INITIATING SANTILLI'S ISO-MATHEMATICS TO TRIPLEX NUMBERS, FRACTALS, AND INOPIN'S HOLOGRAPHIC RING: PRELIMINARY ASSESSMENT AND NEW LEMMAS

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Abstract

In a preliminary assessment, we begin to apply Santilli's iso-mathematics to triplex numbers, Euclidean triplex space, triplex fractals, and Inopin's 2-sphere holographic ring (HR) topology. In doing so, we successfully identify and define iso-triplex numbers for iso-fractal geometry in a Euclidean iso-triplex space that is iso-metrically equipped with an iso-2-sphere HR topology. As a result, we state a series of lemmas that aim to characterize these emerging iso-mathematical structures. These initial outcomes indicate that it may be feasible to engage this encoding framework to systematically attack a broad range of problems in the disciplines of science and mathematics, but a thorough, rigorous, and collaborative investigation should be in order to challenge, refine, upgrade, and implement these ideas.

Keywords: Geometry and topology; Iso-number; Triplex number; Iso-triplex number; Euclidean triplex space; Euclidean iso-triplex space; Inopin holographic ring; Iso-sphere holographic ring; Fractal; Iso-fractal; Mandelbrot set; Chaos theory.

1 Introduction

A number is a mathematical object that is used to count, label, and measure. Number systems are fundamental to all quantitative sciences because they are used to encode the state space and transition space of experimental and observable features in nature. Such systems are tools that let scientists explore ideas and quantify experimental results. Historically, advances in science, engineering, and technology have pushed the mathematical definition of a number to include additional structures such as 0, negative, integer, rational, irrational, real, complex, and quaternion numbers in order to satisfy the imposed representational demands of these disciplines.

A number is an element, so a set of numbers is a set of elements, where the number set is equipped with addition and multiplication operators to establish a number field that complies with five number field axioms [1]. Let

$$X = \{x_1, x_2, x_3, ...\} \tag{1}$$

be a number field, where operators can be applied to numbers for addition $(+)$ and multiplication $(\times)$ to produce a sum $(x_1 + x_2)$ and product $(x_1 \times x_2)$, respectively, because X satisfies the five number field axioms [1, 2, 3]

1. X permits an element 1, namely the *multiplicative unit*, such that $1 \times x_A = x_A \times 1$, $\forall x_A \in X$;

2. X permits an element 0, namely the *additive unit*, such that $0 + x_A = x_A + 0$, $\forall x_A \in X$;

3. X is *closed* under addition and multiplication, which indicates that the respective sums $(x_A + x_B)$ and products $(x_A \times x_B)$ between elements $x_A, x_B \in X$ produce all possible elements of X;

4. X's addition and multiplication are *associative*, such that $(x_A + x_B) + x_C = x_A + (x_B + x_C)$ and $(x_A \times x_B) \times x_C = x_A \times (x_B \times x_C)$, respectively; and

5. the combination of X's addition and multiplication is *distributive*, such that $(x_A + x_B) \times x_C = x_A \times x_C + x_B \times x_C$ and $x_A \times (x_B + x_C) = x_A \times x_B + x_A \times x_C$.

A dominant problem of pure mathematics, in the context of number theory, is to establish a *universal number classification*, such as the identification of *all* sets that exist with numeric field axioms. To attack this *gigantic* classification problem, extensive and rigorous studies have been conducted over the course of history [1, 2, 4, 5, 6, 7]. A major result of these studies is the creation of real numbers [8, 9, 10], complex numbers [11, 12], and quaternion numbers [13, 14] with all possible numeric fields [2]. Moreover, it is known that these encoding frameworks bare *enormous* scientific application to quantifiable and computational implementations of disciplines such as physics, chemistry, biology, and more.

Fractal and chaotic patterns are *abundant* in the physical, chemical, and biological expressions of nature [15, 16, 17]. Fractal geometry—the language of chaos theory [18]—is a relatively new discipline of mathematics that was largely popularized by pioneer B. Mandelbrot [19, 20]. Chaos theory studies the behavior of dynamical systems that are *highly sensitive* to initial conditions [21, 15]. In a chaotic dynamical system, *miniscule* differences in initial conditions yield *widely diverging* outcomes, thereby generally rendering long-term predictions impossible [21, 15]. In addition to particle and astro physics, examples of chaos and fractals are also observed in lightning discharges [22, 23, 24, 25], weather patterns [26, 27, 28], aquatic ecosystems [29, 30], population biology [31], the biological allometric scaling laws [32, 33, 34, 35, 36], cancers and genetics [37, 38], viruses and pathogens [39, 40], the human brain [41, 42, 43], earthquakes [44, 45, 46], volcanoes [47, 48, 49], the global stock market [50, 51], and more. Certainly, iso-mathematics must play a fundamental role in classifying and demystifying such complex systems—but how?

As scientists and mathematicians, it is imperative to continue to investigate, scrutinize, challenge, develop, and test cutting-edge theories and ideas, such as the *triplex numbers* [52, 53, 54, 55], *Inopin's HR* [56], and *Santilli's iso-mathematics* [57, 58, 59, 60]. At the basis of this is the Scientific Method. Thus, in this preliminary paper, we hunt this "universal number classification beast" with an initial application of the Santillian iso-mathematics framework [2, 57, 58, 59, 60] to the triplex numbers [52], the Mandelbrot Set [19, 20], and the Inopin HR topology [56]. In Section 2, we prepare for the expedition by presenting a brief outline of Santilli's framework [2, 57, 58, 59, 60], the triplex numbers [52], Mandelbrot's set [19, 20], and Inopin's HR topology [56]. Subsequently, in Section 3, we begin to apply Santilli's framework [2, 57, 58, 59, 60] to the triplex framework [52], Mandelbrot's set [19, 20], and Inopin's HR framework [56]. The paper concludes with Sections 4–5, where we briefly recapitulate the results and provide a thankful acknowledgment, respectively.

2 Alignment and background

Here, we prepare by aligning the reader with a background that highlights some aspects of Santilli's framework [2, 57, 58, 59, 60], the triplex numbers [52], Mandelbrot's set [19, 20], and Inopin's HR [56] that pertain to the application of Section 3.

2.1 Santilli's iso-mathematics framework

In [57, 58], Santilli reinspected the historical classification of sets verifying the numeric field axioms [1] and discovered that they equally authorize solutions for an arbitrary unit $\hat{r}$, generally outside the original fields [1], under the sole condition of being non-singular, and therefore invertible, such that $\hat{r} = \frac{1}{\hat{\kappa}}$, provided that the conventional associative number multiplication $x_A \times x_B$ is replaced with the associativity-preserving form [2, 57, 58]

$$x_A \hat{\times} x_B = x_A \times \hat{\kappa} \times x_B \tag{2}$$

under which $\hat{r}$ is indeed the right and left multiplicative unit. In [57, 58], Santilli then classified the new numbers depending on the main topological features of $\hat{r}$, such as:

1. $\hat{r}$ is single-valued and Hermitean for the case of *Santilli iso-numbers*;

2. $\hat{r}$ is single-valued and non-Hermitean for the case of *right Santilli geno-numbers*, characterized by $\hat{r}$, and *left Santilli geno-numbers* characterized by $\hat{r}^\dagger$; and

3. $\hat{r}$ is multi-valued and non-Hermitean for the case of *right Santilli hyper-numbers*, characterized by $\hat{r}$, and *left Santilli hyper-numbers* characterized by $\hat{r}^\dagger$.

Moreover, in [57, 58], Santilli identified the additional number series characterized by the anti-Hermitean image of the preceding generalized numbers via the iso-duality map that is denoted with the upper index $\hat{d} = \hat{r}^{\dagger}$ called *Santilli iso-dual iso-, geno-, and hyper-numbers.*

In general, Santilli [57, 58] successfully demonstrated that, in addition to conventional numbers, such axioms authorize the existence of four distinct iso-number classes [2]:

1. **Santilli iso-topic numbers ("iso-numbers")**

    - Iso-numbers exist because Santilli [57, 58] showed that the number field axioms [1] *do not* require that the multiplicative unit $\hat{r}$ is the number 1, so $\hat{r}$ can be any value provided that [2]:

        (a) the new *Santilli iso-unit* $\hat{r}$ is positive ($\hat{r} > 0$) to permit the inverse $\hat{r} = \frac{1}{\hat{\kappa}} > 0$,

        (b) the multiplication $x_A \times x_B$ is changed to the *Santilli iso-multiplication*

        $$x_A \hat{\times} x_B = x_A \times \hat{\kappa} \times x_B = x_A \times \frac{1}{\hat{r}} \times x_B, \tag{3}$$

        which is always associative for the *iso-topic liftings*

        $$\begin{array}{rcl} x_A & = & x_A \times \hat{r} \\ x_B & = & x_B \times \hat{r}, \end{array} \tag{4}$$

        and

        (c) the additive unit and its sum are kept unchanged.

    - The number elements comprising X in eq. (1) are iso-topically "lifted" via the iso-topic lifting $X \rightarrow \hat{X}$ to become the *liftings* of the new iso-topic number set $\hat{X}$ $\hat{X}$ [2], which exist for the number field axioms [1] and are therefore numbers that are applicable to quantitative science [2]. Hence, the axiom of the multiplicative units is confirmed by the expression [2]

    $$1 \hat{\times} x_A = 1 \times \hat{\kappa} \times x_A = x_A \times \frac{1}{\hat{r}} \times 1 = x_A \hat{\times} 1 \tag{5}$$

    is valid $\forall x_A \in \hat{X}$.

- The original number field axioms [1] are preserved for $\hat{X}$ equipped with the iso-unit $\hat{r}$ for the iso-multiplication $x_A \hat{\times} x_B$ [2].
- The iso-multiplication led to additional refinement, identification, and usage [57, 58, 59] of the *iso-real numbers*, *iso-complex numbers*, and *iso-quaternion numbers* [2].
- Note the importance that if $x_A = 2$ and $x_B = 3$, then in general the iso-multiplication $2 \hat{\times} 3$ yields a product that is *different* than 6 [2].

2. **Santilli geno-topic numbers ("geno-numbers")**

   - Geno-numbers exist because, in addition to iso-numbers, Santilli [57, 58] showed that the number field axioms [1] *do not* require that the iso-multiplication operates on both the right *and* left directions because the axioms are also tested when all the multiplications (and sums) are restricted to operate right $x_A \hat{\times}_> x_B$ *or* to operate left $x_A \hat{\times}_< x_B$ [2].
   - When Santilli [57, 58] restricted all operations to act on the right or left, he was able to construct two different sets $\hat{X}_>$ and $\hat{X}_<$ with corresponding *Santilli geno-units* $\hat{r}_>$ and $\hat{r}_<$ for the compatible *Santilli geno-multiplication* operators $\hat{\times}_>$ and $\hat{\times}_<$, respectively [2].
   - The geno-multiplication led to additional refinement, identification, and usage [57, 58, 59] of the *geno-real numbers*, *geno-complex numbers*, and *geno-quaternion numbers* [2].
   - Note the importance that if $x_A = 2$ and $x_B = 3$, then in general the geno-multiplications $2 \hat{\times}_> 3$ and $2 \hat{\times}_< 3$ yield distinct products that both differ from 6 because $2 \hat{\times}_> 3 \neq 2 \hat{\times}_< 3$ and $\hat{r}_> \neq \hat{r}_<$ [2].

3. **Santilli hyper-topic numbers ("hyper-numbers" or "hyper-Santilli numbers")**

   - Hyper-numbers (which are not to be confused with so-called hyper-mathematical structures that generally do not have units) exist because, in addition to the geno-numbers, Santilli [57, 58] showed that a geno-multiplicative unit is not limited to a unique value because it can comprise a set of values for *Santilli hyper-multiplication*, such as the *Santilli hyper-units* $\hat{r}_{>} = \{1, 1, 2, 3, 5, ...\}$ or $\hat{r}_{>} = \{2, \frac{4}{5}, 7, ...\}$, if the set is ordered and defined as being applicable to right or left [2].

- The hyper-multiplication led to additional refinement, identification, and usage [57, 58, 59] of the *hyper-real numbers*, *hyper-complex numbers*, and *hyper-quaternion numbers* [2].
- Note the importance that, in general, the hyper-multiplications $2 \hat{\times}_{>} 3$ and $2 \hat{\times}_{<} 3$ yield two distinct result sets that both differ from 6 [2].

4. **Santilli iso-dual numbers**

   - *Iso-dual numbers* exist because, in addition to iso-numbers, geno-numbers, and hyper-numbers, Santilli [57, 58] showed that the multiplicative unit $\hat{r}$ can be any (positive or negative) value except for zero (i.e. $-\hat{r}$) [2].
   - The "iso-dual" term identifies a duality between positive and negative units in accordance to the original number field axioms [2].
   - The *iso-dual multiplication* led to additional refinement, identification, and usage [57, 58, 59] *iso-dual iso-numbers*, *iso-dual geno-numbers*, and *iso-dual hyper-numbers* [2].

So in total, Santilli's axiomatic iso-mathematics framework [57, 58, 59, 60] reveals *eleven* new data structures [2]:

- iso-numbers,
- geno-numbers (right and left),
- hyper-numbers (right and left),
- conventional iso-dual numbers,

- iso-dual iso-numbers,
- iso-dual geno-numbers (right and left), and
- iso-dual hyper-numbers (right and left);

each data structure is applicable to the real [8, 9, 10], complex [11, 12], and quaternion numbers [13, 14], where each application bares an infinite number of possible units [2].

Subsequently, in a series of physical implementations, Santilli [57, 58, 59, 60] then utilized the generalized iso-numbers, geno-numbers, and hyper-numbers to characterize the increasing complexities of matter with regard to non-linearity, non-Hamiltonian features, irreversibility, multi-valuedness, etc., while the iso-dual images were used by Santilli to characterize anti-matter under the corresponding increase of complexity. For an in-depth explanation of this framework, we recommend a technical study of the original publications [57, 58, 59, 60].

2.2 Complex numbers, Euclidean complex space, Mandelbrot set, and Inopin 1-sphere HR topology

Chronologically, the establishment of the Inopin HR [56] came before the triplex numbers [52]. Inopin's HR was initially introduced in the analytic quark confinement and baryon-antibaryon duality proof of [56]. The HR is a powerful tool because it is topological sphere with an "amplitude-radius" (or "amplitude-modulus") that serves as an *iso-metric* to construct Inopin's HR topology, which may be utilized to attack a wide range of mathematical and physical problems [52, 56]. In this section, we recall the synchronized complex number and 2D coordinate-vector system for Inopin's 1-sphere HR topology, which are fundamental prerequisites of the synchronized triplex number and 3D coordinate-vector system for Inopin's 2-sphere HR topology [52, 56] in the upcoming Section 2.3.

First, starting from eq. (7) in [52], we engage the complex number, 2D polar coordinate-vector, and 2D Cartesian coordinate-vector synchronization form

$$x = \vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}} = (\vec{x}) = (|\vec{x}|, \langle\vec{x}\rangle)_P = (\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C, \quad \forall \vec{x} \in X, \tag{6}$$

where the complex number $\vec{x}$ is a *dual 2D Cartesian-polar coordinate-vector state* in the *dual 2D Cartesian-polar coordinate-vector state space* and Euclidean complex space X, such that

$$X = \mathbb{C} \tag{7}$$

is also the set of all complex numbers (a field): $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C$ is a *2D Cartesian coordinate-vector state* in the *2D Cartesian coordinate-vector state space* X_C so $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C \in X_C$, while $(|\vec{x}|, \langle\vec{x}\rangle)_P$ is a *2D polar coordinate-vector state* in the *2D polar coordinate-vector state space* X_P so $(|\vec{x}|, \langle\vec{x}\rangle)_P \in X_P$, where X_C and X_P are iso-morphic, dual, synchronized, and interlocking in X [52, 56]. To prove this for eq. (6), the transformations between X_C and X_P, $\forall(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C \in X_C$ and $\forall(|\vec{x}|, \langle\vec{x}\rangle)_P \in X_P$, are identified by the complex number-coordinate-vector constraints of eqs. (2–6) in [52], which define:

1. the $|\vec{x}| \in [0, \infty)$ *amplitude* ("radius" or "modulus" or "Euclidean distance") polar component
$$|\vec{x}| = \sqrt{\vec{x}_{\mathbb{R}}^{\,2} + \vec{x}_{\mathbb{I}}^{\,2}}\ ; \tag{8}$$

2. the $\langle\vec{x}\rangle \in [0, 2\pi]_{\mathbb{R}\mathbb{I}}$ *phase* ("azimuth" or "$\mathbb{R}\mathbb{I}$-direction") polar component (for the $\mathbb{R}\mathbb{I}$-plane)
$$\langle\vec{x}\rangle = \arctan\left(\frac{\vec{x}_{\mathbb{I}}}{\vec{x}_{\mathbb{R}}}\right); \tag{9}$$

3. the $\vec{x}_{\mathbb{R}} \in (-\infty_{\mathbb{R}}, \infty_{\mathbb{R}})$ *real* ("x-axis") Cartesian component
$$\vec{x}_{\mathbb{R}} = \frac{\vec{x}_{\mathbb{I}}}{\tan\langle\vec{x}\rangle} = |\vec{x}|\cos\langle\vec{x}\rangle\ ; \tag{10}$$

   and

4. the $\vec{x}_{\mathbb{I}} \in (-\infty_{\mathbb{I}}, \infty_{\mathbb{I}})$ *imaginary* ("y-axis") Cartesian component
$$\vec{x}_{\mathbb{I}} = i\vec{x}_{\mathbb{R}}\tan\langle\vec{a}\rangle = i|\vec{x}|\sin\langle\vec{x}\rangle. \tag{11}$$

Hence, for X, the built-in iso-morphic duality between X_C and X_P is defined as

$$\delta_{2D} : X_C \to X_P \tag{12}$$

from the transformation of eqs. (8–9) and

$$\delta_{2D}^{-1} : X_P \to X_C \tag{13}$$

from the transformation of eqs. (10–11). Therefore, X clearly satisfies the five number field axioms of [1]. For a depiction of eqs. (6–13) see Figure 1, where observe that $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ are treated as vectors (with axis-dependent magnitude and direction) so the vector sum is $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$ with amplitude $|\vec{x}|$ and direction $\langle \vec{x} \rangle$.

Second, given that X is the set of complex numbers, we note that X can be populated with the Mandelbrot Set by systematically iterating Mandelbrot's complex quadratic polynomial [19, 20]

$$\vec{x}_{n+1} = \vec{x}_n{}^2 + \vec{c} = \vec{x}_{n+1_{\mathbb{R}}} + \vec{x}_{n+1_{\mathbb{I}}} = (\vec{x}_{n_{\mathbb{R}}} + \vec{x}_{n_{\mathbb{I}}})^2 + (\vec{c}_{\mathbb{R}} + \vec{c}_{\mathbb{I}}), \tag{14}$$

where $\vec{x}_n, \vec{x}_{n+1}, \vec{c} \in X$ are complex numbers that satisfy eqs. (6–13)—see Figure 2.

Third, following [52, 56], we equip X with Inopin's 1-sphere HR, namely $T^1 \subset X$, by iso-metrically embedding T^1 in X of eqs. (10–11) of [56] to construct Inopin's 1-sphere HR topology; T^1 is a topological circle that is centered on the origin

$$O \in X = 0 + 0i \tag{15}$$

with the *positive-definite* amplitude-radius r and the corresponding curvature $\kappa = \frac{1}{r}$. Hence, T^1 is employed to topologically encode the 1-sphere version of Inopin's "non-linear time dimension" and "temporal distance scale", where eq. (15) of [56] defines it as

$$T^1 = \{\vec{x} \in X : |\vec{x}| = r\} \tag{16}$$

for Inopin's 1-sphere HR topology, which is the multiplicative group of all non-zero complex numbers with the amplitude-radius r.

Now T^1 was equipped with topological deformation order parameter fields of fractional statistics for quasi-particles [52, 56]. The *inside* of T^1

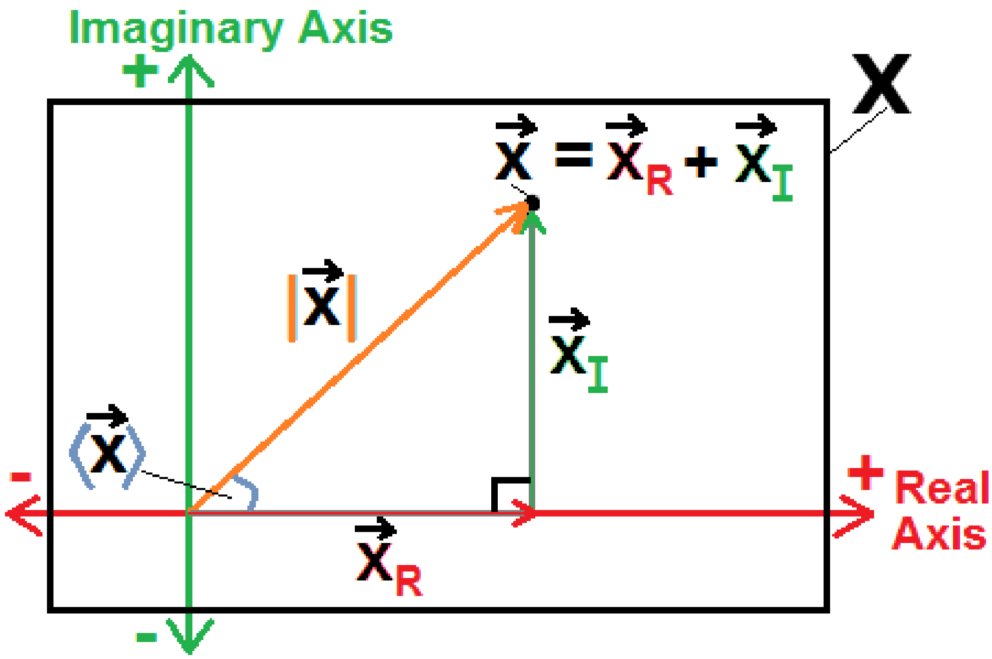

Fig. 1: Complex components for the dual 2D Cartesian-polar coordinate-vector state $\vec{x}$ in the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X, such that $\vec{x} \in X$, where $\vec{x}$ is simultaneously treated as a complex number, 2D polar coordinate-vector, and 2D Cartesian coordinate-vector. Specifically, $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C$ is a 2D Cartesian coordinate-vector state in the 2D Cartesian coordinate-vector state space X_C so $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C \in X_C$, while $(|\vec{x}|, \langle\vec{x}\rangle)_P$ is a 2D polar coordinate-vector state in the 2D polar coordinate-vector state space X_P so $(|\vec{x}|, \langle\vec{x}\rangle)_P \in X_P$, where X_C and X_P are iso-morphic, dual, synchronized, and interlocking in X. Note that $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ are treated as vectors (with axis-dependent magnitude and direction) so the vector sum is $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$ with amplitude $|\vec{x}|$ and direction $\langle\vec{x}\rangle$.

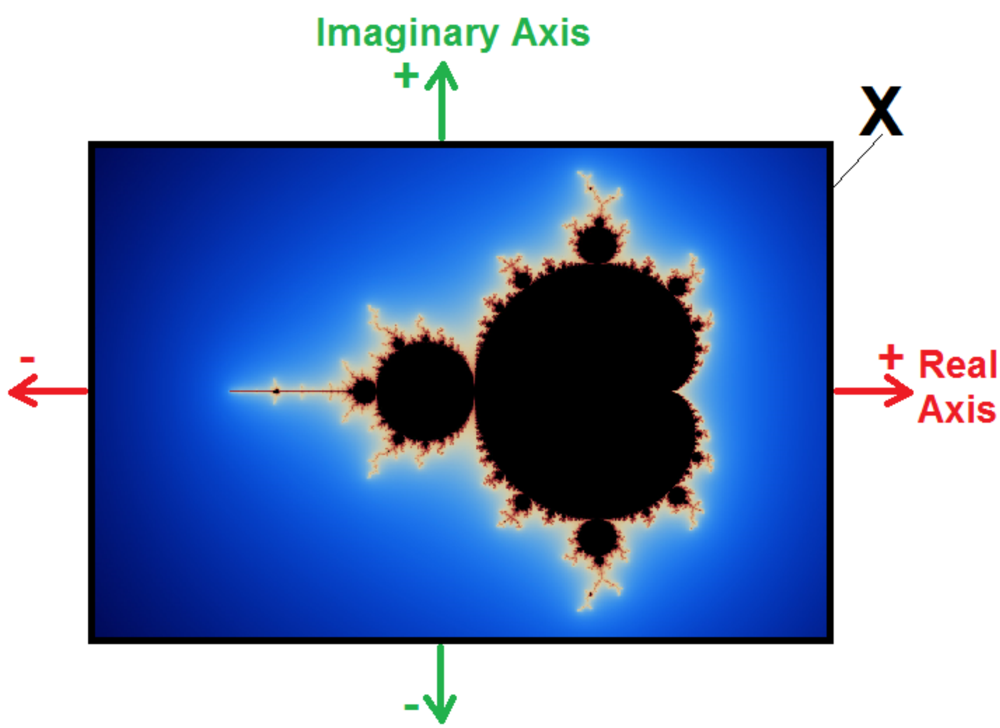

Fig. 2: The Mandelbrot Set [19, 20], which is often considered to be the most famous fractal, can populate the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X.

corresponds to an *interior* dynamical system of *superluminal quasi-particle spatial excitations*, namely the *2-brane micro sub-space zone* (or "short spatial distance scale") $X_- \subset X$ of eq. (16) in [56], while the *outside* of T^1 corresponds to an *exterior* dynamical system of *luminal or sub-luminal quasi-particle spatial excitations*, namely the *2-brane macro sub-space zone* (or "long spatial distance scale") $X_+ \subset X$ of eq. (17) in [56]—T^1 itself is populated with *luminal quasi-particle temporal excitations*, which is simultaneously dual to both X_- and X_+ as in eqs. (20–21) in [56]. Here, $X_-, T^1, X_+ \subset X$ are disjoint and comprise the complete X, such that $X = X_- \cup T^1 \cup X_+$ [56]—see Figure 3. For this, each complex location $\vec{x} \in X$ was equipped with one or more complex order parameter field states in the generic form $\vec{\psi}(\vec{x})$ within a complex order parameter field state space $\Phi(\vec{x})$ for topological deformations, such that $\vec{\psi}(\vec{x}) \in \Phi(\vec{x})$—see eq. (20) in [52].

For the quark confinement proof of [56], the three distinct quark-antiquark pairs for a baryon-antibaryon pair are confined to T^1 in the upgraded Gribov vacuum on a six-coloring kagome lattice antiferromagnet with correlated order parameters. The transforming wavefunction states of T^1 are directly related to the states of X_- and X_+, which are 2-branes within 3-branes, so their 3D state space is directly inferred from the 2D state space of T^1. But at the time of writing [56], the triplex numbers with triplex multiplication did not yet exist, therefore it was not possible to fully encode the 3D state space of the 3-branes—this genuine need to employ 3D numbers to encode 3D states (with a pertinent 3D number multiplication) fueled the motivation for the triplex implementation of [52], which served as a topological upgrade to [56].

2.3 Triplex numbers, Euclidean triplex space, Mandelbulb set, and Inopin 2-sphere HR topology

The triplex work of [52] was inspired by the emerging "3D hyper-complex number" framework of pioneers D. White and P. Nylander [53, 54, 55]. White and Nylander originally developed their 3D hyper-complex number system in order to generate magnificent 3D fractals with computer graphics [53, 54, 55]. However, the White-Nylander 3D hyper-complex approach [53, 54] was still considered incomplete primarily because their 3D spherical

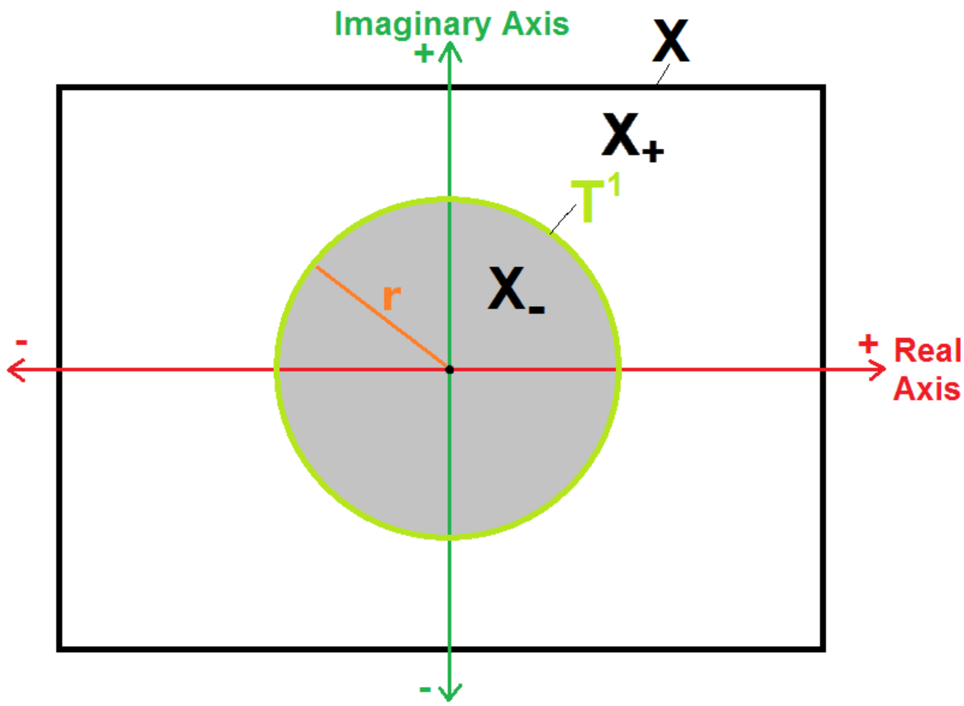

Fig. 3: Inopin's 1-sphere HR topology for the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X, where the topological 1-sphere HR $T^1 \subset X$ is simultaneously dual to two spatial 2-branes [52, 56]: the micro sub-space zone $X_- \subset X$ and the macro sub-space zone $X_+ \subset X$ for interior and exterior dynamical systems, respectively.

form (extended polar form) was not unique while the system did not form a well-behaved algebra [55, 52]. Fortunately, after a thorough and rigorous investigation, this great "White-Nylander mythical beast" of [53, 54, 55] was destroyed via the creation of a well-behaved triplex algebra equipped with a unique 3D spherical form and triplex multiplication—observe the work of [52] and the upcoming eqs. (17–28). Thus, the triplex structure Y was introduced to supersede the complex structure X by upgrading it with a third axis for 3D representation, such that $X \subset Y$, to establish Inopin's 2-sphere HR topology [52].

First, eq. (35) in [52] defines the triplex number, 3D spherical coordinate-vector, and 3D Cartesian coordinate-vector synchronization form

$$y = \vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_Z = (\vec{y}) = (|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S = (\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C, \quad \forall \vec{y} \in Y, \tag{17}$$

where the triplex number $\vec{y}$ is a *dual 3D Cartesian-spherical coordinate-vector state* in the *dual 3D Cartesian-spherical coordinate-vector state space* and Euclidean triplex space Y, such that

$$Y \equiv \mathbb{T} \tag{18}$$

is also defined as the set of all triplex numbers (a field): $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C$ is a *3D Cartesian coordinate-vector state* in the *3D Cartesian coordinate-vector state space* Y_C so $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C \in Y_C$, while $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S$ is a *3D spherical coordinate-vector state* in the *3D spherical coordinate-vector state space* Y_S so $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S \in Y_S$, where Y_C and Y_S are iso-morphic, dual, synchronized, and interlocking in Y [52]. To prove this for eq. (17), the transformations between Y_C and Y_S, $\forall(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C \in Y_C$ and $\forall(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S \in Y_S$, are identified by the triplex number and 3D coordinate-vector constraints of eqs. (29–34) in [52], which define:

1. the $|\vec{y}| \in [0, \infty)$ *amplitude* ("radius" or "modulus" or "Euclidean distance") spherical component

$$|\vec{y}| = \sqrt{\vec{y}_{\mathbb{R}}^{\,2} + \vec{y}_{\mathbb{I}}^{\,2} + \vec{y}_Z^{\,2}}\ ; \tag{19}$$

2. the $\langle\vec{y}\rangle \in [0, 2\pi]_{\mathbb{RI}}$ *phase* ("azimuth" or "$\mathbb{RI}$-direction") spherical component (for the $\mathbb{RI}$-plane)

$$\langle\vec{y}\rangle = \arctan\left(\frac{\vec{y}_{\mathbb{I}}}{\vec{y}_{\mathbb{R}}}\right); \tag{20}$$

3. the $[\vec{y}] \in [0, 2\pi]_{\mathbb{R}Z}$ *inclination* ("zenith" or "$\mathbb{R}Z$-direction") spherical component (for the $\mathbb{R}Z$-plane)

$$[\vec{y}] = \arctan\left(\frac{\vec{y}_Z}{\vec{y}_{\mathbb{R}}}\right); \tag{21}$$

4. the $\vec{y}_{\mathbb{R}} \in (-\infty_{\mathbb{R}}, \infty_{\mathbb{R}})$ *real* ("x-axis") Cartesian component

$$\vec{y}_{\mathbb{R}} = \frac{\vec{y}_{\mathbb{I}}}{\tan\langle\vec{y}\rangle} = |\vec{y}|_{\mathbb{R}\mathbb{I}} \cos\langle\vec{y}\rangle \ ; \tag{22}$$

5. the $\vec{y}_{\mathbb{I}} \in (-\infty_{\mathbb{I}}, \infty_{\mathbb{I}})$ *imaginary* ("y-axis") Cartesian component

$$\vec{y}_{\mathbb{I}} = i\vec{y}_{\mathbb{R}} \tan\langle\vec{y}\rangle = i|\vec{y}|_{\mathbb{R}\mathbb{I}} \sin\langle\vec{y}\rangle, \tag{23}$$

   where

$$|\vec{y}|_{\mathbb{R}\mathbb{I}} = \sqrt{\vec{y}_{\mathbb{R}}^{\,2} + \vec{y}_{\mathbb{I}}^{\,2}} \tag{24}$$

   is the 2D limited amplitude radius for the $\mathbb{R}\mathbb{I}$ plane; and

6. the $\vec{y}_Z \in (-\infty_Z, \infty_Z)$ *projected* ("z-axis") Cartesian component

$$\vec{y}_Z = j\vec{y}_{\mathbb{R}} \tan[\vec{y}] = j|\vec{y}|_{\mathbb{R}Z} \sin[\vec{y}], \tag{25}$$

   where

$$|\vec{y}|_{\mathbb{R}Z} = \sqrt{\vec{y}_{\mathbb{R}}^{2} + \vec{y}_Z^{2}} \tag{26}$$

   is the 2D limited amplitude radius for the $\mathbb{R}Z$ plane.

Hence, for Y, the built-in iso-morphic duality between Y_C and Y_S is defined as

$$\delta_{3D} : Y_C \to Y_S \tag{27}$$

from the transformation of eqs. (19–21) and

$$\delta_{3D}^{-1} : Y_S \to Y_C \tag{28}$$

from the transformation of eqs. (22–26). For a depiction of eqs. (17–28) see Figures 4–5, where observe that $\vec{y}_{\mathbb{R}}$, $\vec{y}_{\mathbb{I}}$, and $\vec{y}_Z$ are treated as vectors

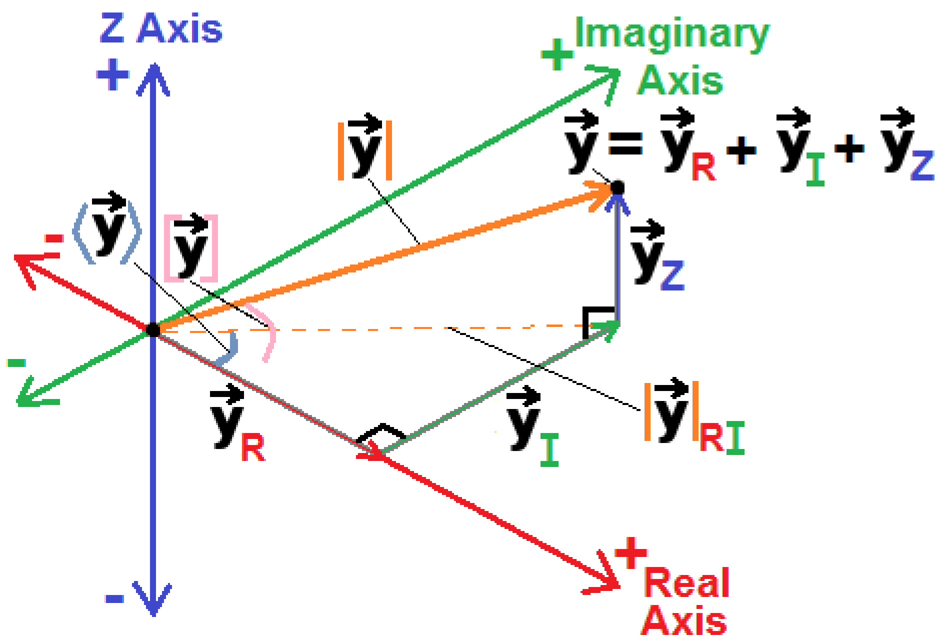

Fig. 4: Triplex components for the dual 3D Cartesian-spherical coordinate-vector state $\vec{y}$ in the dual 3D Cartesian-polar coordinate-vector state space (and Euclidean triplex space) Y, such that $\vec{y} \in Y$, where $\vec{y}$ is simultaneously treated as a triplex number, 3D spherical coordinate-vector, and 3D Cartesian coordinate-vector. Specifically, $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C$ is a 3D Cartesian coordinate-vector state in the 3D Cartesian coordinate-vector state space Y_C so $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C \in Y_C$, while $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S$ is a 3D spherical coordinate-vector state in the 3D spherical coordinate-vector state space Y_S so $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S \in Y_S$, where Y_C and Y_S are iso-morphic, dual, synchronized, and interlocking in Y. Note that $\vec{y}_{\mathbb{R}}$, $\vec{y}_{\mathbb{I}}$, and $\vec{y}_Z$ are treated as vectors (with axis- dependent magnitude and direction) so the vector sum is $\vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_Z$ with amplitude $|\vec{y}|$ and two directions $\langle\vec{y}\rangle$ and $[\vec{y}]$.

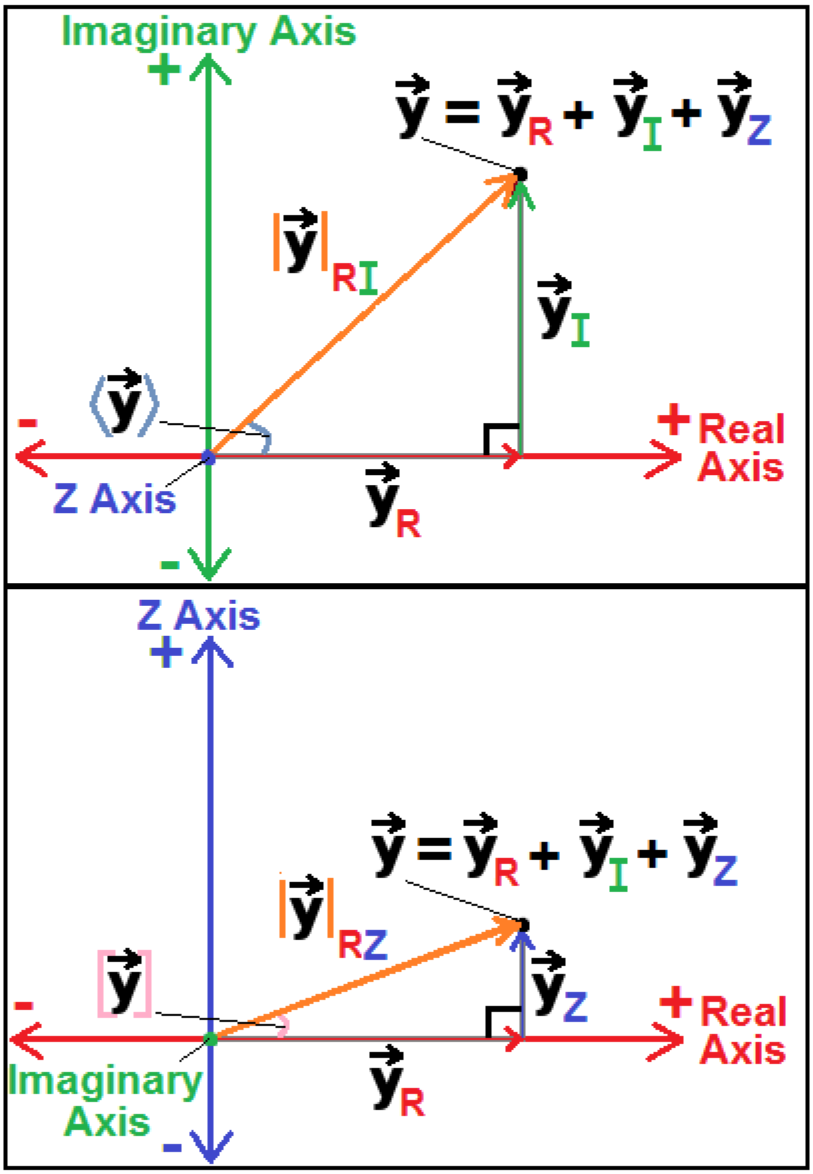

Fig. 5: Aligned perspectives of $\vec{y} \in Y$ from the $\mathbb{RI}$-plane (top) and the $\mathbb{R}Z$-plane (bottom).

(with axis-dependent magnitude and direction) so the vector sum is $\vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_Z$ with amplitude $|\vec{y}|$ and two directions $\langle\vec{y}\rangle$ and $[\vec{y}]$.

Second, using the notation of eq. (17), the triplex multiplication between the two distinct triplex numbers $\vec{y}_A, \vec{y}_B \in Y$ to yield the product $\vec{y}_C \in Y$ is defined in eqs. (70–71) of [52] as

$$\vec{y}_C = \vec{y}_A \times \vec{y}_B \iff \begin{array}{ccccc} |\vec{y}_C| & = & |\vec{y}_A| & \times & |\vec{y}_B| \\ \langle\vec{y}_C\rangle & = & \langle\vec{y}_A\rangle & + & \langle\vec{y}_B\rangle \\ [\vec{y}_C] & = & [\vec{y}_A] & + & [\vec{y}_B] \end{array}, \quad \forall \vec{y}_A, \vec{y}_B, \vec{y}_C \in Y. \tag{29}$$

In other words, the amplitude-radius values are multiplied while the corresponding directional values are added to yield the triplex spherical form of the product in eq. (29). Hence, given that division is defined as the inverse, opposite, and reverse operation of multiplication, then we engage eq. (29) to incorporate this fundamental relation and thereby define the *triplex division* as

$$\vec{y}_C \equiv \vec{y}_A \;/\; \vec{y}_B \iff \begin{array}{ccccc} |\vec{y}_C| & \equiv & |\vec{y}_A| & / & |\vec{y}_B| \\ \langle\vec{y}_C\rangle & \equiv & \langle\vec{y}_A\rangle & - & \langle\vec{y}_B\rangle \\ [\vec{y}_C] & \equiv & [\vec{y}_A] & - & [\vec{y}_B] \end{array}, \quad \forall \vec{y}_A, \vec{y}_B, \vec{y}_C \in Y, \tag{30}$$

where $\vec{y}_A$ is the dividend, $\vec{y}_B$ is the divisor, and $\vec{y}_C$ is the quotient because eq. (30) comprises operations that are the inverse, opposite, and reverse of eq. (29). In other words, the amplitude-radius dividend is divided by the amplitude-radius divisor while the corresponding directional values are subtracted to yield the triplex spherical form of the quotient in eq. (30). Therefore, Y clearly satisfies the five number field axioms of [1].

Third, we summarize the pertinent aspects of P. Rowlands' dual 3D vector space [61, 62, 63], which is a quaternion-based, nilpotent implementation of our Y. According to [61, 62, 63], a triplex structure based on three orthogonal axes—i.e. the $\mathbb{R}$-axis, the $\mathbb{I}$-axis, and the Z-axis of Y—can be constructed with a dual, interlocking 3D vector system that generates a double 3D Clifford algebra to encode spatial states of dynamical systems [61, 62, 63]. An overview of this process is as follows [61, 62, 63]:

1. At the start, two vector spaces are defined in the Clifford algebra formulation, where the vector spaces are *distinct but identical* in every

respect [61, 62, 63]. Both vector spaces have, in addition to one scalar unit, three vector units to encode lengths, three bivector units to encode areas, and one pseudo-scalar unit to encode volume [61, 62, 63]. Vectors in this algebra have a complete product and well-behaved multiplication—see eq. (3) of [61].

2. Once the dual vector spaces are assembled, a commutative combination of the two spaces (allowing for + and 0 signs) generates an algebra structured on 64 units so the Clifford algebra vectors can also be written as complexified quaternions [61, 62, 63]. Therefore, the complete double algebra is encoded as either a complexified double quaternion algebra or as a product of a vector and a quaternion algebra [61, 62, 63]—this yields a total of three unit sets of order 64 that are iso-morphic to, for example, the Dirac algebra [61, 62, 63].

3. Now each unit set can be created by five generators [61, 62, 63]. There are many options for the generators, but they all have the same structure in which the symmetry of one 3D unit set is preserved, while the other is broken [61, 62, 63]. If six generators are used then both symmetries are preserved, but not in the minimum case of using five generators [61, 62, 63]: there are *three* instances of the five generators that preserve one symmetry but not the other [61, 62, 63].

4. These three resulting instances permit the creation of a triplex representation whose symmetry is not preserved as the basis of the three triplex axes [61, 62, 63]. The character of each axis is then determined by the type of quantity in which it is multiplied by [61, 62, 63].

5. Next, the middle three terms of a given five generator are used to encode the Z-axis as a vector, and the three vector units in each case are combined to be a single vector unit [61, 62, 63]. This yields a triplex structure where the axes are encoded as quaternions [61, 62, 63].

6. Interestingly, Rowlands' triplex structure can be thought of as being an endless fractal sequence as one moves up the scale [63]. The first triplex structure, which is originally unbroken, becomes broken

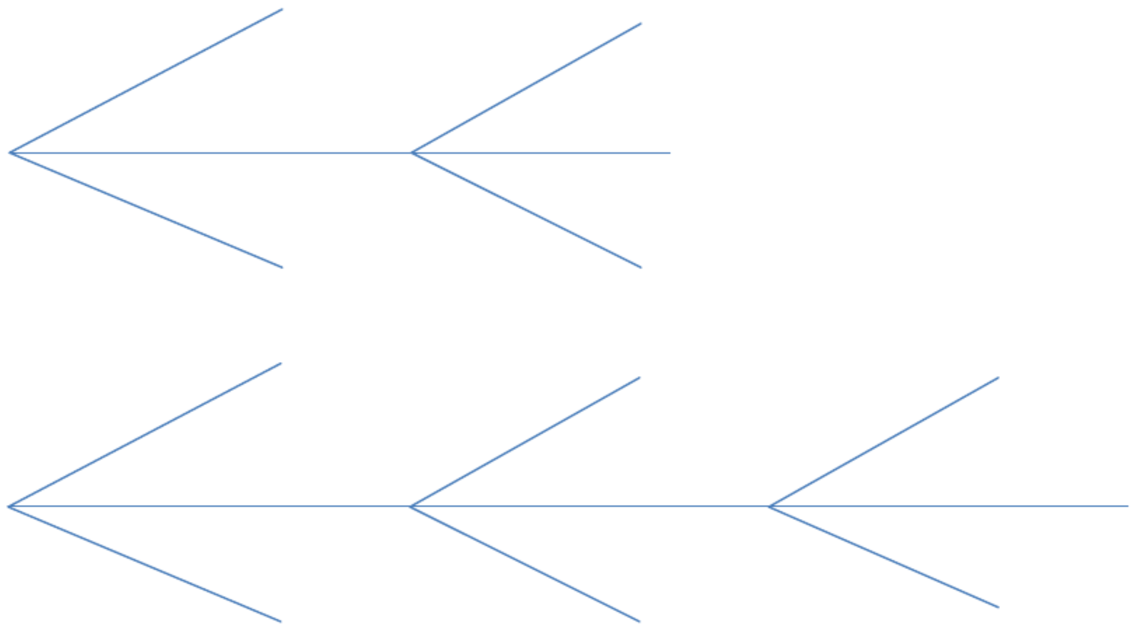

Fig. 6: Rowlands' triplex structure can be thought of as being an endless fractal sequence as one moves up the scale [63]. The first triplex structure, which is originally unbroken, becomes broken by having a second triplex structure attached (top). Consequently, the second triplex structure is now unbroken, but becomes broken by attaching another triplex structure (bottom), and so on indefinitely.

by having a second triplex structure attached [63]. Consequently, the second triplex structure is now unbroken, but becomes broken by attaching another triplex structure, and so on indefinitely [63]. See Figure 6 for a depiction of this process [63].

Fourth, given that Y is the set of triplex numbers equipped with a well-behaved triplex algebra and unique triplex spherical form [52] that is similar to Rowlands' approach [61, 62, 63] and extends the White-Nylander approach [53, 54, 55], we note that Y can be populated with the Mandelbulb Set—a fundamental 3D fractal—by systematically iterating Mandelbrot's triplex quadratic polynomial

$$\vec{y}_{n+1} = \vec{y}_n^{\,2} + \vec{c} = \vec{y}_{n+1_{\mathbb{R}}} + \vec{y}_{n+1_{\mathbb{I}}} + \vec{y}_{n+1_Z} = (\vec{y}_{n_{\mathbb{R}}} + \vec{y}_{n_{\mathbb{I}}} + \vec{y}_{n_Z})^2 + (\vec{c}_{\mathbb{R}} + \vec{c}_{\mathbb{I}} + \vec{c}_Z), \quad (31)$$

where $\vec{y}_n, \vec{y}_{n+1}, \vec{c} \in \mathbb{Y}$ are triplex numbers that satisfy eqs. (17–30).

Fifth, following [52], we equip Y with Inopin's 2-sphere HR, namely $T^2 \subset Y$, by iso-metrically embedding T^2 in Y of eqs. (40–41) of [52] to construct Inopin's 2-sphere HR topology; T^1 is a topological 2-sphere that is centered on the origin

$$O \in Y = 0 + 0i + 0j \quad (32)$$

with the amplitude-radius r and the corresponding curvature $\kappa = \frac{1}{r}$ (the same as T^1). Hence, T^2 is employed to topologically encode the 2-sphere version of Inopin's "non-linear time dimension" and "temporal distance scale", where eq. (40) of [52] defines it as

$$T^2 = \{\vec{y} \in Y : |\vec{y}| = r\} \quad (33)$$

for Inopin's 2-sphere HR topology, which is the multiplicative group of all non-zero triplex numbers with the amplitude-radius r [52]—$T^1 \subset X$ is generalized to $T^2 \subset Y$ [52]. Hence, given that $X \subset Y$ and $T^1 \subset X \subset Y$, then

$$T^1 = T^2 \cap X, \quad (34)$$

so T^1 is the great (topological) circle of T^2 [52, 56], such that $T^1 \subset T^2$, where both T^1 and T^2 share the same positive-definite amplitude-radius r and curvature $\kappa = \frac{1}{r}$. So given $X \to Y$ and $X \subset Y$, T^2 delineates the dual interconnected spatial 3-branes $Y_- \subset Y$ and $Y_+ \subset Y$ in eq. (41) of [52], which supercede the spatial 2-branes $X_- \subset Y_- \subset Y$ and $X_+ \subset Y_+ \subset Y$, where T^2 is simultaneously dual to the *3-brane micro sub-space zone* Y_- and the *3-brane macro sub-space zone* Y_+ in eq. (44) of [52] for $Y = Y_- \cup T^2 \cup Y_+$, recalling that $Y_-, T^2, Y_+ \subset Y$ are disjoint [52]—see Figure 7. For this, we observe the analogy: T^1 *is to* $X_-, X_+ \subset X$ *just as* T^2 *is to* $Y_-, Y_+ \subset Y$, *respectively.* Here, we recall from [52] that each triplex location $\vec{y} \in Y$ was equipped with one or more triplex order parameter field states in the generic form $\vec{\psi}(\vec{y})$ within a triplex order parameter field state space $\Phi(\vec{y})$ for topological deformations, such that $\vec{\psi}(\vec{y}) \in \Phi(\vec{y})$—see eq. (50) in [52].

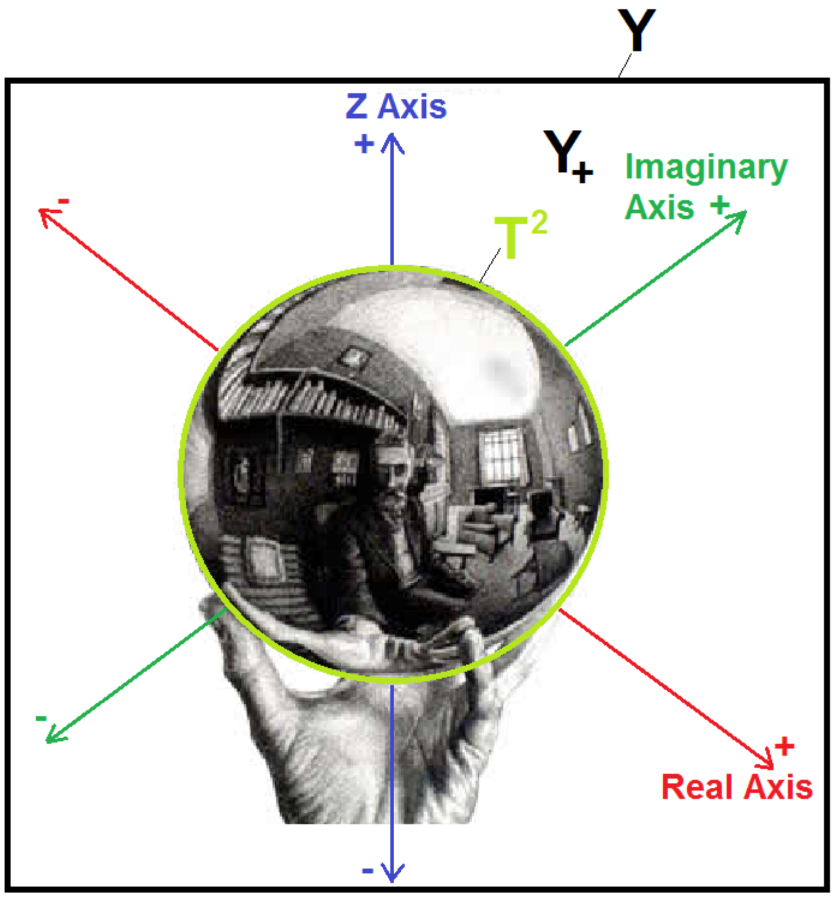

Fig. 7: Inopin's 2-sphere HR topology in the dual 3D Cartesian-spherical coordinate-vector state space (and Euclidean triplex space) Y, where the topological 2-sphere HR $T^2 \subset Y$ is simultaneously dual to two spatial 3-branes [52, 56]: the micro sub-space zone $Y_{-} \subset Y$ and the macro sub-space zone $Y_{+} \subset Y$ for interior and exterior dynamical systems, respectively. Here, T^2 is depicted as M. C. Escher's famous reflecting sphere [64].

Therefore, given that Santilli applied his iso-mathematics [57, 58] to the real [8, 9, 10], complex [11, 12], and quaternion numbers [13, 14], it seems important that his breakthroughs [2, 57, 58, 59, 60] should also be applied to the triplex numbers [52], where the amplitude-radius r of both T^1 and T^2 serves as an iso-metric [52, 56].

3 Application

In this section, we begin to apply Santilli's iso-mathematics framework [2, 57, 58, 59, 60] to the triplex numbers [52] and Inopin's 2-sphere HR topology [52, 56]. In Section 3.1, our first attack focuses on iso-topically lifting the triplex number space Y—*excluding* T^2—and subsequently gives a preliminary assessment on the feasibility of iso-morphing Y via geno-topic, hyper-topic, and iso-dual-topic liftings. Next, we advance to our second engagement in Section 3.2, where we assemble a preliminary explanation on how one can upgrade the Y—*including* T^2—of Inopin's 2-sphere HR topology [52, 56] with Santilli's iso-numbers [2, 57, 58, 59, 60]. In both phases, we report the initial results.

3.1 Iso-triplex numbers, Euclidean iso-triplex space, and iso-fractal Mandelbulb Set

Here, in the first phase of engagement, the initial and primary objective is to iso-topically lift Y in accordance to Santilli's iso-numbers [2, 57, 58, 59, 60], while the secondary objective is to assess the possibility of iso-morphing Y via geno-topic, hyper-topic, and iso-dual-topic liftings.

First, for Santilli's iso-topic numbers [2, 57, 58, 59, 60], we select some $\hat{r} > 0$ with corresponding inverse $\hat{\kappa} = \frac{1}{\hat{r}}$, such that $\hat{\kappa} > 0$. Next, following [2, 57, 58, 59, 60], Y's triplex numbers $\vec{y}_A, \vec{y}_B \in Y$ are redefined in the form

$$\begin{array}{rcl} \vec{y}_A & \equiv & \vec{y}_A \times \hat{r} \\ \vec{y}_B & \equiv & \vec{y}_B \times \hat{r} \end{array}, \quad \forall \vec{y}_A, \vec{y}_B \in Y \to \forall \vec{y}_A, \vec{y}_B \in \hat{Y}, \tag{35}$$

where the triplex number set and Euclidean triplex space Y of eq. (18) is iso-topically lifted via $Y \to \hat{Y}$ to become $\hat{Y}$—defined as both the *iso-triplex number set* and the *Euclidean iso-triplex space*—such that the triplex multiplication between $\vec{y}_A, \vec{y}_B \in Y$ of eq. (29) is upgraded to define the *iso-*

triplex multiplication as

$$\vec{y}_A \hat{\times} \vec{y}_B \equiv \vec{y}_A \times \hat{\kappa} \times \vec{y}_B = \vec{y}_A \times \frac{1}{\hat{r}} \times \vec{y}_B \tag{36}$$

for $\hat{Y}$, which is always associative, such that the additive unit and its sum remain unmodified. Hence, from [2, 57, 58, 59, 60], the axiom of multiplicative units is confirmed by the expressions

$$\begin{array}{ccccccc} 1 \hat{\times} \vec{y}_A & \equiv & 1 \times \hat{\kappa} \times \vec{y}_A & \equiv & \vec{y}_A \times \frac{1}{\hat{r}} \times 1 & \equiv & \vec{y}_A \hat{\times} 1 \\ 1 \hat{\times} \vec{y}_B & \equiv & 1 \times \hat{\kappa} \times \vec{y}_B & \equiv & \vec{y}_B \times \frac{1}{\hat{r}} \times 1 & \equiv & \vec{y}_B \hat{\times} 1, \end{array} \tag{37}$$

which are valid $\forall \vec{y}_A, \vec{y}_B \in Y \rightarrow \forall \vec{y}_A, \vec{y}_B \in \hat{Y}$. Therefore, the operation $\vec{y}_A \hat{\times} \vec{y}_B$ of eq. (36) is known as the triplex version of Santilli iso-multiplication, so we've employed the work of [57, 58, 59, 60] to identify the Santilli iso-triplex number set in accordance to [2]. Therefore, the results of eqs. (36–37) indicate that the iso-topic lifting and its inverse are

$$\begin{array}{rccc} f(\hat{r}): & Y & \rightarrow & \hat{Y} \\ f^{-1}(\hat{r}): & \hat{Y} & \rightarrow & Y, \end{array} \tag{38}$$

respectively, which define an iso-morphism (or bijective homo-morphism), where $\hat{Y}$ is the Euclidean iso-triplex space. At this point, we've successfully applied Santilli's iso-numbers [57, 58, 59, 60] to the triplex numbers [52] so we identify:

- **Lemma 1.1**: A Euclidean triplex space Y (equipped with the multiplication operator $\times$) can be iso-topically lifted via the transition $f(\hat{r}) : Y \rightarrow \hat{Y}$ (with iso-unit $\hat{r}$) to become a Euclidean iso-triplex space $\hat{Y}$ (equipped with the iso-multiplication operator $\hat{\times}$), where Y and $\hat{Y}$ are locally iso-morphic.

- **Lemma 1.2**: A Euclidean triplex space Y and its corresponding (iso-topically lifted) Euclidean iso-triplex space $\hat{Y}$ are locally iso-morphic because:

  1. Y and $\hat{Y}$ share the same origin $O \in Y, \hat{Y}$ for $O = 0 + 0i + 0j$; and

2. there is a one-to-one mapping of mathematical location elements between Y and $\hat{Y}$.

Second, for Santilli's geno-topic numbers [57, 58], we see that the right $\vec{y}_A \hat{\times}_> \vec{y}_B$ and left $\vec{y}_A \hat{\times}_< \vec{y}_B$ geno-multiplication operations may establish the two different sets $\hat{Y}_>$ and $\hat{Y}_<$ from $\hat{Y}$ with corresponding geno-units $\hat{r}_>$ and $\hat{r}_<$ to possibly identify *geno-triplex numbers* in accordance to [2]. Hence, Santilli's geno-multiplication applies to the triplex numbers of [52].

Third, for Santilli's hyper-topic numbers [57, 58], we clarify that the iso-multiplicative unit for $\vec{y}_A \hat{\times} \vec{y}_B$ is not limited to a unique value because it can comprise an ordered set of values (i.e. recall $\hat{r}_> = \{1, 1, 2, 3, 5, ...\}$ or $\hat{r}_> = \{2, \frac{4}{5}, 7, ...\}$) since the set is applicable to right and left [2]. Thus, it is evident that Santilli's hyper-multiplication may apply to the triplex numbers of [52] and possibly identify the pertinent *hyper-triplex numbers* in accordance to [2].

Fourth, for Santilli's iso-dual numbers [57, 58], the iso-multiplicative unit $\hat{r}$ may be any value except for zero, so Santilli's method [57, 58] may apply to the triplex numbers of [52] to possibly identify the relevant *iso-dual iso-triplex numbers* in accordance to [2].

And finally, at this point, these initial results of our preliminary assessment suggest the existence of geno-triplex, hyper-triplex, and iso-dual-triplex numbers, with direct application to *iso-fractals*, *geno-fractals*, *hyper-fractals*, and *iso-dual-fractals*—certainly, the iso-triplex numbers and iso-triplex space of eqs. (35–38) engage an iso-fractal version of the Mandelbulb Set in eq. (31). Therefore, these topics will be subject for future investigation beyond the limited scope of this paper.

3.2 Iso-2-sphere HR topology

Here, in the second phase of engagement, the primary objective is to apply Santilli's framework [2, 57, 58, 59, 60] to construct a preliminary explanation on how the geometry of $T^2 \subset Y$ in Inopin's 2-sphere HR topology [52, 56] can be reduced to the *one single geodesic* T^1, where $T^1 \subset T^2 \subset Y$, over the iso-triplex numbers.

First, given that the T^1 of eq. (16) and T^2 of eq. (33) are both triplex sub-spaces of Y, then Lemmas 1.1–1.2 apply to $T^1, T^2 \subset Y$ so they can be iso-topically lifted to $\hat{T}^1, \hat{T}^2 \subset \hat{Y}$, respectively, where we define $\hat{T}^1$ as the *iso-1-sphere HR* and $\hat{T}^2$ as the *iso-2-sphere HR*. This enables us to apply eq. (38) to define the iso-topic liftings (and the inverses) for $T^1, T^2 \subset Y$ with some iso-unit $\hat{r}$ as

$$\begin{array}{rccc} f(\hat{r}): & T^1 & \rightarrow & \hat{T}^1 \\ f^{-1}(\hat{r}): & \hat{T}^1 & \rightarrow & T^1 \end{array} \tag{39}$$

and

$$\begin{array}{rccc} f(\hat{r}): & T^2 & \rightarrow & \hat{T}^2 \\ f^{-1}(\hat{r}): & \hat{T}^2 & \rightarrow & T^2, \end{array} \tag{40}$$

respectively. At this point, we identify:

- **Lemma 2.1**: An Inopin n-sphere HR T^n that is iso-metrically embedded in a triplex space Y, such that $n \in \{1, 2\}$, can be iso-topically lifted via the transition $f(\hat{r}): T^n \rightarrow \hat{T}^n$ (with iso-unit $\hat{r}$) to become an iso-n-sphere HR $\hat{T}^n$ when the superseding Y is simultaneously lifted to $\hat{Y}$, where T^n and $\hat{T}^n$ are locally iso-morphic.

- **Lemma 2.2**: An Inopin n-sphere HR T^n and its corresponding (iso-topically lifted) iso-n-sphere HR $\hat{T}^n$ are locally iso-morphic because:

    1. T^n and $\hat{T}^n$ share the same center $O \in Y, \hat{Y}$ for $O = 0 + 0i + 0j$;
    2. there is a one-to-one mapping of mathematical location elements between Y and $\hat{Y}$; and
    3. Y and $\hat{Y}$ are locally iso-morphic.

Now, the iso-morphic characterization of Y, T^1, and T^2 in Lemmas 1.1–1.2 and 2.1–2.2, which is one initial outcome of applying Santilli's iso-number framework [2, 57, 58, 59, 60], is *fundamentally* compliant with the fact that Inopin's 2-sphere HR topology can be equipped with various configurations of complex and triplex order parameter fields to encode topological deformations for a wide spectrum of physical features [52]. In other words, the order parameter field is a powerful tool that enables us to define topological quantities and functions for states and transitions associated with, for example, iso-morphism and/or stereo-graphic projections [52, 56]. Additionally, Santilli [57, 58] *proved* that the infinite set of all possible ellipsoids arising from topologically-preserving deformations of a sphere in conventional Euclidean space over a conventional field are mapped to a corresponding Santilli iso-sphere in a Euclidean-Santillian iso-real space. More specifically, Santilli [57, 58] demonstrated that the mechanism for this is quite simple: the procedure consists of embedding all deformations $\hat{\kappa}\delta$ of Euclid's metric in the iso-unit $\hat{r} = \frac{1}{\hat{\kappa}} > 0$, such that $\delta = Diag.(1,1,1)$—any observer who looks at the resulting image of the ellipsoid under this map will see only *one* object: the perfect sphere with an amplitude-radius of $\hat{r}$. Therefore, given Lemmas 1.1–1.2 and 2.1–2.2, the infinite set of all possible ellipsoids arising from topologically-preserving deformations of $T^2 \subset Y$ are mapped to the corresponding $\hat{T}^2 \subset \hat{Y}$, where the positive-definite amplitude-radius is $r = \hat{r}$ and the corresponding curvature is $\kappa = \hat{\kappa} = \frac{1}{\hat{r}}$. To interpret this enlightening result in the case of, for example, Riemann geometry, we select a geodesic of Y with the 3x3 iso-metric $\hat{g}$ for Inopin's 2-sphere HR topology, then all infinitely possible Riemannian metrics G can be factorized in the form

$$G = \hat{\kappa} \times \hat{g}. \tag{41}$$

Thus, for T^2's great circle geodesic of T^1, the reformulation of Y in terms of $\hat{Y}$ with iso-unit $\hat{r}$ can be reduced to the analysis of only *one* circular element—namely Inopin's T^1 for $\hat{T}^1$—with a geodesic characterized by $\hat{g}$ because all infinitely possible circular elements in Y are obtained via (stereographic) projection! Immediately, we observe the resemblance of Santilli's *interior* and *exterior* dynamical systems [57, 58, 59, 60] to Y_- and Y_+, respectively, in terms of the iso-multiplicative unit $\hat{r}$ for the corresponding $\hat{Y}_-$ and $\hat{Y}_+$. Thus, we've defined an *iso-2-sphere HR topology* of $\hat{Y}$. At this point, we identify:

- **Lemma 3.1**: An Inopin 1-sphere HR $T^1 \subset Y$ (the great circle of the Inopin 2-sphere HR $T^2 \subset Y$) that is iso-metrically embedded in the Euclidean triplex space Y is one non-linear structure that satisfies the strict iso-curvature and iso-duality constraints of the iso-2-sphere HR topology, where $Y_- \subset Y$ and $Y_+ \subset Y$ correspond to Santilli-Inopin interior and exterior dynamical systems, respectively, such that the complete Y (with its said topological sub-spaces) are iso-morphic to their respective iso-topic liftings.

- **Lemma 3.2**: If a Euclidean triplex space Y that is iso-metrically equipped with the Inopin 2-sphere HR $T^2 \subset Y$ and its great circle, namely the Inopin 1-sphere HR $T^1 \subset Y$, is locally iso-morphic to its iso-topic lifting $\hat{Y}$ (iso-metrically equipped with the iso-2-sphere HR $\hat{T}^2$), then all forms of the iso-1-sphere HR $\hat{T}^1$ may be unified into one single iso-circular form on $\hat{Y}$, where $\hat{Y}_- \subset \hat{Y}$ and $\hat{Y}_+ \subset \hat{Y}$ are iso-dual because they are simultaneously dual to $\hat{T}^2$ for which $\hat{T}^1$ is the great circle.

The said geometric and non-linear unifications are parallel to that for Lie algebras because all possible, simple Lie algebras (compact and non-compact) of dimension d can be unified into one single *Lie-Santilli* iso-tope of dimension d (with the sole exclusion of exceptional algebra currently being investigated from other perspectives). Moreover, we achieve the capability of dealing with non-linear, non-local, and non-Hamiltonian systems at the Lie-Santilli level, which currently remains unattainable for the conventional Lie.

4 Conclusion

In this work, we started by briefly discussing the importance and development of number systems in terms of science. We touched on the five original number field axioms [2, 3], the abundance of fractals and chaos in nature, and acknowledged the significance of identifying a universal number classification system. Subsequently, we identified Santilli's four distinct data structure classes, namely the iso-numbers, the geno-numbers, the hyper-numbers, and the iso-dual numbers, [2, 57, 58, 59, 60], which are pertinent to an application assessment of the triplex numbers [52], triplex fractals, and Inopin's HR topology [52, 56].

Next, we conducted a preliminary upgrade to the triplex numbers and Euclidean triplex space of Inopin's 2-sphere HR topology [52] by engaging Santilli's four iso-number classes [2, 57, 58, 59, 60]. In doing so, we identified and defined the Santilli iso-triplex numbers to construct the Euclidean iso-triplex space for 3D iso-fractals, and subsequently demonstrated the existence of geno-triplex numbers, hyper-triplex numbers, and iso-dual-triplex numbers in a preparatory context. Afterwards, we provided a preliminary explanation on how Santilli's iso-mathematics [57, 58, 59, 60] apply to the

Inopin HR [52, 56] by defining the iso-2-sphere HR (with the iso-1-sphere HR as its great circle) to assemble the iso-2-sphere HR topology. Consequently, we stated an array of lemmas that aim to characterize these emerging mathematical structures.

In total, the resulting constructions of this venture are significant because they are cutting-edge, and therefore advance the frontiers of iso-mathematics to new realms of exploration and application. Hence, with the objective of further implementing these developments in the disciplines of science, technology, and engineering, we propose that a thorough and rigorous iso-mathematical investigation should be conducted along this research trajectory to challenge, upgrade, and generalize this emerging framework.

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