\operatorname{Re} z|\leqslant 1/2\}$.

Here we shall be interested only in one private case of the function $f(x)$, namely, the function $f_0(x)=2\omega(x)$, where $\omega(x)$ is the function used by Riemann when studying of zeta-function $\zeta(s)$ and which is equal to [1]:

$$\omega(x)=\sum_{n=1}^{\infty} e^{-\pi n^2 x}. \tag{2.3}$$

First of all, let's find the connection of $F_{2\omega}(z)$ and the Riemann zeta-function $\zeta(s)$.

Lemma 2.1 *In any finite part of the plane $\mathbb{C}$ the following ratio is true:*

$$F_{2\omega}(z)=\frac{\Gamma(\frac14+\frac z2)}{\pi^{\frac14+\frac z2}}\zeta\left(\tfrac12+z\right), \quad z\in\mathbb{C}.$$

$\lhd$ From definitions (2.1) and (2.2) for function $F_f(z)$ when $f=f_0=2\omega$ we find the general expression for $F_{2\omega}$:

$$F_{2\omega}(z)=\frac{1}{z^2-\frac14}+\int\limits_1^{\infty}\omega(x)x^{-3/4}(x^{z/2}+x^{-z/2})\,dx, \tag{$*$.1}$$

where z is a complex number from some finite subregion $U\subset\mathbb{C}$.

Let's replace the variable $z=it$, where $t\in\mathbb{C}$ in ($*$.1). Elementary transformations give the expression for $F_{2\omega}(z)$:

$$\left(-\frac12\right)\left(t^2+\frac14\right)F_{2\omega}(it)=\frac12-\left(t^2+\frac14\right)\int\limits_1^{\infty}\omega(x)x^{-3/4}\cos\left(\frac t2\ln x\right)\,dx. \tag{$*$.2}$$

Following [1] and introducing functions

$$\xi(s)=\Xi(t)=\frac12 s(s-1)\Gamma(s/2)\pi^{-s/2}\zeta(s), \tag{$*$.3}$$

where $s=\frac12+it$, while

$$\Xi(t)=\frac12-\left(t^2+\frac14\right)\int\limits_1^{\infty}\omega(x)\,x^{-3/4}\cos\left(\frac t2\ln x\right)\,dx, \tag{$*$.4}$$

at once we find, that

$$\left(-\frac12\right)\left(t^2+\frac14\right)F_{2\omega}(it)=\Xi(t), \quad t\in\mathbb{C}, \tag{$*$.5}$$

or

$$F_{2\omega}(it)=-2\frac{\xi(\frac12+it)}{t^2+\frac14}, \tag{$*$.6}$$

since $\Xi(t)=\xi(\frac12+it)$. For the last function (when $s=\frac12+it$) we have representation:

$$\xi(\tfrac12+it)=-\frac12\left(t^2+\tfrac14\right)\pi^{-\frac14-i\frac t2}\Gamma\left(\tfrac14+i\tfrac t2\right)\zeta(\tfrac12+it), \tag{$*$.7}$$

which being substituted in (∗.6) gives the required result

$$F_{2\omega}(it) = \frac{\Gamma(\frac{1}{4} + i\frac{t}{2})}{\pi^{\frac{1}{4}+i\frac{t}{2}}}\zeta\left(\tfrac{1}{2} + it\right), \quad t \in \mathbb{C}. \tag{$*.8$}$$

Returning to the initial designation $z = it$ in (∗.8) we get the statement of lemma. ▷

Using the principle of analytical continuation, it is easy to show, that lemma 2.1. is true everywhere on the complex plane $\mathbb{C}$, but not only in some of its finite subregion. In particular, the statement of lemma 2.1. is true in the closed zone $-\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}$, $\sigma = \operatorname{Re} z$, where zeros of function $F_{2\omega}(z)$ coincide with zeros of zeta-function because the function $\pi^{-\frac{1}{4}-\frac{z}{2}}\Gamma(\frac{1}{4}+\frac{z}{2})$ has no zeros. To be exact, it follows from lemma 2.1. that zeros of the Riemann zeta-function $\zeta(s)$ in the critical region $0 \leqslant \sigma \leqslant 1$ fully coincide with zeros of the $F_{2\omega}(z)$ function in the closed zone $-\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}$, $\sigma = \operatorname{Re} z$. Hence, we can formulate the following

Condition 2.1 For the Riemann hypothesis on zeros of zeta-function $\zeta(s)$ in the critical zone $0 \leqslant \sigma \leqslant 1$, to be true, it is necessary and sufficient, the equation $F_{2\omega}(z)$ to be true only when $\sigma = 0$ for the $F_{2\omega}(z) = 0$ function in the closed zone $-\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}$, $\sigma = \operatorname{Re} z$.

In other words, for the Riemann hypothesis to be true a strict imagination of roots of the equation $F_{2\omega}(z) = 0$ in the zone $-\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}$ is required.

3 Diverging equations

In this item we shall demonstrate difficulties typical for the theory of zeta- function emerging when working with it in the critical region. Such situation appears in the attempt of the direct use of the condition 2.1 for the Riemann hypothesis proof.

Let's consider condition 2.1 in details.

Lemma 3.1 *The equation $F_{2\omega}(s) = 0$, $s = \sigma + i\lambda$ in the closed zone*

$$\Pi_0 \equiv \{s \in \mathbb{C}\ :\ -\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2},\ -\infty < \lambda < \infty\}$$

is equivalent to the system of two equations

$$\int_0^\infty \omega(e^{2t})\, e^{t/2} \cosh \sigma t \cos \lambda t\, dt = \frac{\lambda^2 + 1/4 - \sigma^2}{4\delta}, \tag{3.1}$$

$$\int_0^\infty \omega(e^{2t})\, e^{t/2} \sinh \sigma t \sin \lambda t\, dt = \frac{\sigma\lambda}{2\delta}, \tag{3.2}$$

in the semi-open zone

$$\Pi_0^+ \equiv \{s \in \mathbb{C}\ :\ 0 \leqslant \sigma < \frac{1}{2},\ 0 \leqslant \lambda < \infty\}.$$

◁ Let the equation $F_{2\omega}(s) = 0$, where $s = \sigma + i\lambda$, $(\sigma = \operatorname{Re} s, \lambda = \operatorname{Im} s)$ be given in the closed zone Π_0, and the $F_{2\omega}(s)$ function be given by the expression (∗.1) of lemma 2.1. Perform technical replacement of variables, $x = e^t$ in the last one to represent $F_{2\omega}(s)$ in more convenient form. As a result of the replacement we get:

$$F_{2\omega}(s) = A_1 + B_1 + i(A_2 + B_2), \tag{$*.1$}$$

where

$$A_1 = 4\int_0^\infty \omega(e^{2t})\, e^{t/2} \cosh\sigma t \cos\lambda t\, dt, \tag{$*.2$}$$

$$A_2 = 4\int_0^\infty \omega(e^{2t})\, e^{t/2} \sinh\sigma t \sin\lambda t\, dt, \tag{$*.3$}$$

$$B_1 = \frac{\sigma^2 - \lambda^2 - 1/4}{\delta}, \quad B_2 = -\frac{2\sigma\lambda}{\delta}, \tag{$*.4$}$$

and

$$\delta = \left[(\sigma - 1/2)^2 + \lambda^2\right] \cdot \left[(\sigma + 1/2)^2 + \lambda^2\right]. \tag{$*.5$}$$

Summing up the right part of $(*.1)$ to zero from independence of real and imaginary parts of $F_{2\omega}$, we get a system of two equations:

$$A_1 + B_1 = 0, \tag{$*.6$}$$

$$A_2 + B_2 = 0. \tag{$*.7$}$$

Hence, by substituting expressions $(*.2)$ – $(*.4)$, in them, and taking into account the character of evenness with respect to σ and λ, and the fact that zeta-function has no zeros [1] on the straight line $\sigma = 1$, we find the truth of lemma. $\triangleright$

It is seen from lemma 3.1., that basic technical difficulty, under such way of consideration is reduced to computation of two integrals in the left side of the equations (3.1) and (3.2) respectively.

To find them we shall prove the following additional statement.

Lemma 3.2 *For the integral $j_n(s;\varepsilon,\eta)$ the following equality is true:*

$$j_n(s;\varepsilon,\eta) \equiv \int_0^\infty e^{-\pi n^2 e^{2t}}\, e^{t/2}\, e^{[(-1)^\varepsilon\sigma + i(-1)^\eta\lambda]t}\, dt =$$

$$= \frac{[1 + (-1)^\eta]}{2} \pi^{-\frac{1}{4} - \frac{\tilde{s}}{2}} \Gamma(\tfrac{1}{4} + \tfrac{\tilde{s}}{2}) n^{-(\frac{1}{2} + \tilde{s})},$$

where n is arbitrary natural; parameters $\varepsilon, \eta = \pm 1$; argument $s = \sigma + i\lambda \in \Pi_0^+$, and finally, $\tilde{s} = (-1)^\varepsilon\sigma + i(-1)^\eta\lambda \in \tilde{\Pi}_0,\ \tilde{\Pi}_0 \equiv \Pi_0 \setminus \{\sigma = \pm\frac{1}{2}\}$.

$\triangleleft$ (a) Let's make replacement of the variable $e^t = y$ in the integral $j_n(s;\varepsilon,\eta)$. After substitution find:

$$j_n(s;\varepsilon,\eta) = \int_1^\infty e^{-\pi n^2 y^2}\, y^{-\frac{1}{2} + \tilde{s}}\, dy. \tag{$*.1$}$$

The integral $(*.1)$ can be rewritlen as well as

$$j_n(s;\varepsilon,\eta) = j_n^\infty(s;\varepsilon,\eta) - j_n^1(s;\varepsilon,\eta), \tag{$*.2$}$$

where addends are obviously equal to integrals:

$$j_n^\infty(\tilde{s}) \equiv \int_0^\infty e^{-\pi n^2 y^2}\, y^{-\frac{1}{2} + \tilde{s}}\, dy, \tag{$*.3$}$$

$$j_n^1(\tilde{s}) \equiv \int_0^1 e^{-\pi n^2 y^2}\, y^{-\frac{1}{2} + \tilde{s}}\, dy, \tag{$*.4$}$$

where $\tilde{s} \in \tilde{\Pi}_0$.

(*b*) The first integral (∗.3) is calculated easily.

According to [6], the integral

$$\int_0^\infty x^{\alpha-1}\, e^{-\rho x^\mu}\, dx = \frac{1}{\mu}\rho^{-\alpha/\mu}\Gamma\left(\frac{\alpha}{\mu}\right),$$

if $\mu, \operatorname{Re}\alpha, \operatorname{Re}\rho > 0$. In our case $\alpha = \frac{1}{2} + \tilde{s}, \mu = 2 > 0, \rho = \pi n^2 > 0$. As it is already known that there are no roots of the equation $F_{2\omega} = 0$ on the straight line $\sigma = \frac{1}{2}$, they will also be out on the line $\sigma = -\frac{1}{2}$ due to symmetry, and thus, it may be supposed that $\operatorname{Re}\alpha > 0$. Then the integral (∗.3) is equal to:

$$j_n^\infty(\tilde{s}) = \int_0^\infty e^{-\pi n^2 y^2}\, y^{(\frac{1}{2}+\tilde{s})-1}\, dy = \frac{1}{2}\pi^{-\frac{1}{4}-\frac{\tilde{s}}{2}}\Gamma(\tfrac{1}{4} + \tfrac{\tilde{s}}{2})n^{-(\frac{1}{2}+\tilde{s})}. \qquad (*.5)$$

Computation of the second integral (∗.4) demands the usage of some facts of theory of functions of complex variable.

(*c*) By virtue of even convergence of the series $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$ to the function e^{-x} on the segment $[0, 1]$, we can perform integrating term by term in the integral (∗.4) which will lead $j_n^1(\tilde{s})$ to representation in the form of a series:

$$j_n^1(\tilde{s}) = \sum_{k=0}^{\infty} \frac{(-\pi n^2)^k}{k!} \int_0^1 y^{2k-\frac{1}{2}+\tilde{s}}\, dy = \sum_{k=0}^{\infty} \frac{(-\pi n^2)^k}{k!} \cdot \frac{1}{2k + (\frac{1}{2} + \tilde{s})}. \qquad (*.6)$$

(*d*) For finding the series (∗.6), let's consider the function of complex variable z:

$$f(z) \equiv \frac{a^z}{z!(z+\mu)}, \qquad (*.7)$$

where $a = -\pi n^2 \in \mathbb{R}, n \in \mathbb{N}, \mu \equiv \frac{1}{2}(\frac{1}{2} + \tilde{s}),\ \tilde{s} \in \tilde{\Pi}_0$. Write the function $f(z)$ in the form of:

$$f(z) = \frac{\varphi_a(z)}{z+\mu}, \qquad (*.8)$$

where $\varphi_a(z) \equiv a^z\Gamma^{-1}(z+1)$ is a whole function everywhere in the plane $\mathbb{C}$. Therefore, the function $f(z)$ has a pole in the point:

$$z_0 = -\mu = -\frac{1}{2}\left(\frac{1}{2} + \tilde{s}\right) = -\left[\frac{1 + 2(-1)^\varepsilon\sigma}{4}\right] - i\frac{(-1)^\eta\lambda}{2}. \qquad (*.9)$$

Since for all $\sigma \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ the inequality $-\frac{1}{2} < \operatorname{Re} z_0 < 0$, where $\operatorname{Re} z_0 = -\frac{1}{4}[1 + 2(-1)^\varepsilon\sigma]$ is true, the region U_{z_0} of the possible state of the pole z_0 always belongs to the initial region $\tilde{\Pi}_0$, i.e. $U_{z_0} \subset \tilde{\Pi}_0$.

Let's consider a module of the function $f(z)$:

$$|f(z)| = \left|\frac{\varphi_a(z)}{z+\mu}\right| \leqslant \frac{|\varphi_a(z)|}{|z| - |\mu|} = |z|^{-1} \cdot \frac{|\varphi_a(z)|}{1 - \dfrac{|\mu|}{|z|}}. \qquad (*.10)$$

Take into account the asymptotical formula [7]:

$$\Gamma(z+1) \sim \sqrt{2\pi}\, e^{-z}\, z^{z+1/2}, \quad z \to \infty,\ |\arg z| < \pi,$$

from which evaluate the module of the function $\varphi_a(z)$ as:

$$|\varphi_a(z)| = \left| \frac{e^{z\ln a}}{\Gamma(z+1)} \right| \sim \frac{1}{\sqrt{2\pi}} \left| z^{-z-\frac{1}{2}} \cdot e^{z(\ln a+1)} \right| = \frac{1}{\sqrt{2\pi}} \left| e^{-(z+\frac{1}{2})\ln z} \right| \cdot \left| e^{z(\ln a+1)} \right|. \qquad (*.11)$$

Noting in $(*.11)$, that $|\,z^z| = e^{x\ln|z| - y\arg z}$, $(x = \operatorname{Re} z,\ y = \operatorname{Im} z)$, and $\left|\, e^{z\ln a} \right| = e^{x\ln t - \pi y}$, since

$$\left|\, e^{z\ln a} \right| = \left| e^{(x+iy)(\ln t + i\pi)} \right| = \left|\, e^{x\ln t - \pi y} \right| \cdot \left| e^{i(\pi x + y\ln t)} \right|,$$

where $t = -a = \pi n^2 > 0,\ n = 1, 2, \ldots,$ for $|\varphi_a(z)|$ we find:

$$|\varphi_a(z)| \sim \frac{1}{\sqrt{2\pi|z|}} \cdot e^{x\left(1+\ln\frac{t}{|z|}\right) + y(\arg z - \pi)}. \qquad (*.12)$$

Substituting $(*.12)$ into $(*.10)$ for the module of the function $f(z)$ we get an asymptotic expression:

$$|f(z)| \lesssim \frac{|z|^{-3/2}}{\sqrt{2\pi}} \cdot \frac{e^{x\left(1+\ln\frac{t}{|z|}\right)}}{\left[1 - \frac{|\mu|}{|z|}\right]} \cdot e^{-y(\pi - \arg z)}, \qquad (*.13)$$

where $|\arg z| < \pi,\ |z| \to \infty,\ t = \pi n^2,\ n \in \mathbb{N}$.

For further reasonings rewrite the expression $(*.13)$ in the form of:

$$|f(z)| \lesssim \varphi^+(z) \cdot e^{-y(\pi - \arg z)} \qquad (*.14)$$

where the function $\varphi^+(z)$, equal to

$$\varphi^+(z) \equiv \frac{|z|^{-3/2}}{\sqrt{2\pi}} \cdot \frac{e^{x\left(1+\ln\frac{t}{|z|}\right)}}{\left[1 - \frac{|\mu|}{|z|}\right]} \qquad (*.15)$$

is introduced. The function $\varphi^+(z)$, defined for all $x \in (0, +\infty)$ as:

$$\varphi^+(x) \equiv \varphi^+(\operatorname{Re} z) = \frac{x^{-3/2}}{\sqrt{2\pi}} \cdot \frac{e^{x\left(1+\ln\frac{t}{x}\right)}}{\left[1 - \frac{|\mu|}{x}\right]} \qquad (*.16)$$

adjoins directly in form to the function $\varphi^+(x)$.

From the comparison $(*.15)$ and $(*.16)$ it is seen, that

$$\varphi^+(z) \leqslant \varphi^+(x) \qquad (*.17)$$

for any z from the right semi-plane $\mathbb{C}$ (i.e. for $\operatorname{Re} z > 0$).

Let's define asymptotical behaviour of the function $\varphi^+(x)$ when $x \to +\infty$. We have:

$$\lim_{x\to+\infty} \varphi^+(x) = \frac{1}{\sqrt{2\pi}} \cdot \lim_{x\to+\infty} \frac{1}{\left[1 - \frac{|\mu|}{x}\right]} \cdot \lim_{x\to+\infty} \frac{e^{x\left(1+\ln\frac{t}{x}\right)}}{x^{3/2}} \sim$$

$$\sim \lim_{x\to+\infty} \frac{e^{x\ln\frac{t}{x}}}{\sqrt{2\pi}\, x^{3/2}} = o\left(e^{-x}\right), \quad x \to +\infty. \qquad (*.18)$$

Thus, the function $\varphi^+(x)$ is limited on the semi-axis $(0, +\infty)$ and hence, there is an integral

$$\int\limits_0^{+\infty} \varphi^+(x)\, dx < \infty, \qquad (*.19)$$

Actually, as it follows from (∗.18):

$$\left|\int_0^{+\infty} \varphi^+(x)\,dx\right| \leqslant \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left|\frac{e^{x\left(1+\ln\frac{t}{x}\right)}}{x^{3/2}\,(1-|\mu|/x)}\right| dx \sim \int_0^{+\infty} o\left(e^{-x}\right)\, dx.$$

Now let's consider an "argumental" part of the inequality (∗.14). From condition $|\arg z| < \pi$ it follows: $\arg z = \pi - |\beta|$, when $\arg z > 0$ and $\arg z = \pi + |\beta|$, when $\arg z < 0$, while $0 < |\beta| < 2\pi$. Hence, $\pi - \arg z = \pi - (\pi - |\beta|) = |\beta| > 0$ when $\arg z > 0$ and $\pi - \arg z = \pi - (\pi + |\beta|) = -|\beta| < 0$ when $\arg z < 0$. Thus; $-y(\pi - \arg z) = -|\beta|\, y$ when $\arg z > 0$ and $-y(\pi - \arg z) = |\beta|\, y$ when $\arg z < 0$. Supposing that $\beta > 0$ when $\arg z < 0$ and $\beta < 0$ when $\arg z > 0$ and taking into account the limit $0 < |\beta| < 2\pi$, let's rewrite the last equalities in a single form:

$$-\, y(\pi - \arg z) = \beta|y| = \beta\, |\mathrm{Im}\, z|\,, \tag{∗.20}$$

where $-2\pi < \beta < 2\pi$.

Taking into account (∗.17), (∗.19) and (∗.20), finally find inequality in sought for the function $f(z)$, proceeding from the expression (∗.14):

$$|f(z)| \leqslant e^{\beta|\,\mathrm{Im}\, z|}\, \varphi^+(\mathrm{Re}\, z), \tag{∗.21}$$

where $-2\pi < \beta < 2\pi$ and $\int\limits_0^{+\infty} \varphi^+(x)\, dx < \infty$, $\varphi^+(x)$ being given by the expression (∗.16).

(*e*) Condition (∗.21), proved in item (*d*), permits to use for the function $f(z)$ the method of summing of the form of [8,9]:

$$\sum_{k=-\infty}^{+\infty} f(k) - \int_{-\infty}^{+\infty} f(x)\, dx = -\pi \sum_{\mathrm{Im}\, a_k > 0} \operatorname*{res}_{z=a_k} f(z)(\cot \pi z + i) + \\ + \pi \sum_{\mathrm{Im}\, a_k < 0} \operatorname*{res}_{z=a_k} f(z)(\cot \pi z - i). \tag{∗.22}$$

For this let's finish to define the function $f(x)$ on the semi-interval $(-\infty, 0)$ as identically equal to zero, i.e. $f(x) \equiv 0$ when $x < 0$. Then it follows from (∗.22):

$$\sum_{k=0}^{+\infty} f(k) = \int_0^{+\infty} f(x)\, dx - \pi \cdot \operatorname*{res}_{\mathrm{Im}\, z_0 > 0} \left[f(z)(\cot \pi z + i)\right] + \\ + \pi \cdot \operatorname*{res}_{\mathrm{Im}\, z_0 < 0} \left[f(z)(\cot \pi z - i)\right]. \tag{∗.23}$$

First of all calculate the integral being a part of (∗.23). From the above, it can be concluded, that the function $f(z)$ is regular in the semi-plane $\mathrm{Im}\, z > 0$ when $\mathrm{Im}\, \tilde{s} > 0$ (i.e. when $\eta = 2$) or in the semi-plane $\mathrm{Im}\, z < 0$ when $\mathrm{Im}\, \tilde{s} < 0$ (i.e. when $\eta = 1$),with the exception of ordinary pole z_0, and it is continuous up to the boundary (with the exception of the same pole). Let's show, that function $f(z)$ satisfies the condition $f(z) = o\left(\frac{1}{z}\right)$, when $z \to \infty$ in the semi-plane $\mathrm{Im}\, z \geqslant 0\, (\mathrm{Im}\, z \leqslant 0)$.

In fact, when $z \to \infty$ we have:

$$f(z) = \frac{e^{z \ln a}}{(z - z_0)\, \Gamma(z+1)} \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{z(1+\ln a)}}{z^{z+1/2}\,(z - z_0)} \sim \frac{e^{z(1+\ln a)}}{\sqrt{2\pi}\; z^{z+3/2}} = \\ = \frac{1}{z}\left\{\frac{1}{\sqrt{2\pi z}}\left[\frac{e^{(1+\ln a)}}{z}\right]^z\right\} = o\left(\frac{1}{z}\right), \tag{∗.24}$$

since it is obvious, that when $|z| \to \infty$

$$\left| \frac{1}{\sqrt{2\pi z}} \left[\frac{e^{(1+\ln a)}}{z} \right]^z \right| \to 0.$$

The asymptotical formula $\Gamma(z+1)$ (see item (d)) is used in $(*.24)$. The property $(*.24)$ permits to use theorem on residues [8,9] for the integral in the left part of the expression $(*.23)$:

$$\int\limits_{-\infty}^{+\infty} f(x)\,dx = \int\limits_{0}^{+\infty} f(x)\,dx = 2\pi i \cdot \operatorname*{res}_{z=z_0} f(z) = 2\pi i \cdot \frac{a^{z_0}}{(z_0)!}. \qquad (*.25)$$

Substituting $(*.25)$ into $(*.23)$ and performing elementary transformations, find an expression for the series $(*.6)$:

$$\sum_{k=0}^{\infty} \frac{(-\pi n^2)^k}{k!} \cdot \frac{1}{2k + (\frac{1}{2} + \tilde{s})} =$$

$$= \frac{\pi}{2} \cdot \frac{(-\pi n^2)^{-\frac{1}{4}-\frac{\tilde{s}}{2}}}{\Gamma\left(\frac{3}{4} - \frac{\tilde{s}}{2}\right)} \cdot \left[i + (-1)^{\eta+1} \cot \frac{\pi}{2} \left(\tfrac{1}{2} + \tilde{s}\right) \right]. \qquad (*.26)$$

(f) Transform and simplify the expression $(*.26)$. For this purpose let's perform a number of elementary transformations. We have [6]:

$$i + (-1)^{\eta+1} \cot \frac{\pi}{2} \left(\frac{1}{2} + \tilde{s} \right) = \frac{(-1)^{\eta+1} \cos \frac{\pi}{2} \left(\frac{1}{2} + \tilde{s}\right) + i \sin \frac{\pi}{2} \left(\frac{1}{2} + \tilde{s}\right)}{\sin \frac{\pi}{2} \left(\frac{1}{2} + \tilde{s}\right)}.$$

Calculating the numerator of the given expression, we get:

$$(-1)^{\eta+1} \cdot \exp \left[i(-1)^{\eta+1} \frac{\pi}{2} \left(\frac{1}{2} + (-1)^{\varepsilon}\, \sigma \right) + (-1)^{\eta} \frac{\pi\lambda}{2} \right].$$

Further:

$$(-1)^{-\frac{1}{4}-\frac{\tilde{s}}{2}} = e^{\pm i \frac{\pi}{2} \left(\frac{1}{2} + \tilde{\sigma}\right)} \cdot e^{\mp \frac{\pi \tilde{\lambda}}{2}}.$$

By virtue of randomness of signs selection let's choose them in this way:

$$(-1)^{-\frac{1}{4}-\frac{\tilde{s}}{2}} = \exp \left[i(-1)^{\eta} \frac{\pi}{2} \left(\frac{1}{2} + (-1)^{\varepsilon}\, \sigma \right) + (-1)^{\eta+1} \frac{\pi\lambda}{2} \right].$$

Multiplication of the last expression by the numerator of the considering fraction, probably leads to the quantity $(-1)^{\eta+1}$. Taking also into account that [7]:

$$\frac{\pi}{\sin \pi \left(\frac{1}{4} + \frac{\tilde{s}}{2}\right)} = \Gamma\left(\tfrac{1}{4} + \tfrac{\tilde{s}}{2}\right) \Gamma\left(\tfrac{3}{4} - \tfrac{\tilde{s}}{2}\right),$$

find the representation of the series $(*.26)$ in the form of:

$$j_n^1(\tilde{s}) = \sum_{k=0}^{\infty} \frac{(-\pi n^2)^k}{k!} \cdot \frac{1}{2k + (\frac{1}{2} + \tilde{s})} =$$

$$= \frac{(-1)^{\eta+1}}{2} \cdot \pi^{-\frac{1}{4}-\frac{\tilde{s}}{2}} \cdot \Gamma\left(\tfrac{1}{4} + \tfrac{\tilde{s}}{2}\right) \cdot n^{-(\frac{1}{2}+\tilde{s})}. \qquad (*.27)$$

And finally, substitution of $(*.5)$ and $(*.27)$ into $(*.2)$ proves the result of lemma. ▷

Lemma 3.3 *The direct application of lemma 3.2. to lemma 3.1. leads to the system of two diverging equations:*

$$\sum_{n=1}^{\infty}\Big\{\pi^{-\frac{1}{4}-\frac{\sigma}{2}-i\frac{\lambda}{2}}\cdot\Gamma\left(\tfrac{1}{4}+\tfrac{\sigma}{2}+i\tfrac{\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}+\sigma+i\lambda\right)}+$$
$$+\pi^{-\frac{1}{4}+\frac{\sigma}{2}-i\frac{\lambda}{2}}\cdot\Gamma\left(\tfrac{1}{4}-\tfrac{\sigma}{2}+i\tfrac{\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}-\sigma+i\lambda\right)}\Big\}=\tfrac{\lambda^2+1/4-\sigma^2}{\delta}, \tag{3.3}$$

$$\sum_{n=1}^{\infty}\Big\{\pi^{-\frac{1}{4}-\frac{\sigma}{2}-i\frac{\lambda}{2}}\cdot\Gamma\left(\tfrac{1}{4}+\tfrac{\sigma}{2}+i\tfrac{\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}+\sigma+i\lambda\right)}-$$
$$-\pi^{-\frac{1}{4}+\frac{\sigma}{2}-i\frac{\lambda}{2}}\cdot\Gamma\left(\tfrac{1}{4}-\tfrac{\sigma}{2}+i\tfrac{\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}-\sigma+i\lambda\right)}\Big\}=2i\cdot\tfrac{\sigma\lambda}{\delta}, \tag{3.4}$$

for all $s=\sigma+i\lambda\in\Pi_0^+$.

$\lhd$ Calculate the integral in the left part of the equation (3.1) with the help of lemma 3.2. proved above:

$$\int_0^{\infty}\omega(e^{2t})\,e^{t/2}\cosh\sigma t\cos\lambda t\,dt=$$
$$=\sum_{n=1}^{\infty}\int_0^{\infty}e^{-\pi n^2e^{2t}}\,e^{t/2}\cdot\frac{1}{4}\Big\{e^{\left[(-1)^2\sigma+i(-1)^2\lambda\right]t}+e^{\left[(-1)^2\sigma+i(-1)^1\lambda\right]t}+$$
$$+e^{\left[(-1)^1\sigma+i(-1)^2\lambda\right]t}+e^{\left[(-1)^1\sigma+i(-1)^1\lambda\right]t}\Big\}\,dt=$$
$$=\frac{1}{4}\sum_{n=1}^{\infty}\Big\{\pi^{-\frac{1}{4}-\frac{1}{2}(\sigma+i\lambda)}\cdot\Gamma\left(\tfrac{1}{4}+\tfrac{\sigma+i\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}+\sigma+i\lambda\right)}+$$
$$+\pi^{-\frac{1}{4}-\frac{1}{2}(-\sigma+i\lambda)}\cdot\Gamma\left(\tfrac{1}{4}+\tfrac{-\sigma+i\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}-\sigma+i\lambda\right)}\Big\}. \tag{$*$.1}$$

Substitution of (∗.1) into the equation (3.1) gives the first equation (3.3). Doing the same with the left part of the equation (3.2), let's find:

$$\int_0^{\infty}\omega(e^{2t})\,e^{t/2}\sinh\sigma t\sin\lambda t\,dt=$$
$$=\frac{1}{4i}\sum_{n=1}^{\infty}\int_0^{\infty}e^{-\pi n^2e^{2t}}\,e^{t/2}\cdot\Big\{e^{\left[(-1)^2\sigma+i(-1)^2\lambda\right]t}-e^{\left[(-1)^2\sigma+i(-1)^1\lambda\right]t}-$$
$$-e^{\left[(-1)^1\sigma+i(-1)^2\lambda\right]t}+e^{\left[(-1)^1\sigma+i(-1)^1\lambda\right]t}\Big\}\,dt=$$
$$=\frac{1}{4i}\sum_{n=1}^{\infty}\Big\{\pi^{-\frac{1}{4}-\frac{1}{2}(\sigma+i\lambda)}\cdot\Gamma\left(\tfrac{1}{4}+\tfrac{\sigma+i\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}+\sigma+i\lambda\right)}-$$
$$-\pi^{-\frac{1}{4}-\frac{1}{2}(-\sigma+i\lambda)}\cdot\Gamma\left(\tfrac{1}{4}+\tfrac{-\sigma+i\lambda}{2}\right)\cdot n^{-\left(\frac{1}{2}-\sigma+i\lambda\right)}\Big\}. \tag{$*$.2}$$

From here by substitution of (∗.2) into the equation (3.2), find the second equation (3.4). Integrating term by term in (∗.1) and (∗.2) is admitable due to uniform convergence of the series $\omega(e^{2t})$, when $t\in[0,\infty)$, that in turn follows from the fact: $\omega(x)=O\left(e^{-\pi x}\right)$, when $x\to+\infty$ [3].

Divergence of series in the left part of (3.3) and (3.4) for all $s = \sigma + i\lambda$ from Π_0^+ is obviously connected with impossibility to represent ζ-function in the critical region as absolutely converged Dirichlet series [1,2]. $\triangleright$

4 Functional equation for $\tilde{\omega}(x)$

Basic idea of this item is concerned with finding of the representation of the function $\omega(x)$ via "related" to it $\tilde{\omega}(x)$ function (which will be defined below) with the help of functional equation. In this case there will not be a problem of divergence in the critical zone for the $\tilde{\omega}(x)$ function.

Let's preliminary state a number of well-known facts which are necessary here.

According to [8,9], when any complex $\alpha = \operatorname{Re}\alpha + i \operatorname{Im}\alpha$, the sum of ascending power series

$$\zeta(\alpha; z) \equiv \sum_{n=1}^{\infty} n^{-\alpha} z^n, \quad (z \in \overline{\mathbb{C}}), \tag{4.1}$$

originally defined in the circle $|z| < 1$, may be analytically continued through the whole complex plane $\overline{\mathbb{C}}$ with the cut along the ray $[1, \infty]$.

Hence, everywhere in $\mathbb{C}$, in particular, the function of complex $\alpha \in \mathbb{C}$ has been defined:

$$\zeta(\alpha; -1) = \sum_{n=1}^{\infty} (-1)^n n^{-\alpha}. \tag{4.2}$$

At the same time the equality [1,7] is true for zeta-function :

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = \left(1 - 2^{1-s}\right) \zeta(s), \quad s \in \mathbb{C}, \tag{4.3}$$

from which we easily find the connectoin with function $\zeta(s; -1)$ by comparison with (4.2):

$$\zeta(s; -1) = \left(2^{1-s} - 1\right) \zeta(s), \tag{4.4}$$

true when any $\operatorname{Re} s > 0$, as well as in the point $s = 1$, in which $\zeta(1; -1) = T_1 = -\ln 2 \approx -0,6931\ldots$ [6].

Remind that function $\omega(x)$, defined for all $x \in [0, +\infty)$ is subjected to functional equation [1,3] by the expression (2.3):

$$\omega(x) = \frac{1}{\sqrt{x}} \left[\omega\left(\frac{1}{x}\right) + \frac{1}{2}\right] - \frac{1}{2}. \tag{4.5}$$

Let's introduce into consideration the function $\tilde{\omega}(x)$, $x \in [0, +\infty)$:

$$\tilde{\omega}(x) = \sum_{n=1}^{\infty} (-1)^n e^{-\pi n^2 x}. \tag{4.6}$$

Lemma 4.1 *For all $x \in [0, +\infty)$ the function $\tilde{\omega}(x)$ satisfies the following functional equation:*

$$\tilde{\omega}(x) = \frac{1}{\sqrt{x}} \left[\omega\left(\frac{1}{4x}\right) - \omega\left(\frac{1}{x}\right)\right] - \frac{1}{2}. \tag{4.7}$$

$\triangleleft$ According to [10] it is known that for a series

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{\beta} e^{-n^{\alpha} x}, \tag{*.1}$$

where $\beta \in \mathbb{R}, \alpha > 0$ is real, $x \in [0, +\infty)$, the integral representation is true:

$$f(x) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s)\, x^{-s} \left(1 - 2^{1+\beta-\alpha s}\right) \zeta(\alpha s - \beta)\, ds, \qquad (*.2)$$

where $\sigma > 0$, $\sigma > \frac{1+\beta}{\alpha}$. Suppose, in $(*.2)$ $\beta = 0, \alpha = 2$. Then

$$f(x) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s)\, x^{-s} \left(1 - 2^{1-2s}\right) \zeta(2s)\, ds, \qquad (*.3)$$

where $\sigma > 1/2$. Replace x by πx^2 and s by $s/2$ in $(*.3)$:

$$f(x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\pi n^2 x^2} = \frac{1}{4\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) \left(1 - 2^{1-s}\right) \zeta(s) x^{-s}\, ds, \qquad (*.4)$$

$\sigma > 1/2$. If we transit in $(*.4)$ from the straight line of integrating $\operatorname{Re} s = \sigma > 1/2$ to the straight line $\operatorname{Re} s = \sigma_0, -\frac{1}{2} < \sigma_0 < 0$, that is possible [10], then it is necessary to add residues in the point $s = 1$ (the pole $\zeta(s)$) and in the point $s = 0$ (the pole $\Gamma(s)$) in the zone $\sigma_0 \leqslant \operatorname{Re} s \leqslant \sigma$ when transiting. Both poles of the first order, exactly for them respectively

$$\lim_{s \to 1} (s-1)\zeta(s) = 1,$$
$$\lim_{s \to 0} s\Gamma(s) = \lim_{s \to 0} \Gamma(s+1) = \Gamma(1) = 1.$$

That's why the residue of sub-integral function in $(*.4)$ in the point $s = 1$ is equal to
$\pi^{-1/2}\, \Gamma(1/2)\, x^{-1}(1 - 2^{1-1}) = 0$, and in the point $s = 0$ it is equal to $2(1-2)\zeta(0) = 1$.

Substituting residues and performing replacement of s by $1 - s$ in $(*.4)$, we get an expression:

$$(-1) + 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\pi n^2 x^2} =$$
$$= \frac{1}{2\pi i} \int\limits_{1-\sigma_0 - i\infty}^{1-\sigma_0 + i\infty} \pi^{-\frac{1-s}{2}}\, \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \left(1 - 2^s\right) x^{-1+s}\, ds. \qquad (*.5)$$

Taking into account in $(*.5)$ functional equation (1.5) for ζ-function, rewrite the first one in the form of:

$$(-1) + 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\pi n^2 x^2} =$$
$$= \frac{1}{2\pi i x} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) \zeta(s) \left(1 - 2^s\right) x^s\, ds, \qquad (*.6)$$

where $\sigma = 1 - \sigma_0 > 1$.

Multiply the expression $(*.4)$ by 2 and consider integrals in its right part separately.

For the first integral, according to [10,4], we know that:

$$\frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) \zeta(s)\, x^{-s}\, ds = 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x^2}. \qquad (*.7)$$

Then for the second integral in (∗.4) we have:

$$\frac{1}{\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)\,(2x)^{-s}\,ds = 4\sum_{n=1}^{\infty}e^{-\pi n^2(2x)^2}. \tag{$*$.8}$$

Expanding integral in the right part of (∗.6) by the same way we find:

$$\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)\,(1-2^s)\,x^s\,ds =$$

$$= 2\sum_{n=1}^{\infty}e^{-\pi\frac{n^2}{x^2}} - 2\sum_{n=1}^{\infty}e^{-\pi\frac{n^2}{4x^2}}, \quad (\sigma>1). \tag{$*$.9}$$

From the expressions (∗.4) and (∗.7), (∗.8) it follows that:

$$\sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi n^2x^2} = \sum_{n=1}^{\infty}e^{-\pi n^2x^2} - 2\sum_{n=1}^{\infty}e^{-4\pi n^2x^2}. \tag{$*$.10}$$

In its turn from expressions (∗.6) and (∗.9) it follows:

$$\sum_{n=-\infty}^{+\infty}(-1)^{n-1}e^{-\pi n^2x^2} = \frac{2}{x}\sum_{n=1}^{\infty}\left[e^{-\pi\frac{n^2}{x^2}} - e^{-\pi\frac{n^2}{4x^2}}\right], \tag{$*$.11}$$

or

$$(-1) - 2\tilde{\omega}(x^2) = \frac{2}{x}\left[\omega\left(\frac{1}{x^2}\right) - \omega\left(\frac{1}{4x^2}\right)\right]. \tag{$*$.12}$$

And finally, performing in (∗.12) replacement $x^2 \to x$, after elementary transformations, we find the equation (4.7). ▷

Lemma 4.2 *For all $x \in [0,+\infty)$ the function $\omega(x)$ satisfies the following functional equation:*

$$\omega(x) = \tilde{\omega}(x) + \frac{1}{\sqrt{x}}\left[\tilde{\omega}\left(\frac{1}{4x}\right) + \frac{1}{2}\right]. \tag{4.8}$$

◁ For proof we shall use the expression (∗.10) of lemma 4.1., which can be written in the form of:

$$\tilde{\omega}(x^2) = 2\omega(4x^2) - \omega(x^2), \tag{$*$.1}$$

or, performing replacement $x^2 \to x$, in the form of:

$$\tilde{\omega}(x) = 2\omega(4x) - \omega(x). \tag{$*$.2}$$

If we perform replacement $x \to \dfrac{1}{4x}$ in (∗.2), we shell come to the expression

$$\omega\left(\frac{1}{4x}\right) - \omega\left(\frac{1}{x}\right) = \omega\left(\frac{1}{x}\right) - \tilde{\omega}\left(\frac{1}{4x}\right). \tag{$*$.3}$$

Substitution of (∗.3) into the equation (4.7) of lemma 4.1. leads to a new functional equation:

$$\tilde{\omega}(x) = \frac{1}{\sqrt{x}}\left[\omega\left(\frac{1}{x}\right) - \tilde{\omega}\left(\frac{1}{4x}\right)\right] - \frac{1}{2}. \tag{$*$.4}$$

Transpose the second term in parentheses into the left side and add the addend $\dfrac{1}{2\sqrt{x}}$ at the left and at the right. Then

$$\tilde{\omega}(x) + x^{-1/2}\,\tilde{\omega}\left(\frac{1}{4x}\right) + \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}\left[\omega\left(\frac{1}{x}\right) + \frac{1}{2}\right] - \frac{1}{2}\ . \qquad (*.5)$$

Comparing right sides of the expression $(*.5)$ and the equations (4.5) we convince in their identity and, hence, in truth of the equation (4.8). $\triangleright$

Therefore, lemma 4.2. is the conversion of lemma 4.1.

5 Zeros of zeta-function

Let's turn to the series (4.3) again:

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}, \qquad s = \mathrm{Re}\, s + i\,\mathrm{Im}\, s \in \mathbb{C}.$$

When $\mathrm{Re}\, s = \sigma > 1$ the series (4.3) converges absolutely, while in the critical region $0 < \mathrm{Re}\, s < 1$ the series (4.3) converges only conditionally [2]. Since when $s \in \mathbb{C}$ the limit of partial sums of the series (4.3) is a complex number in any transposition of its terms, then for it, the Riemann theorem [11] on complex conditionally converging series is true. According to the Riemann theorem, a set of values M_Σ of all possible sums of the series (4.3) is either a straight line $\mathbb{T}_0$, passing through the origin, or the whole plane $\mathbb{C}$, when every $s \in \mathbb{C}$, $\mathrm{Re}\, s \in (0, 1)$.

So,

$$M_\Sigma(s_0) \subseteq \{\mathbb{T}_0\,(s_0)\,, \mathbb{C}\}\,, \qquad (5.1)$$

where s_0 is some complex number from the critical region. In this connection let's denote a set of series, formed from the series (4.3) by arbitrary transposition of its terms as $\Sigma_I \equiv \{\Sigma_q\}_{q\in I}$, where I is some index set. Prove the statement on one of the subsets of the set of conditionally converging series $\{\Sigma_q\}_{q\in I}$ operating in the critical region which will be necessary for us further.

Lemma 5.1 *Any converging series Σ_q, $q \in I$, taken from the set Σ_I, and proportional to the Riemann zeta-function in every point of critical semi-zone*
$\mathbb{K}^+ \equiv \{s \in \mathbb{C} : 0 < \mathrm{Re}\, s < 1, 0 \leq \mathrm{Im}\,\lambda < \infty\}$*, can be presented in this area in the form of:*

$$\Sigma_q(s) = \left(1 - 2^{1-s}\Delta_q\right)\zeta(s), \qquad s \in \mathbb{K}^+, \qquad (5.2)$$

where Δ_q is a real value, the inequality $|\Delta_q| \leq \rho_0 < \infty$ being true for all $q \in I$.

$\triangleleft$ (a) Instead of the set $M_\Sigma\,(s_0)$ we shall consider the set $M_{\tilde{\Sigma}}\,(s_0)$, consisting of sums meanings, forming by all possible transposition of auxiliary series:

$$\tilde{\Sigma}_0\,(s_0) \equiv \zeta\,(s_0) - \sum_{n=1}^{\infty}(-1)^{n-1}\,n^{-s_0}, \qquad (*.1)$$

when every fixed $s_0 \in \mathbb{K}^+$. Then by virtue of (5.1) for the set $M_{\tilde{\Sigma}}\,(s_0)$ we can write at once, that

$$M_{\tilde{\Sigma}}\,(s_0) \subseteq \left\{\tilde{\mathbb{T}}_0\,(s_0)\,, \mathbb{C}\right\}, \qquad s_0 \in \mathbb{K}^+. \qquad (*.2)$$

Since in any case, a complex plane $\mathbb{C}$ can be considered as projective plane $\mathbb{PC}^1$, i. e. as a totality of all straight lines passing through the origin, we can consider below a straight line $\mathbb{T}_{\tilde{\Sigma}_0}\,(s_0)$ (that is, a straight line $\tilde{\mathbb{T}}_0\,(s_0)$, passing through the "frame" point $\tilde{\Sigma}_0\,(s_0)$, when any arbitrary $s_0 \in \mathbb{K}^+$) which, at least, is exactly in $M_{\tilde{\Sigma}}\,(s_0)$.

(b) At first we shall consider $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$ as a straight line, passing into $\mathbb{C}$ through the middle of the segment, connecting points z_1 and z_2, perpendicular to this segment. According to, for example [12,9], for any $z \in \mathbb{T}_{\tilde{\Sigma}_0}(s_0)$ the equation is true :

$$|z - z_1| = |z - z_2|, \qquad (*.3)$$

which can be rewritten for arbitrary $z = \operatorname{Re} z + i \operatorname{Im} z \in \mathbb{T}_{\tilde{\Sigma}_0}(s_0)$, $s_0 \in \mathbb{K}^+$ in the form of:

$$\frac{2 \operatorname{Re} z}{(y_2 - y_1)} + \frac{2 \operatorname{Im} z}{(x_2 - x_1)} = (x_2 + x_1) + (y_2 + y_1), \qquad (*.4)$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ are some fixed points from $\mathbb{C}$. A pair of numbers $\left\{\frac{(x_1+x_2)}{2}, \frac{(y_1+y_2)}{2}\right\}$ is coordinates of the middele of the segment $[z_1, z_2]$, which can conveniently be taken as an origin in our case. Then the equation $(*.4)$ can be reduced to the simplest form:

$$\frac{\operatorname{Re} z}{\operatorname{Im} z} = -\gamma, \qquad (*.5)$$

where $\gamma \equiv \frac{(y_2 - y_1)}{(x_2 - x_1)}$ is a real value, constant (when z_1 and z_2 are fixed) for all points of the straight line $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$. In particular, applying the equation $(*.5)$ to "frame" point $\tilde{z}_0 = \tilde{\Sigma}_0(s_0)$, which according to (4.3) and $(*.1)$ is equal to $\tilde{z}_0 = 2^{1-s_0}\zeta(s_0), s_0 \in \mathbb{K}^+$, find the equation:

$$\frac{x}{y} = \frac{\tilde{\alpha}_0 \operatorname{Re} \zeta_0 - \tilde{\beta}_0 \operatorname{Im} \zeta_0}{\tilde{\alpha}_0 \operatorname{Im} \zeta_0 + \tilde{\beta}_0 \operatorname{Re} \zeta_0}, \qquad (*.6)$$

where $x = \operatorname{Re} z, y = \operatorname{Im} z$ and $\tilde{\alpha}_0 = \operatorname{Re}\left(2^{1-s_0}\right), \tilde{\beta}_0 = \operatorname{Im}\left(2^{1-s_0}\right)$.

If we now consider only those conditional sums $\tilde{\Sigma}_{q'}(s_0), q' \in I' \subset I$ on the straight line $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$ that

$$z = \tilde{\Sigma}_{q'}(s_0) = \left(\tilde{\alpha} + i\tilde{\beta}\right)\zeta(s_0), \qquad (*.7)$$

and $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$ for all $s_0 \in \mathbb{K}^+$, then we shall find an equality from $(*.6)$:

$$\frac{\tilde{\alpha}}{\tilde{\beta}} = \frac{\tilde{\alpha}_0}{\tilde{\beta}_0} = (-1)\cot\left(\ln 2^{\operatorname{Im} s_0}\right). \qquad (*.8)$$

Let's denote sets of sums of the form of $(*.7)$, when arbitrary $s_0 \in \mathbb{K}^+$, via $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$. Thus, we have an embedding

$$\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s_0) \subset \mathbb{T}_{\tilde{\Sigma}_0}(s_0). \qquad (*.9)$$

(c) Specify the following necessary moments here:

$c.1)$ The quation of the straight line $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$ for any $s_0 \in \mathbb{K}^+$ is the equation of the form of $(*.6)$ and only such;

$c.2)$ Equations $(*.6)$ for different $s_0 \in \mathbb{K}^+$ uniquely define, generally speaking, different straight lines $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$;

$c.3)$ The equation $(*.8)$, obtaining from the equation $(*.6)$ under condition $(*.7)$, is the only condition for defining a set of points $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s_0) \subset \mathbb{T}_{\tilde{\Sigma}_0}(s_0)$ on every straight line $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$, $s_0 \in \mathbb{K}^+$;

$c.4)$ In this connection the set $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$ is not uniquely defined by the condition $(*.8)$. Actually, the condition $(*.8)$ uniquely follows from the equation $(*.6)$, but the uniqueness is broken in the reverse side;

$c.5)$ Condition $(*.8)$ shows, that one and the same fixed equality $\tilde{\alpha}\tilde{\beta}_0 = \tilde{\beta}\tilde{\alpha}_0$ is true for the whole (indefinite) set $\zeta\mathbb{T}_{\tilde{\Sigma}_0}\left(\mathbb{K}^+_{\lambda_0}\right)$ of subsets $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s)$, where $s \in \mathbb{K}^+_{\lambda_0} \equiv \{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1, \operatorname{Im} s = \lambda_0\}$, located, generally speaking, on different straight lines $\mathbb{T}_{\tilde{\Sigma}_0}(s), s \in \mathbb{K}^+_{\lambda_0}$ (i.e. from the fact that $s', s'' \in \mathbb{K}^+_{\lambda_0}$ and $s' \neq s''$ it follows, that $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s') \neq \zeta\mathbb{T}_{\tilde{\Sigma}_0}(s'')$ even when combining the corresponding straight lines $\mathbb{T}_{\tilde{\Sigma}_0}(s')$ and $\mathbb{T}_{\tilde{\Sigma}_0}(s'')$).

(d) From the condition $(*.8)$, when any $\lambda_0 = \operatorname{Im} s_0 \geq 0$, find that

$$\tilde{\alpha} = \Delta_{q'}\tilde{\alpha}_0, \tag{$*.10a$}$$

$$\tilde{\beta} = \Delta_{q'}\tilde{\beta}_0, \tag{$*.10b$}$$

where initially $\Delta_{q'}$ is a complex value for all $q' \in I' \subset I$, depending upon the method of summing up the series (4.3) (or the same series $(*.1)$).

Substituting $(*.10a,b)$ into $(*.7)$ (and returning for convenience to previous designations $q' \to q$ and $I' \to I$) find the representation for the arbitrary series $\tilde{\Sigma}_q(s), q \in I$, or otherwise, for the arbitrary element of the set $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s)$, where $s \in \mathbb{K}^+_{\lambda_0}$, in the form of:

$$\tilde{\Sigma}_q(s) = \Delta_q\left(\tilde{\alpha}_0 + i\tilde{\beta}_0\right)\zeta(s). \tag{$*.11$}$$

From here we find for initial series

$$\Sigma_q(s) = \left(1 - 2^{1-s}\Delta_q\right)\zeta(s), \qquad q \in I, \qquad s \in \mathbb{K}^+. \tag{$*.12$}$$

(e) Reality of Δ_q for all $q \in I$ is easily determined for three points on the straight line $\mathbb{T}_{\tilde{\Sigma}_0}(s)$, $s \in \mathbb{K}^+$. For this it is sufficient to take $z_1 = 0, z_2 = \tilde{z}_0 = 2^{1-s}\zeta(s)$ and $z_3 = z = \tilde{\Sigma}_q(s) = 2^{1-s}\Delta_q\zeta(s)$ (the last is possible by virtue of $(*.11)$). Since according to condition for finding three given points on the line, the value $\frac{z_3-z_1}{z_2-z_1} \in \mathbb{R}$ [12,9], it is obvious that:

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_3}{z_2} = \frac{z}{\tilde{z}_0} = \Delta_q \in \mathbb{R}$$

for all $q \in I$.

(f) Now let's consider a rule for finding four different points z_1, z_2, z_3, z_4 $(z_i \in \mathbb{C}, i = 1,2,3,4)$ on the line $\mathbb{T}_{\tilde{\Sigma}_0}(s)$, $s \in \mathbb{K}^+_{\lambda_0}$. According for example to [12,9], for this purpose there must be true the condition, under which the value $\frac{z_2-z_1}{z_3-z_1} : \frac{z_2-z_4}{z_3-z_4}$ - is real.

Suppose that, $z_1 = 0, z_2 = \tilde{z}_0 = 2^{1-s}\zeta(s), z_3 = \tilde{z}_{q_1} = \tilde{\Sigma}_{q_1}(s) = 2^{1-s}\Delta_{q_1}\zeta(s), z_4 = \tilde{z}_{q_2} = \tilde{\Sigma}_{q_2}(s) = 2^{1-s}\Delta_{q_2}\zeta(s)$, where $q_1, q_2 \in I, s \in \mathbb{K}^+_{\lambda_0}$ and $\Delta_{q_1} \neq \Delta_{q_2} \neq 1, 0 \leq \lambda_0 < \infty$.

In this case the condition takes the form of:

$$\frac{z_2 - z_1}{z_3 - z_1} : \frac{z_2 - z_4}{z_3 - z_4} = \frac{\Delta_{q_1} - \Delta_{q_2}}{\Delta_{q_1}\left(1 - \Delta_{q_2}\right)} \in \mathbb{R}. \tag{$*.13$}$$

Supposing that $\operatorname{Re}\Delta_{q_1} = x_1, \operatorname{Re}\Delta_{q_2} = x_2$ and $\operatorname{Im}\Delta_{q_1} = y_1, \operatorname{Im}\Delta_{q_2} = y_2$ let's bring the condition $(*.13)$ after elementary transformations to the equation:

$$\left(x_1^2 + y_1^2 - x_1\right)y_2 = \left(x_2^2 + y_2^2 - x_2\right)y_1. \tag{$*.14$}$$

If in the last equation $y_1 = y_2 = y$, no matter what the value of y is (is equal or not equal to zero) the condition $(*.14)$ is reduced to the simple form:

$$x_1 + x_2 = 1. \tag{$*.15$}$$

(g) Imagine every real value Δ_q in the form of the sum of two complex values Δ_q^+ and Δ_q^-:

$$\Delta_q = \Delta_q^+ + \Delta_q^- \in \mathbb{R}, \qquad q \in I, \tag{$*.16$}$$

where $\Delta_q^{\pm} = \left|\Delta_q^{\pm}\right| e^{i\varphi_q}, \varphi_q \in \mathbb{R}, q \in I$. In this connection modules of value $\Delta_q^{+}, \Delta_q^{-}$, necessary for the real Δ_q formation must be, probably, equal between each other:

$$\left|\Delta_q^{+}\right| = \left|\Delta_q^{-}\right| = \rho_q/2, \qquad 0 < \rho_q < \infty, \qquad q \in I. \tag{$*.17$}$$

Therefore, from ($*$.16) and ($*$.17) it follows that:

$$\Delta_q = \rho_q \cos\varphi_q, \qquad 0 \le \varphi_q \le \pi, \qquad q \in I. \tag{$*.18$}$$

Consider the elements of the set $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$, located on the straight line $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$ when some fixed $s_0 \in \mathbb{K}_{\lambda_0}^{+}$. The equation ($*$.15), rewritten here in the form of

$$\rho_{q_1} \cos\varphi_{q_1} + \rho_{q_2} \cos\varphi_{q_2} = 1, \qquad q_1, q_2 \in I_{\lambda_0}, (q_1 \ne q_2), \tag{$*.19$}$$

allows to draw a deduction on the limitation of the set of all ρ_q:$\{\rho_q\}_{q \in I_{\lambda_0}}$ $\equiv \{\rho_q(\lambda_0)\}_{q \in I_{\lambda_0}}, I_{\lambda_0} \subset I$ - is an index subset in I. From the limitation of the set $\{\rho_q(\lambda_0)\}_{q \in I_{\lambda_0}}$ there follows the existence of two constant positive values $\rho_1(\lambda_0)$, $\rho_2(\lambda_0) > 0$, such that: $\rho_1(\lambda_0) \le \rho_q(\lambda_0) \le \rho_2(\lambda_0)$ for all $q \in I_{\lambda_0}$.

Let $\rho(\lambda_0) = max(\rho_1(\lambda_0), \rho_2(\lambda_0))$. Then the set of numbers $\left\{\frac{\Delta_q(\lambda_0)}{\rho(\lambda_0)}\right\}_{q \in I_{\lambda_0}}$ $\equiv \left\{\frac{\rho_q(\lambda_0)}{\rho(\lambda_0)} \cos\varphi_q(\lambda_0)\right\}_{q \in I_{\lambda_0}}$ may be mapped in a one-to-one manner into the segment $[-1, 1]$ (as always $\left|\frac{\Delta_q}{\rho}\right| = \left|\frac{\rho_q}{\rho}\cos\varphi_q\right| \le \left|\frac{\rho_q}{\rho}\right| \le 1$ for $\forall q \in I_{\lambda_0}$). In other words, for all $\frac{\Delta_q(\lambda_0)}{\rho(\lambda_0)}$ there will be found such number $\psi_q(\lambda_0) \in [0, \pi]$, that the equality $\frac{\Delta_q}{\rho} = \cos\psi_q$ will be true for any $q \in I_{\lambda_0}$.

(h) Now consider the set of all straight lines $\mathbb{T}_{\tilde{\Sigma}_0}(s_0), s_0 \in \mathbb{K}^{+}$:

$$\mathbb{T}_{\tilde{\Sigma}_0}(\mathbb{K}^{+}) = \bigcup_{s_0 \in \mathbb{K}^{+}} \mathbb{T}_{\tilde{\Sigma}_0}(s_0), \tag{$*.20$}$$

when the parameter s_0 runs through the whole region $\mathbb{K}^{+}$. Since every straight line $\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$, $s_0 \in \mathbb{K}^{+}$ contains the corresponding subset $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(s_0)$, then the set of all the subsets $\zeta\mathbb{T}_{\tilde{\Sigma}_0}$ may be represented (similarly to ($*$.20)) as:

$$\zeta\mathbb{T}_{\tilde{\Sigma}_0}(\mathbb{K}^{+}) = \bigcup_{0 \le \lambda_0 < \infty} \zeta\mathbb{T}_{\tilde{\Sigma}_0}(\mathbb{K}_{\lambda_0}^{+}). \tag{$*.21$}$$

In its turn, the set $\zeta\mathbb{T}_{\tilde{\Sigma}_0}(\mathbb{K}^{+})$ (by the ratio ($*$.11)) generates the common set of all Δ_q:

$$\{\Delta_q\}_{q \in I}^{G} = \bigcup_{0 \le \lambda_0 < \infty} \{\Delta_{q'}(\lambda_0)\}_{q' \in I_{\lambda_0}} \tag{$*.22$}$$

In this connection, for convenience, we shall think that in the set $\{\Delta_q\}_{q \in I}^{G}$ indexation of elements was performed so that for different q there correspond different Δ_q, i.e. from $q', q'' \in I$ and $q' \ne q'' \Rightarrow \Delta_{q'} \ne \Delta_{q''}$. In item ($g$) it is shown, that for any $\lambda_0 \in [0, +\infty)$ the equality $\left|\frac{\Delta_{q'}(\lambda_0)}{\rho(\lambda_0)}\right| \le 1$ is true, where $q' \in I_{\lambda_0}$. Choose from the whole set $\{\rho(\lambda_0)\}, \lambda_0 \in [0, +\infty)$ a maximum element:

$$\rho_0 = \max_{0 \le \lambda_0 < \infty} \{\rho(\lambda_0)\}. \tag{$*.23$}$$

Then, in complete similarity with reasonings in item (g), for every $\Delta_q \in \{\Delta_q\}_{q \in I}^{G}$ there will be found such number $f_q \in [0, \pi]$, that the equality

$$\Delta_q = \rho_0 \cos f_q \tag{$*.24$}$$

will be true for all $q \in I$. $\triangleright$

Lemma 5.2 *The equation $F_{2\omega}(\tilde{s}) = 0$ in the zone $\tilde{\Pi}_0$ is equivalent to the system of two equations:*

$$\chi_q\left(\frac{1}{2} - \sigma - i\lambda\right) \cdot \varphi\left(\frac{1}{2} + \sigma + i\lambda\right) + \\ + \chi_q\left(\frac{1}{2} + \sigma - i\lambda\right) \cdot \varphi\left(\frac{1}{2} - \sigma + i\lambda\right) = 0, \qquad (5.3)$$

$$\chi_q\left(\frac{1}{2} - \sigma - i\lambda\right) \cdot \varphi\left(\frac{1}{2} + \sigma + i\lambda\right) - \\ - \chi_q\left(\frac{1}{2} + \sigma - i\lambda\right) \cdot \varphi\left(\frac{1}{2} - \sigma + i\lambda\right) = 0, \qquad (5.4)$$

where the coefficients χ_q are equal by definition:

$$\chi_q(a) \equiv \frac{2^{2a-1} + 1 - 2^a \Delta_q}{2^a}, \qquad a \in \mathbb{C}, \qquad q \in I. \qquad (5.5)$$

The system (5.3),(5.4) *is given for the argument $s = \sigma + i\lambda$ from the zone Π_0^+.*

$\triangleleft$ We shall start from the statement of the equivalent system of equations (3.1) and (3.2) of lemma 3.1., given in the zone Π_0^+. Consider the first equation (3.1). Substituting the functional equation (4.8) of lemma 4.2. into the integral in the left part of (3.1) we shall get the sums of three integrals:

$$\int_0^\infty \omega(e^{2t})\, e^{t/2} \cosh \sigma t \cos \lambda t \, dt = \\ = \int_0^\infty \tilde{\omega}(e^{2t})\, e^{t/2} \cosh \sigma t \cos \lambda t \, dt \; + \\ + \frac{1}{2} \int_0^\infty e^{-t/2} \cosh \sigma t \cos \lambda t \, dt \; + \\ + \int_0^\infty \tilde{\omega}\left(\frac{1}{4e^{2t}}\right) e^{-t/2} \cosh \sigma t \cos \lambda t \, dt. \qquad (*.1)$$

Let's do calculation of each of three integrals $(*.1)$ separately.

(a) For finding the first integral $(*.1)$, use the scheme of calculation of the expression $(*.1)$ in the proof of lemma 3.3., and also the definition (4.2) and the property (4.4). We have:

$$\int_0^\infty \tilde{\omega}(e^{2t})\, e^{t/2} \cosh \sigma t \cos \lambda t \, dt = \\ = \frac{1}{4}\left\{\pi^{-\frac{1}{4} - \frac{\sigma}{2} - i\frac{\lambda}{2}}\, \Gamma\left(\frac{1}{4} + \frac{\sigma}{2} + i\frac{\lambda}{2}\right) \zeta\left(\frac{1}{2} + \sigma + i\lambda; -1\right) + \right. \\ \left. + \pi^{-\frac{1}{4} + \frac{\sigma}{2} - i\frac{\lambda}{2}}\, \Gamma\left(\frac{1}{4} - \frac{\sigma}{2} + i\frac{\lambda}{2}\right) \zeta\left(\frac{1}{2} - \sigma + i\lambda; -1\right)\right\} = \\ = \frac{1}{4}\left[\left(2^{\frac{1}{2} - \sigma - i\lambda} - 1\right) \varphi\left(\frac{1}{2} + \sigma + i\lambda\right) + \left(2^{\frac{1}{2} + \sigma - i\lambda} - 1\right) \varphi\left(\frac{1}{2} - \sigma + i\lambda\right)\right]. \qquad (*.2)$$

Show here the assumption of integration term by term, used by default during the integral (∗.2) calculation.

Let $\varphi(x)$ be a non-negative continuous function on the interval $[0,+\infty)$. Then $\tilde{\omega}(\varphi) = \tilde{\omega}(\varphi(x)) = \sum_{n=1}^{\infty} (-1)^n e^{-\pi n^2 \varphi(x)}, x \in [0,+\infty)$.

On the basis of convergence of series, it follows, that a series $\tilde{\omega}(\varphi)$ is absolutely converging for all $x \in [0,+\infty)$. In fact, the limit of ratio of terms of the sum $\tilde{\omega}(\varphi)$ is equal to:

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = e^{-\pi\varphi(x)} \lim_{n\to\infty} e^{-2\pi n \varphi(x)} = 0,$$

since $\varphi(x) \geq 0$ for all $x \in [0,+\infty)$.

Therefore, the integral

$$I_{\tilde{\omega}}^{\alpha}(\varphi, f) \equiv \int_0^{\infty} \tilde{\omega}(\varphi(x)) f_\alpha(x)\, dx = \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} e^{-\pi n^2 \varphi(x)} f_\alpha(x)\, dx,$$

for each continuous, parametrical from $\alpha \in \mathbb{C}$ ($\operatorname{Re}\alpha \in (0,1)$), function $f_\alpha(x)$, given on the interval $x \in [0,+\infty)$.

(b) The second integral in (∗.1) has an elementary character, that's why we write the result for it at once:

$$\frac{1}{2}\int_0^{\infty} e^{-t/2} \cosh\sigma t \cos\lambda t\, dt = -\frac{B_1}{4\delta} = \frac{\lambda^2 + \frac{1}{4} - \sigma^2}{4\delta}. \tag{∗.3}$$

From (∗.3) it is seen, that the given integral fully coincides with the right side of the equation (3.1).

(c) Now find the most complicated integral in (∗.1). Using term by term integration on the same basis, as in item (a), we get:

$$\int_0^{\infty} \tilde{\omega}\left(\frac{1}{4e^{2t}}\right) e^{-t/2} \cosh\sigma t \cos\lambda t\, dt =$$

$$= \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} e^{-\frac{1}{4}\pi n^2 e^{-2t}}\, e^{-t/2} \cosh\sigma t \cos\lambda t\, dt. \tag{∗.4}$$

Let's consider an auxiliary integral:

$$j_n^-(\tilde{s}) \equiv j_n^-(\tilde{s}; \varepsilon, \eta) = \int_0^{\infty} e^{-\frac{1}{4}\pi n^2 e^{-2t}}\, e^{-t/2}\, e^{[(-1)^\varepsilon \sigma + i(-1)^\eta \lambda]t}\, dt, \tag{∗.5}$$

where, as in lemma 3.2., $\tilde{s} = (-1)^\varepsilon \sigma + i(-1)^\eta \lambda$ and $\tilde{s} \in \tilde{\Pi}_0,\ \varepsilon, \eta = 1, 2$.

Performing in (∗.5) two successive replacements of variables $y = e^t$ and $u = \frac{1}{y}$, and also of the parameter $n = 2m$ (where m is rational), reduce the integral $j_n^-(\tilde{s})$ to the form of:

$$j_{2m}^-(\tilde{s}) = \int_0^{1} e^{-\pi m^2 u^2}\, u^{-\frac{1}{2}-\tilde{s}}\, du. \tag{∗.6}$$

From the comparison of (∗.6) and the integral $j_n^1(\tilde{s})$, defined by the expression (∗.4) of lemma 3.2. it is seen, that $j_{2m}^-(\tilde{s}) = j_m^1(-\tilde{s})$ (here rationality of m has no sufficient meaning). Returning again to the parameter $n = 2m$, find that (see (∗.27) of lemma 3.2.):

$$j_n^1(\tilde{s}) = 2^{-\left(\frac{1}{2}+\tilde{s}\right)} (-1)^\eta\, \pi^{-\frac{1}{4}+\frac{\tilde{s}}{2}} \cdot \Gamma\left(\tfrac{1}{4} - \tfrac{\tilde{s}}{2}\right) \cdot n^{-\left(\frac{1}{2}-\tilde{s}\right)}. \tag{∗.7}$$

(d) Let's give the following reasonings.

Let the series of the form

$$\Sigma_0(\alpha) \equiv \sum_{n=1}^{\infty} (-1)^{n-1} n^{-\alpha},$$

be given to us, where $\alpha \in \mathbb{C}$ belongs to the critical region:

$$\mathbb{K} \equiv \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, -\infty < \operatorname{Im} z < \infty\}.$$

According to [2] a series $\Sigma_0(\alpha)$ is conditionally converging for all $\alpha \in \mathbb{K}$. Hence, the sum of series "Σ" depends on the transposition p of the series $\Sigma_0(\alpha)$. Exactly, let p be an operator of terms transformation of the initial series $\Sigma_0(\alpha)$, such as:

$$p\Sigma_0(\alpha) \equiv \sum_{p(n)} (-1)^{p(n)-1} [p(n)]^{-\alpha}, \qquad \alpha \in \mathbb{K},$$

where there is a symbolical designation at the left, and a concrete representation in the form of a series at the right, p being an arbitrary transposition from the set of all transpositions $\mathbb{P}$, defined on the set of natural numbers $\{1, 2, \ldots\}$.

As it is shown above in item (c), the integral $(*.4)$ is reduced to the sums of the form:

$$\sum_{n=1}^{\infty} (-1)^n j_n^-(\tilde{s}) = \sum_{m\in\mathbb{Q}^*} (-1)^{2m} j_m^1(-\tilde{s}) = \sum_{m\in\mathbb{Q}^*} (-1)^{2m} \sum_{k=0}^{\infty} \frac{\left(-\pi m^2\right)^k}{k!} \frac{1}{2k + \left(\frac{1}{2} - \tilde{s}\right)} = \frac{1}{2}\pi^{-\frac{1}{4}+\frac{\tilde{s}}{2}} \times$$

$$\times \Gamma\left(\tfrac{1}{4} - \tfrac{\tilde{s}}{2}\right) \textstyle\sum_{m\in\mathbb{Q}^*} (-1)^{2m+\eta} m^{-\left(\frac{1}{2}-\tilde{s}\right)},$$

where $\mathbb{Q}^* \subset \mathbb{Q}^+$ is an isolated subset in the set of positive rational numbers, equal to $\mathbb{Q}^* = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots\right\}$. The latter expression is decomposed into two independent cases when $\eta = 1$ and $\eta = 2$ respectively. It is easy to see, that in the second case, when $\eta = 2$ "previous" equality (4.4) is formally preserved. That is, for all

$$\alpha = \left(\frac{1}{2} - \tilde{s}\right) = \left[\frac{1}{2} + (-1)^{\varepsilon+1} \sigma\right] - i\lambda \in \mathbb{K}^-,$$

where $\mathbb{K}^- \equiv \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, -\infty < \operatorname{Im} z \leq 0\}$ - is a lower critical semi-zone, the series

$$\sum_{n=1}^{\infty} (-1)^n j_n^-(\tilde{s})$$

is reduced to the sum of the form

$$\sum_{m\in\mathbb{Q}^*} (-1)^{2m} m^{-\alpha},$$

which is uniquely expressed through the corresponding equality (4.4).

When $\eta = 1$, i.e. in the upper critical semi-zone $\mathbb{K}^+$, the situation with the series $\sum_{n=1}^{\infty} (-1)^n j_n^-(\tilde{s})$, generally speaking, sufficiently changes. It is connected with ambiguity of representation of the series

$$\sum_{m\in\mathbb{Q}^*} (-1)^{2m+1} m^{-\alpha}, \qquad \alpha \in \mathbb{K}^+,$$

by means of formal sums

$$\sum_{m\in\mathbb{Q}^*} (-1)^{2m} m^{-\alpha}, \qquad \alpha \in \mathbb{K}^+.$$

Note, that the series $\sum_{m\in\mathbb{Q}^*}(-1)^{2m+1}m^{-\alpha}, \alpha\in\mathbb{K}^+$, may be interpreted as a result of analytical continuation of the series

$$\sum_{m\in\mathbb{Q}^*}(-1)^{2m+1}m^{-\alpha}, \qquad \operatorname{Re}\alpha>1,$$

from the right semi-plane (where it absolutely converges) into upper critical semi-zone $\mathbb{K}^+$. In this connectoin a whole equality (4.4) analytically continues into the critical semi-zone $\mathbb{K}^+$. Since the right side of the equality (4.4) regularly continues in $\mathbb{K}^+$, then any representation of the series $\sum_{m\in\mathbb{Q}^*}(-1)^{2m+1}m^{-\alpha}$ by formal sums $\sum_{m\in\mathbb{Q}^*}(-1)^{2m}m^{-\alpha}$ must always be proportional to zeta-function in every point $\alpha\in\mathbb{K}^+$.

Let's perform formal decomposition:

$$\sum_{m\in\mathbb{Q}^*}(-1)^{2m+1}m^{-\alpha}=\sum_{m\in\mathbb{Q}^*}(-1)^{2\left(m+\frac{1}{2}\right)}\left[\left(m+\frac{1}{2}\right)-\frac{1}{2}\right]^{-\alpha}=$$

$$=(-2)^{\alpha}\sum_{k=0}^{\infty}\sum_{q=0}^{k}(-1)^k 2^q C_{-\alpha}^{k}C_k^q\sum_{m\in\mathbb{Q}^*}(-1)^{2(m+1/2)}m^q=$$

$$=(-2)^{\alpha}\sum_{k=0}^{\infty}(-1)^k C_{-\alpha}^{k}\sum_{p_k(q)\in[0,k]}2^{p_k(q)}C_k^{p_k(q)}\times$$

$$\times\sum_{m\in\mathbb{Q}^*}(-1)^{2(m+1/2)}m^{p_k(q)}\to\ p\sum_{m\in\mathbb{Q}^*}(-1)^{2m+1}m^{-\alpha}=\sum_{p(m)\in p(\mathbb{Q}^*)}(-1)^{2p(m)+1}\left[p\left(m\right)\right]^{-\alpha},$$

where $p\in P_\zeta$ is an arbitrary transposition from the set $P_\zeta\subset\mathbb{P}$, that is, the set of all transpositions, preserving proportionality of the series of zeta-function; C_β^α are binomial coefficients, p_k is some finite transposition over the set $[0,1,\ldots,k], k=0,1,2,\ldots$. As it is seen from the given decompositions, the transposition p appears because of the second sum in the second equality, due to finiteness of which, when every $k\in\mathbb{N}$, an arbitrary transposition p_k of inner terms is possible.

(e) Unite in this item the results of reasonings in item (d) and calculations in item (c) for finding the integral $(*.4)$.

Substituting $(*.7)$ in $(*.4)$, using lemma 5.1 (see the expression (5.2)) and performing transformations identical to those used in the conclusion $(*.2)$ (see item (a)), we get finally for $(*.4)$:

$$\int_0^\infty\tilde{\omega}\left(\frac{1}{4e^{2t}}\right)e^{-t/2}\cosh\sigma t\cos\lambda t\,dt=$$

$$=\frac{1}{4}\left\{\left[1-2^{-\left(\frac{1}{2}+\sigma+i\lambda\right)}\right]\varphi\left(\frac{1}{2}-\sigma-i\lambda\right)-\right.$$

$$-\left[\Delta_q-2^{-\left(\frac{1}{2}+\sigma-i\lambda\right)}\right]\varphi\left(\frac{1}{2}-\sigma+i\lambda\right)+$$

$$+\left[1-2^{-\left(\frac{1}{2}-\sigma+i\lambda\right)}\right]\varphi\left(\frac{1}{2}+\sigma-i\lambda\right)-$$

$$\left.-\left[\Delta_q-2^{-\left(\frac{1}{2}-\sigma-i\lambda\right)}\right]\varphi\left(\frac{1}{2}+\sigma+i\lambda\right)\right\}, \qquad (*.8)$$

where $\Delta_q=\rho_0\cos f_q, f_q\in\mathbb{R}, q\in I$. Remind, that $\varphi(z)=\pi^{-z/2}\Gamma(z/2)\zeta(z)$ (see introduction). So, the expression $(*.8)$ leads to the whole set of values of the integral $(*.4)$, emerging when Δ_q are different , $q\in I$. In this connection the capacity of the set of values $(*.8)$ evidently coincides, with the capacity of index set I, i. e. it is equal to $card(I)$.

(f) Substituting ($*$.2), ($*$.3) and ($*$.8) into the integral ($*$.1) and then the last one into the left side of the equation (3.1), we shall get the equation (5.3) after elementary transformations. In this connection coefficients $\chi_q(a)$ appear to be equal:

$$\chi_q(a) = (2^a - 1) - 2^{-a}\,(2^a \Delta_q - 1) + 2^{-(1-a)}\,\left(2^{1-a} - 1\right), \qquad (*.9)$$

or

$$\chi_q(a) \equiv \frac{2^{2a-1} + 1 - 2^a \Delta_q}{2^a}, \qquad q \in I.$$

(g) Thinking over the equation (3.2) of lemma 3.1, fully similar to reasonings given above in items (a) – (f), we shall get the second equation (5.4) in sought. That's why, in this item we shall write out, for the sake of completeness, final expressions for corresponding integrals.

$$\int_0^\infty \omega(e^{2t})\, e^{t/2} \sinh \sigma t \sin \lambda t\, dt = J_1 + J_2 + J_3, \qquad (*.10)$$

where

$$J_1 \equiv \int_0^\infty \tilde{\omega}(e^{2t})\, e^{t/2} \sinh \sigma t \sin \lambda t\, dt =$$

$$= \frac{1}{4i}\left\{\left(2^{\frac{1}{2}-\sigma-i\lambda} - 1\right)\varphi\left(\tfrac{1}{2} + \sigma + i\lambda\right) - \left(2^{\frac{1}{2}+\sigma-i\lambda} - 1\right)\varphi\left(\tfrac{1}{2} - \sigma + i\lambda\right)\right\}; \qquad (*.11)$$

$$J_2 \equiv \frac{1}{2}\int_0^\infty e^{-t/2} \sinh \sigma t\, \sin \lambda t\, dt = \frac{\sigma\lambda}{2\delta}; \qquad (*.12)$$

$$J_3 \equiv \int_0^\infty \tilde{\omega}\left(\frac{1}{4e^{2t}}\right) e^{-t/2} \sinh \sigma t\, \sin \lambda t\, dt =$$

$$= \frac{1}{4i}\left\{\left[1 - 2^{-\left(\frac{1}{2}+\sigma+i\lambda\right)}\right]\varphi\left(\frac{1}{2} - \sigma - i\lambda\right) + \right.$$

$$+ \left[\Delta_q - 2^{-\left(\frac{1}{2}+\sigma-i\lambda\right)}\right]\varphi\left(\frac{1}{2} - \sigma + i\lambda\right) -$$

$$- \left[1 - 2^{-\left(\frac{1}{2}-\sigma+i\lambda\right)}\right]\varphi\left(\frac{1}{2} + \sigma - i\lambda\right) -$$

$$\left. - \left[\Delta_q - 2^{-\left(\frac{1}{2}-\sigma-i\lambda\right)}\right]\varphi\left(\frac{1}{2} + \sigma + i\lambda\right)\right\}, \qquad q \in I. \qquad (*.13)$$

$\triangleright$

Lemma 5.3 *Let* $s_0^\pm = \frac{1}{2} \pm \sigma_0 + i\lambda_0$ *be two arbitrary zeros of zeta-function given symmetrically with respect to central line* $\operatorname{Re} s = \frac{1}{2}$ *in the semi-zone* $\mathbb{K}^+$. *Then modules* $\left|\frac{\varphi(s_0^+)}{\varphi(s_0^-)}\right|$ *of the ratio of the function* $\varphi(s)$ *in points* s_0^+ *and* s_0^- *will be finite, not equal to zero value, for all* $0 \le \sigma_0 < \frac{1}{2}, 0 < \lambda_0 < \infty$.

$\triangleleft$ For the proof of lemma let's use the representation (1.6) for the whole function $\xi(s)$, which we shall rewrite here in the form of [1]:

$$\xi(s) = \xi(0)\, e^{b_0 s} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \qquad (*.1)$$

where $\xi(0) = -\zeta(0) = \frac{1}{2}$, $b_0 = b - \frac{1}{2}\ln\pi = \left(\ln 2\pi - 1 - \frac{1}{2}\gamma\right) - \frac{1}{2}\ln\pi$, γ is the Euler constant.

Taking into account the connection between functions $\varphi(s)$ and $\xi(s)$ for all $0 < \sigma_0 < \frac{1}{2}, 0 < \lambda_0 < \infty$, we find from $(*.1)$:

$$\frac{\varphi\left(s_0^+\right)}{\varphi\left(s_0^-\right)} = \lim_{\substack{s_1\to s_0^+\\ s_2\to s_0^-}} \frac{\varphi(s_1)}{\varphi(s_2)} = e^{2\sigma_0 b_0}\frac{s_0^-\left(s_0^- - 1\right)}{s_0^+\left(s_0^+ - 1\right)}\frac{\prod_{\rho\neq s_0^+}\left(1-\frac{s_0^+}{\rho}\right)e^{s_0^+/\rho}}{\prod_{\rho\neq s_0^-}\left(1-\frac{s_0^-}{\rho}\right)e^{s_0^-/\rho}}\times$$

$$\times \lim_{\substack{s_1\to s_0^+\\ s_2\to s_0^-}} \frac{\left(1 - s_1/s_0^+\right)^{m_1} e^{m s_1/s_0^+}}{\left(1 - s_2/s_0^-\right)^{m_2} e^{m s_2/s_0^-}} \qquad (*.2)$$

Here $\prod_{\rho\neq a}$ means that ρ runs through all the complex zeros of ζ - function except $\rho = a$; m_1, m_2 are multiplicities of zeros s_0^+ and s_0^- respectively. Since $\zeta\left(\frac{1}{2}+\sigma_0+i\lambda_0\right)$ and $\zeta\left(\frac{1}{2}-\sigma_0-i\lambda_0\right)$ are connected by functional equation (1.5) and a pair of values $\zeta\left(\frac{1}{2}-\sigma_0+i\lambda_0\right)$ and $\zeta\left(\frac{1}{2}-\sigma_0-i\lambda_0\right)$ is complexly conjugate, then multiplicities of zeros s_0^+ and s_0^- coincide: $m_1 = m_2 = m$, m always being finite natural number.

Note that the products of $\prod_{\rho\neq s_0^\pm}\left(1 - s_0^\pm/\rho\right)e^{s_0^\pm/\rho}$ consist of finite, not equal to zero factors, and this leads, in combination with property of ζ - function regularity everywhere in the open critical zone, to the conclusion of finiteness of products themselves.

And, finally, writing variables $s_1, s_2 \in \mathbb{K}^+$ in the form of $s_i = \frac{1}{2}+\sigma_i+i\lambda_i$, where $-\frac{1}{2} < \sigma_i < \frac{1}{2}, 0 < \lambda_i < \infty, i = 1,2$ and calculating the limit in the right side $(*.2)$, we find once and for all

$$\frac{\varphi\left(s_0^+\right)}{\varphi\left(s_0^-\right)} = (-1)^m e^{2\sigma_0 b_0}\left(\frac{s_0^-}{s_0^+}\right)^{m+1}\frac{\left(s_0^- - 1\right)}{\left(s_0^+ - 1\right)}\frac{\prod_{\rho\neq s_0^+}\left(1-\frac{s_0^+}{\rho}\right)e^{s_0^+/\rho}}{\prod_{\rho\neq s_0^-}\left(1-\frac{s_0^-}{\rho}\right)e^{s_0^-/\rho}} \qquad (*.3)$$

when $\sigma_0 \in (0, 1/2)$, and

$$\frac{\varphi\left(s_0^+\right)}{\varphi\left(s_0^-\right)} = 1 \qquad (*.4)$$

when $\sigma_0 = 0$.

The statement of lemma follows directly from $(*.3)$ and $(*.4)$. $\triangleright$

Theorem 5.1 *All complex zeros of the Riemann ζ - function in the critical region $0 < \mathrm{Re}\, z < 1$ are located on the straight line $\mathrm{Re}\, z = \dfrac{1}{2}$ and described by general expression:*

$$z_0^\pm(q) = \frac{1}{2} \pm i\left(\log_2 e\right) \mathit{Arccos}^+\left(\frac{\Delta_q}{\sqrt{2}}\right), \qquad (5.6)$$

in which $q \in I$; $\mathit{Arccos}^+ x \equiv 2k\pi \pm$ arccos x, where arccos x - is the main value of arc cosine, k is any natural number.

$\triangleleft$ First of all find complex roots of the equation $\chi_q(a) = 0$ for arbitrary $q \in I$, where $\chi_q(a)$ is defined by the expression (5.5) of lemma 5.2.

If we denote via $x = 2^a$, then the equation $\chi_q(a) = 0$ with respect to the unknown x will have the form of:

$$x^2 - 2\Delta_q x + 2 = 0, \qquad q \in I, \qquad (*.1)$$

where, according to $(*.24)$ of lemma 5.1 $\Delta_q = \rho_0 \cos f_q$, $\rho_0 > 0$ is a constant independent of q, $f_q \in [0, \pi]$ for all $q \in I$.

Roots of the equation $(*.1)$ are equal to:

$$x_q^\pm = \Delta_q\left[1 \pm \sqrt{1-\beta_q^2}\right], \qquad q \in I, \qquad (*.2)$$

where $\beta_q \equiv \frac{\sqrt{2}}{\Delta_q}$. As we are only interested in complex roots of (∗.1), then for all $q \in I$ it should be supposed $\beta_q^2 > 1$, i. e. $|\beta_q| > 1$ and $\sqrt{\beta_q^2 - 1} > 0$. Thus, the roots of (∗.2) should be written in the form of:

$$x_q^{\pm} = \Delta_q \left[1 \pm i\sqrt{\beta_q^2 - 1}\right], \qquad q \in I. \tag{$*.3$}$$

Transforming the expression in brackets of (∗.3) to the form of

$$1 \pm i\sqrt{\beta_q^2 - 1} = \beta_q \exp\left(\pm i Arctan^{+}\sqrt{\beta_q^2 - 1}\right)$$

and taking into account that $\beta_q \Delta_q \equiv \sqrt{2}$ for $\forall q \in I$, we get:

$$x_q^{\pm} = \sqrt{2}\exp\left(\pm i Arctan^{+}\sqrt{\beta_q^2 - 1}\right), \qquad q \in I. \tag{$*.4$}$$

The designation $Arctan^{+}x \equiv\ k\pi + arctan\ x$, where $x > 0, k \in \mathbb{N}, arctan\ x$ - is the main value of the arctangent (concluded for $x \in (0, \infty)$ in the limits $(0, \pi/2)$) is introduced here for convenience.

Returning to the variable a from (∗.4) we find:

$$a_0^{\pm}(q) = \log_2 x_q^{\pm} = \frac{1}{2} \pm i\frac{Arctan^{+}\sqrt{\beta_q^2 - 1}}{\ln 2}, \qquad q \in I. \tag{$*.5$}$$

Substituting into (∗.5) concrete expressions of arguments of coefficients χ_q from the equations (5.3), (5.4) of lemma 5.2, find an equation with respect to $\sigma_0(q)$ and $\lambda_0(q)$:

$$\frac{1}{2} \pm \sigma_0(q) - i\lambda_0(q) = a_0^{\pm}(q), \qquad q \in I. \tag{$*.6$}$$

From (∗.6) it follows that $\sigma_0(q) \equiv 0$ for all $q \in I$, and

$$\lambda_0^{\pm}(q) = \pm\frac{Arctan^{+}\sqrt{\beta_q^2 - 1}}{\ln 2}, \qquad q \in I. \tag{$*.7$}$$

So, the roots of equations $\chi_q\left(\frac{1}{2} \pm \sigma - i\lambda\right) = 0$ coincide and are equal to $\frac{1}{2} + i\lambda_0^{\pm}(q)$, when every given $q \in I$.

Now let's turn to the system of equations (5.3), (5.4) of lemma 5.2. Rewrite the given system in a short form as

$$A_q\varphi_1 + B_q\varphi_2 = 0, \tag{$*.8a$}$$

$$A_q\varphi_1 - B_q\varphi_2 = 0, \tag{$*.8b$}$$

where $A_q, B_q = \chi_q\left(\frac{1}{2} \mp \sigma - i\lambda\right)$ and $\varphi_1, \varphi_2 = \varphi\left(\frac{1}{2} \pm \sigma - i\lambda\right)$ respectively. In this connection the system of (∗.8a), (∗.8b) is defined for every given $q \in I$.

Let $A_q = 0$, for some $q \in I$; then $B_q = 0$ too, and therefore the system (∗.8a), (∗.8b) becomes an identity. According to lemma (5.2) the truth of equations (∗.8a), (∗.8b), i.e. equations (5.3), (5.4) means the truth of the equation $F_{2\omega}(\tilde{s}) = 0$ in the zone $\tilde{\Pi}_0$. Hence, by virtue of lemma 2.1. and the equation $\zeta\left(\frac{1}{2} + \tilde{s}\right) = 0$ in the critical zone $0 < \mathrm{Re}\, z < 1,\ z = \frac{1}{2} + \tilde{s}$. Thus, from the fact that $A_q = B_q = 0$, $q \in I$ and from expressions (∗.5), (∗.6) it follows that $\varphi_1 = \varphi_2 = 0$.

Inversely, let $\varphi_1 = 0$ and let $A_q, B_q \neq 0$ when no $q \in I$. Show, that these conditions are incompatible. According to the symmetry [1,3] of zeros of ζ - function with respect to the straight line $\mathrm{Re}\, z = \frac{1}{2}$, from $\varphi_1 = 0$ it follows that $\varphi_2 = 0$ too. According to lemma 2.1. and lemma 5.2. the equality $\varphi_1 = \varphi_2 = 0$ is possible only when corresponding equations (∗.8a),(∗.8b) are true. Hence, there exists such $q \in I$, that when $\varphi_1 = \varphi_2 = 0$ the system (∗.8a),(∗.8b) becomes an identity.

From the given system it formally follows: $A_q = -B_q \frac{\varphi_2}{\varphi_1} = B_q \frac{\varphi_2}{\varphi_1}$. Considering $\frac{\varphi_2}{\varphi_1}$ as a limit ratio of two functions equally tending to zero under some $\sigma_0\left(q\right)$, $\lambda_0\left(q\right)$ and using the result of lemma 5.3. we find, that $B_q = -B_q = 0$ from which $A_q = 0$ too when the given $q \in I$.

All the reasonings carried out here can be stated as follows.

For the functions φ_1, φ_2 to become zero it is necessary and sufficient the condition $A_q = B_q = 0$ when corresponding $q \in I$ is true. In other words, the Riemann zeta - function $\zeta(z) = \zeta\left(\frac{1}{2} + \tilde{s}\right)$, $\tilde{s} \in \tilde{\Pi}_0$, in the critical zone $0 < \operatorname{Re} z < 1$ has no other zeroes z_0 except the roots of the equations $\chi_q\left(\frac{1}{2} \pm \sigma - i\lambda\right) = 0, q \in I$. Since according to $(*.6)$ the roots of the equation $\chi_q\left(\frac{1}{2} \pm \sigma - i\lambda\right) = 0$ for all $q \in I$ in the zone $0 < \operatorname{Re} z < 1$ are located on the straight line $\operatorname{Re} z = \dfrac{1}{2}$, then according to lemma 2.1. all the roots of the equation $F_{2\omega}(z) = 0$ in the zone $\tilde{\Pi}_0$ are located on the straight line $\operatorname{Re} z = 0$, i.e. are clearly imaginary.

Hence, it is shown, that the condition 2.1 for the function $F_{2\omega}(z)$ in $\tilde{\Pi}_0$ is true and thus, the Riemann hypothesis on zeros of zeta - function in the critical zone may be considered to be fully proved.

At last, let's find the final expression (within the frameworks of given constructions) for zeros $\left(z_0^{\pm}(q), q \in I\right)$ of ζ - function, located on the straight line $\operatorname{Re} z = \frac{1}{2}$. By virtue of the symmetry property of zeros of ζ - function on the straight line $\operatorname{Re} z = \frac{1}{2}$ with respect to the line $0x$ (see $(*.5)$), it is sufficient to consider zeros only in the upper part of the critical zone, i.e. for $\operatorname{Im} z > 0$.

It is easy to note, that solutions of the equation $(*.1)$ with respect to constant ρ_0, are generally speaking, decomposed in form into two cases. In the first of them $\rho_0 = \sqrt{2}$ and complex solution $(*.3)$ are reduced to the simple form:

$$x_q^{\pm} = \sqrt{2} e^{\pm i f_q}, \qquad q \in I, \tag{$*.9$}$$

the expression:

$$f_q = Arccos^{+}\left(\frac{\Delta_q}{\sqrt{2}}\right) \equiv 2k\pi \pm \; arccos\left(\frac{\Delta_q}{\sqrt{2}}\right), \tag{$*.10$}$$

being conveniently used for f_q, where $q \in I, k \in \mathbb{N}$ and arccos x - is the main value of the arc cosine , enclosed in boundaries $[0, \pi]$.

In the second case $\rho_0 \neq \sqrt{2}$, i. e. ρ_0 can acquire any value from the region $\left(0, \sqrt{2}\right) \bigcup \left(\sqrt{2}, +\infty\right)$, but for all these values distribution of zeros will be given in the form of the general expression $(*.5)$, that is, it is always expressed via $Arctan^{+}$. It is obvious, that only one of these cases truely describes zeros of ζ - function in the critical zone.

Suppose that $\rho_0 \neq \sqrt{2}$, from which it follows that zeros of ζ - function are given by $(*.5)$. As the values of the expression $Arctan^{+}\sqrt{\beta_q^2 - 1}$ for all $q \in I$ belong to the region:

$$\bigcup_{k=0}^{\infty} \left(k\pi, \frac{(2k+1)}{2}\pi\right), \tag{$*.11$}$$

the region

$$\bigcup_{k=0}^{\infty} \left(\frac{(2k+1)}{2}\pi, (k+1)\,\pi\right). \tag{$*.12$}$$

leaves "uncovered" by the distribution of $(*.5)$ on the straight line $\operatorname{Re} z = \frac{1}{2}$, $\operatorname{Im} z > 0$. Therefore, it is sufficient to give an example, at least, of one zero of ζ - function, exactly located in the region $(*.12)$ in units $\ln 2$, in order the distribution $(*.5)$ to be false. Such example already supplies the second zero of ζ - function $\alpha_2 \cong 21,0220\ldots$ [1]. Actually, $\alpha_2 \ln 2 \cong 14,5703 \in \left(\frac{9}{2}\pi, 5\pi\right)$.

Well, it is proved, that $\rho_0 = \sqrt{2}$ and all complex zeros of ζ - function $z_0^{\pm}(q)$, $q \in I$, in the region $\operatorname{Re} z \in (0,1)$ are given by the expression (5.6). $\triangleright$

Note that theorem 5.1. doesn't fully solve the question on vertical distribution of zeros of ζ - function in the critical zone, as the order of zeros sequence on the straight line $\operatorname{Re} z = \frac{1}{2}$

is not clear in its frameworks. Formula (5.6) only gives the general representation about this distribution, connecting zeros of ζ - function on the straight line $\operatorname{Re} z = \frac{1}{2}$ with the set of sums $\{\Sigma_0(q)\}_{q\in I}$ - conditionally converging when $\operatorname{Re} z \in (0,1)$ of the Dirichlet series.

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