Effective state, Hawking radiation and quasi-normal modes for Kerr black holes

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Abstract: The non-strictly continuous character of the Hawking radiation spectrum generates a natural correspondence between Hawking radiation and black hole (BH) quasi-normal modes (QNM). In this work, we generalize recent results on this important issue to the framework of Kerr BHs (KBH). We show that also for the KBH, QNMs can be naturally interpreted in terms of quantum levels. Thus, the emission or absorption of a particle is in turn interpreted in terms of a transition between two different levels. At the end of the paper, we also generalize some concepts concerning the "effective state" of a KBH.

Contents

1 Introduction

The non-strictly thermal character [1, 2] of the Hawking radiation spectrum [3] shows that the emission of Hawking quanta is also a non-strictly continuous process, by enabling a natural correspondence between Hawking radiation and BH QNMs [4, 5].

Working with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units), in a strictly thermal approximation the probability of emission is [1–3]

$$\Gamma \sim \exp\left(-\frac{\omega}{T_H}\right), \tag{1.1}$$

where $T_H \equiv \frac{1}{8\pi M}$ is the Hawking temperature and ω is the energy-frequency of the emitted radiation.

By considering the important deviation from the strictly thermal character, the correct probability of emission is indeed [1, 2]

$$\Gamma \sim \exp\left[-\frac{\omega}{T_H}\left(1 - \frac{\omega}{2M}\right)\right]. \tag{1.2}$$

The additional term $\frac{\omega}{2M}$ takes into due account the conservation of energy, which arises from the fact that the BH contracts during the process of radiation [1, 2].

By introducing the *effective temperature* [4, 5]

$$T_E(\omega) \equiv \frac{2M}{2M - \omega}T_H = \frac{1}{4\pi(2M - \omega)} \,, \tag{1.3}$$

eq. (1.2) can be rewritten in Boltzmann-like form [4, 5]

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp\left(-\frac{\omega}{T_E(\omega)}\right), \tag{1.4}$$

where $\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}$ and $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor* appropriate for an object with inverse effective temperature $T_E(\omega)$ [4, 5]. The effective temperature represents the temperature of a strictly thermal body that would emit the same total

amount of radiation [4, 5] and the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ represents the deviation of the radiation spectrum of a BH from the strictly thermal feature [4, 5]. In other words, M is called the initial mass of the BH *before* the emission and $M - \omega$ is the final mass of the BH *after* the emission, where eqs. (1.2) and (1.3) permit the BH *effective mass* and *effective horizon* definitions *during* the particle emission, i.e. during the BH's contraction phase [4, 5]

$$M_E(\omega) \equiv M - \frac{\omega}{2}, \ r_E(\omega) \equiv 2M_E(\omega). \tag{1.5}$$

The *effective quantities* introduced above are *average quantities* between the two states b*efore and after the emission* [4, 5]. T_E is the inverse of the average value of the inverses of the initial and final Hawking temperatures *before* the emission $T_{H \text{ initial}} = \frac{1}{8\pi M}$ and after the emission $T_{H \text{ final}} = \frac{1}{8\pi(M-\omega)}$, respectively, while M_E is the average of the initial and final masses, and r_E is the average of the initial and final horizons [4, 5].

The interpretation of the particle emission is in terms of a *quantum transition* of frequency ω between the two discrete states before and after the emission [4, 5]. From the tunneling point of view, two separated classical turning points are joined by a trajectory in imaginary or complex time when a tunneling happens [1, 4, 5]. As a consequence, the radiation spectrum is also discrete. The reason is that, even if the statistical probability distribution (1.2) and the statistical energy distribution are continuous functions at a fixed Hawking temperature, such a Hawking temperature varies in time with a discrete character in (1.2). The size of the forbidden region that the tunneling particle traverses is finite [1] and this issue enables the introduction of the effective temperature (1.3). Indeed, in a strictly thermal approximation the turning points look to have null separation [1]. In that case, we do not know which joining trajectory needs to be considered [1]. In fact, there is not any barrier [1]. When the spectrum is instead not strictly thermal the tunneling particle traverses a finite forbidden region from $r_{\text{initial}} = 2M$ to $r_{\text{final}} = 2(M-\omega)$, which works like a barrier [1]. As a consequence, the Hawking temperature and the energy emissions are also discrete quantities.

We recall that the emitted energies are not only discrete, but also countable. In fact, they have been counted in [6, 7], where non-trivial correlations among radiations have been found in energies governed by the spectrum (1.2). The occurrence probability for a specific sequence of n subsequent energies $E_i = (E_1, E_2, \ldots, E_n)$ is [6, 7]

$$\Gamma(E_1, E_2, \ldots, E_n) = \Gamma\left(\sum_1^n E_i\right). \tag{1.6}$$

If one considers two emissions with energies E_1 and E_2, or one emission with energy $E_1 + E_2$, the function[1]

$$C\left[(E_1 + E_2), E_1, E_2\right] = \ln \Gamma(E_1 + E_2) - \ln \left[\Gamma(E_1)\Gamma(E_2)\right] = 8\pi E_1 E_2 \tag{1.7}$$

represents the statistical correlation between the emissions [6, 7].

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[1]Notice that refs. [23, 24] are the first papers where the Parikh-Wilczek method has been used in order to check if there are correlations between emitted quanta.

On the other hand, we recall that there are interesting proposals on the non-strictly continuous character of Hawking radiation in the literature [8, 9]. In general, quantum systems of finite size are inclined to have a discrete energy spectrum instead of a continuous one [8]. In fact, the dynamics of a BH responsible for the spectrum's character refer to both of the finite region enclosed by the horizon [8] and the finite size of the forbidden region that the tunneling particle traverses [1]. It is exactly such a finite size which makes the process of tunneling to be discrete instead of continuous [4, 5]. Hence, the BH energy spectrum is discrete [4, 5, 8, 9].

The discrete character of the emission process and of the emission spectrum implies a natural correspondence between Hawking radiation and BH QNMs [4, 5]. Hence, QNMs can be naturally interpreted in terms of quantum levels for both the emission and absorption of particles [4, 5, 10].

By calling l the angular momentum quantum number, the QNMs are usually labeled as ω_{nl}. For each l ($l \geq 2$ for gravitational perturbations), there is a second quantum number, namely the "overtone" one n ($n = 1, 2, \ldots$), which labels the countable sequence of QNMs [4, 5, 11–13]. For large n the QNMs of the Schwarzschild BH (SBH) become independent of l, and, in a strictly thermal approximation, have the following structure [4, 5, 11–13]

$$
\begin{aligned}
\omega_n &= \ln 3 \times T_H + 2\pi i \left(n + \frac{1}{2} \right) \times T_H + \mathcal{O}\left(n^{-\frac{1}{2}} \right) \\
&= \frac{\ln 3}{8\pi M} + \frac{2\pi i}{8\pi M} \left(n + \frac{1}{2} \right) + \mathcal{O}\left(n^{-\frac{1}{2}} \right).
\end{aligned}
\tag{1.8}
$$

Eq. (1.8) was originally obtained numerically in [14, 15]. It was re-obtained through an analytic proof in [16, 17].

The non-strictly thermal character of the BH spectrum permits us to replace eq. (1.8) with [4, 5, 10]

$$
\begin{aligned}
\omega_n &= \ln 3 \times T_E(|\omega_n|) + 2\pi i \left(n + \frac{1}{2} \right) \times T_E(|\omega_n|) + \mathcal{O}\left(n^{-\frac{1}{2}} \right) \\
&= \frac{\ln 3}{4\pi(2M - |\omega_n|)} + \frac{2\pi i}{4\pi(2M - |\omega_n|)} \left(n + \frac{1}{2} \right) + \mathcal{O}\left(n^{-\frac{1}{2}} \right).
\end{aligned}
\tag{1.9}
$$

The Hawking temperature in eq. (1.8) has been replaced by the effective temperature in eq. (1.9) [4, 5, 10]. The physical interpretation is that the deviation of the spectrum of BH QNMs from the strictly thermal feature implies that the spacing of the poles in (1.9) coincides with the spacing $2\pi i T_E(\omega)$ expected for a *non-thermal* Green's function as a dependence on the frequency is present, while in eq. (1.8) the spacing of the poles coincides with the spacing $2\pi i T_H$ for a *thermal* Green's function, see [4, 5, 10] for details.

The physical solutions for the absolute values of the frequencies in eq. (1.9) is [4, 5, 10]

$$
(\omega_0)_n \equiv |\omega_n| = M - \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 \left(n + \frac{1}{2} \right)^2}}.
\tag{1.10}
$$

In [4, 5, 10] this solution has been used to analyze important properties and quantities of the SBH, like the horizon area quantization, the area quanta number, the Bekenstein-Hawking entropy, its sub-leading corrections and the number of micro-states, i.e. quantities which are considered fundamental to realize the underlying unitary quantum gravity theory. In this work, we generalize the SBH results in [4, 5, 10] to the framework of KBHs.

2   Quasi-normal modes in Kerr black holes

Following [18], the strictly thermal approximation of KBH QNMs is given by [18–21]

$$\omega(n) = \tilde{\omega}_0 - i \left[4\pi T_0 \left(n + \frac{1}{2} \right) \right], \tag{2.1}$$

where $\tilde{\omega}_0$ is a function of the BH parameters [18]. Although T_0 is called "effective temperature" in [18], it is *not* the same concept of effective temperature introduced in [4, 5, 10] that we discuss in the present paper. Indeed, T_0 is a quantity introduced in [20] within the framework of Boltzmann weights and resonances. The two concepts must not be confused.

Calling J the angular momentum of the BH and assuming

$$M^2 \gg J \tag{2.2}$$

gives

$$T_0(J) \approx -\frac{T_H(J=0)}{2}, \tag{2.3}$$

where $T_H(J=0)$ is the Hawking temperature of the SBH. If one wants to go beyond the strictly thermal approximation, then the replacement $T_H \to T_E$ is needed, as T_E (instead of T_H) is the quantity associated to the emitted particle, i.e. the inverse of the average value of the inverses of the initial and final Hawking temperatures (*before* the emission and *after* the emission, respectively). Hence, eq. (2.3) becomes

$$T_0(J) \approx -\frac{T_E(J=0)}{2}, \tag{2.4}$$

where $T_E(J=0)$ is the effective temperature of the SBH given by eq. (1.3).

As we are interested in highly excited BHs, i.e. n is large, the imaginary part of eq. (2.1) becomes dominant. Thus, setting $(\omega_0)_n \equiv |\omega(n)|$, by using eqs. (2.1) and (2.4) we immediately get

$$\triangle M_n = (\omega_0)_{n-1} - (\omega_0)_n = 4\pi T_0 = -2\pi T_E(J=0), \tag{2.5}$$

for an emission involving the quantum levels n and $n-1$.

The result (2.5) is totally consistent with the results in [4, 5, 10] for the SBH. In fact, in [4, 5, 10] we find

$$\triangle M_n = (\omega_0)_{n-1} - (\omega_0)_n$$
$$= \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 \left(n + \frac{1}{2} \right)^2}} - \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 \left(n - \frac{1}{2} \right)^2}}, \tag{2.6}$$

which for large n becomes

$$\triangle M_n \approx \sqrt{M^2 - \frac{1}{2}\left(n + \frac{1}{2}\right)} - \sqrt{M^2 - \frac{1}{2}\left(n - \frac{1}{2}\right)}. \tag{2.7}$$

On the other hand, by recalling that, as the BH's mass is decreasing due to emissions of Hawking quanta, the BH's mass becomes [4, 5, 10]

$$M_{n-1} \equiv \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2\left(n - \frac{1}{2}\right)^2}} \tag{2.8}$$

and

$$M_n \equiv \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2\left(n + \frac{1}{2}\right)^2}}, \tag{2.9}$$

at the levels $n - 1$ and n, respectively. By using eq. (1.3), we find that the BH's effective temperature for an emission involving the quantum levels n and $n - 1$ is given by

$$
\begin{aligned}
T_E(\omega_n) &= \frac{1}{4\pi(2M - \omega_n)} = \frac{1}{8\pi M_E(\omega_n)} \\
&= \frac{1}{4\pi\left[\sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2\left(n - \frac{1}{2}\right)^2}} + \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2\left(n + \frac{1}{2}\right)^2}}\right]}.
\end{aligned}
\tag{2.10}
$$

For large n eq. (2.10) becomes

$$T_E(\omega_n) \approx \frac{1}{4\pi\left[\sqrt{M^2 - \frac{1}{2}\left(n + \frac{1}{2}\right)} + \sqrt{M^2 - \frac{1}{2}\left(n - \frac{1}{2}\right)}\right]}. \tag{2.11}$$

Thus, by combining eqs. (2.5), (2.7) and (2.11), we see that the result (2.5) is consistent with the results in [4, 5, 9] for the SBH if

$$\sqrt{M^2 - \frac{1}{2}\left(n + \frac{1}{2}\right)} - \sqrt{M^2 - \frac{1}{2}\left(n - \frac{1}{2}\right)} = -\frac{\frac{1}{2}}{\sqrt{M^2 - \frac{1}{2}\left(n + \frac{1}{2}\right)} + \sqrt{M^2 - \frac{1}{2}\left(n - \frac{1}{2}\right)}}. \tag{2.12}$$

By multiplying each side of eq. (2.12) by $\sqrt{M^2 - \frac{1}{2}\left(n + \frac{1}{2}\right)} + \sqrt{M^2 - \frac{1}{2}\left(n - \frac{1}{2}\right)}$ one easily obtains the identity $-\frac{1}{2} = -\frac{1}{2}$.

3 Effective states of Kerr black holes

The introduction of the BH's *effective state* enables the establishment of additional effective quantities that should be important in the framework of BH physics. Here, for a KBH of original mass M, we define the effective state after a transition with QNM frequency ω.

Following [18], we define the KBH's *effective angular momentum* $J_E(\omega) \equiv M_E(\omega)\alpha_E(\omega)$, where $\alpha_E(\omega)$ is the KBH's *effective specific angular momentum*. Thus, the KBH's *outer and inner effective horizons* can be defined as

$$r_{E+}(\omega) \equiv M_E(\omega) + \sqrt{M_E^2(\omega) - \alpha_E^2(\omega)}$$
$$r_{E-}(\omega) \equiv M_E(\omega) - \sqrt{M_E^2(\omega) - \alpha_E^2(\omega)}. \tag{3.1}$$

Eq. (3.1) generalizes the results of eq. (1.2) in [18] to the non-strictly thermal case. The two expressions in eq. (3.1) are the roots of the KBH's *effective quantity*

$$\triangle_E(\omega) \equiv r^2 - 2M_E(\omega)r + \alpha_E^2(\omega). \tag{3.2}$$

If we also define

$$\Sigma_E(\omega) \equiv r^2 + \alpha_E^2(\omega)\cos^2\theta, \tag{3.3}$$

then we can introduce the KBH's effective line element

$$\left(ds^2\right)_E \equiv -\left(1 - \frac{2M_E(\omega)r}{\Sigma_E(\omega)}\right)dt^2 - \frac{4M_E(\omega)\alpha_E(\omega)r\sin^2\theta}{\Sigma_E(\omega)}dtd\varphi + \frac{\Sigma_E(\omega)}{\triangle_E(\omega)}dr^2$$
$$+ \Sigma_E(\omega)d\theta^2 + \left(r^2 + \alpha_E^2(\omega) + 2M_E(\omega)\alpha_E^2(\omega)r\sin^2\theta\right)\sin^2\theta d\varphi^2. \tag{3.4}$$

Eq. (3.4) takes into due account the dynamical geometry of the KBH which emits and/or absorbs particles.

The introduced effective quantities permit us to define the KBH's *effective angular velocity*

$$\Omega_E(\omega) \equiv \frac{\alpha_E(\omega)}{r_{E+}^2(\omega) + \alpha_E^2(\omega)}$$
$$= \frac{J_E(\omega)}{2M_E(\omega)\left(M_E^2(\omega) + \sqrt{M_E^4(\omega) - J_E^2(\omega)}\right)}. \tag{3.5}$$

Therefore, we can define the KBH's *effective horizon area*

$$A_E(\omega) \equiv 4\pi\left(r_{E+}^2(\omega) + \alpha_E^2(\omega)\right) = 8\pi\left(M_E^2(\omega) + \sqrt{M_E^4(\omega) - J_E^2(\omega)}\right), \tag{3.6}$$

which permits us to define the KBH's *effective temperature*

$$\left(T_{\text{KBH}}\right)_E(\omega) \equiv \frac{r_{E+}(\omega) - r_{E-}(\omega)}{A_E(\omega)}$$
$$= \frac{\sqrt{M_E^4(\omega) - J_E^2(\omega)}}{4\pi M_E(\omega)\left(M_E^2(\omega) + \sqrt{M_E^4(\omega) - J_E^2(\omega)}\right)}. \tag{3.7}$$

Now, the adiabatically invariant integral is written as [18, 22]

$$I = \int \frac{dM - \Omega dJ}{\omega}. \tag{3.8}$$

So how do we adjust Vagenas' eq. (3.8) to integrate with our KBH effective scenario? To answer this question we must rewrite eq. (3.8) to establish an effective formula that accepts the transition frequency ω as input. Then, the transition frequency given by eq. (2.5) permits to define the KBH's *effective adiabatic invariant* as

$$\begin{aligned}
I_E(\omega) &\equiv \int \frac{dM_E(\omega) - \Omega_E(\omega)dJ_E(\omega)}{2\pi T_E(\omega)} \\
&= 2\left(M_E^2(\omega) + \sqrt{M_E^4(\omega) - J_E^2(\omega)}\right) \\
&\quad - 2M_E^2(\omega)\log\left(M_E^2(\omega) + \sqrt{M_E^4(\omega) - J_E^2(\omega)}\right),
\end{aligned} \tag{3.9}$$

which generalizes the results of eqs. (3.3) and (3.4) in [18] to the non-strictly thermal case.

Using eq. (3.6), we can also generalize the result of eq. (3.5) in [18] to the non-strictly thermal case

$$I_E(\omega) = \frac{A_E(\omega)}{4\pi} - 2M_E^2(\omega)\log\left(\frac{A_E(\omega)}{8\pi}\right). \tag{3.10}$$

Let us consider a KBH of original mass M with the assumption (2.2). After a high number of emissions (and potential absorptions as the BH can capture neighboring particles), the mass of the BH changes from M to the quantity M_{n-1} of eq. (2.8) [10]. In the transition from the state with $n-1$ to the state with n the mass of the BH changes again from M_{n-1} to the quantity M_n of eq. (2.9) [10]. Now, the BH is excited at the level n. We define the effective state for an emission from the level $n-1$ to the level n, with emission frequency $\triangle M_n$. Therefore, the BH's effective mass is defined as

$$M_E(\triangle M_n) \equiv \frac{M_{n-1} + M_n}{2} = \frac{2M_n - \triangle M_n}{2} = M_n - \frac{\triangle M_n}{2}, \tag{3.11}$$

where the BH's effective horizon is defined as

$$r_E(\triangle M_n) \equiv 2M_E(\triangle M_n). \tag{3.12}$$

Clearly, an absorption from the level n to the level $n-1$ is now potentially possible. In that case, the BH's effective mass and the BH's effective horizon are the same.

Second, the KBH's effective angular momentum components are defined as

$$\alpha_E(\triangle M_n) \equiv \frac{J_E(\triangle M_n)}{M_E(\triangle M_n)} \tag{3.13}$$

and using eqs. (3.2)–(3.3) to obtain

$$\Delta_E(\triangle M_n) \equiv r^2 - 2M_E(\triangle M_n)r + \alpha_E^2(\triangle M_n) \tag{3.14}$$

and

$$\Sigma_E(\triangle M_n) \equiv r^2 + \alpha_E^2(\triangle M_n)\cos^2\theta. \tag{3.15}$$

Hence, eqs. (3.1), (3.11), and (3.13) enable us to construct the KBH's effective outer and inner horizons

$$\begin{aligned}
r_{E+}(\triangle M_n) &\equiv M_E(\triangle M_n) + \sqrt{M_E^2(\triangle M_n) - \alpha_E^2(\triangle M_n)} \\
r_{E-}(\triangle M_n) &\equiv M_E(\triangle M_n) - \sqrt{M_E^2(\triangle M_n) - \alpha_E^2(\triangle M_n)},
\end{aligned} \tag{3.16}$$

which permit us to rewrite eq. (3.4) as

$$
(ds^2)_E \equiv -\left(1 - \frac{2M_E(\triangle M_n)r}{\Sigma_E(\triangle M_n)}\right) dt^2 - \frac{4M_E(\triangle M_n)\alpha_E(\triangle M_n)r\sin^2\theta}{\Sigma_E(\triangle M_n)}dtd\varphi + \frac{\Sigma_E(\triangle M_n)}{\triangle_E(\triangle M_n)}dr^2
$$
$$
+ \Sigma_E(\triangle M_n)d\theta^2 + \left(r^2 + \alpha_E^2(\triangle M_n) + 2M_E(\triangle M_n)\alpha_E^2(\omega)r\sin^2\theta\right)\sin^2\theta d\varphi^2,
$$

$$(3.17)$$

that takes into due account the dynamical geometry of the KBH which emits or absorbs particles and the neighbouring quantum levels which are involved in the transition.

Thus far, the introduced effective quantities authorize us to rewrite the KBH's effective angular velocity in eq. (3.5) as

$$
\Omega_E(\triangle M_n) \equiv \frac{\alpha_E(\triangle M_n)}{r_{E+}^2(\triangle M_n) + \alpha_E^2(\triangle M_n)}
$$
$$
= \frac{J_E(\triangle M_n)}{2M_E(\triangle M_n)\left(M_E^2(\triangle M_n) + \sqrt{M_E^4(\triangle M_n) - J_E^2(\triangle M_n)}\right)},
$$

$$(3.18)$$

to rewrite the KBH's effective horizon area in eq. (3.6) as

$$
A_E(\triangle M_n) \equiv 4\pi\left(r_{E+}^2(\triangle M_n) + \alpha_E^2(\triangle M_n)\right)
$$
$$
= 8\pi\left(M_E^2(\triangle M_n) + \sqrt{M_E^4(\triangle M_n) - J_E^2(\triangle M_n)}\right)
$$

$$(3.19)$$

and to rewrite the KBH's effective temperature in eq. (3.7) as

$$
(T_{\mathrm{KBH}})_E(\triangle M_n) \equiv \frac{r_{E+}(\triangle M_n) - r_{E-}(\triangle M_n)}{A_E(\triangle M_n)}
$$
$$
= \frac{\sqrt{M_E^4(\triangle M_n) - J_E^2(\triangle M_n)}}{4\pi M_E(\triangle M_n)\left(M_E^2(\triangle M_n) + \sqrt{M_E^4(\triangle M_n) - J_E^2(\triangle M_n)}\right)}.
$$

$$(3.20)$$

The KBH's effective adiabatic invariant in eq. (3.10) can be rewritten as

$$
I_E(\triangle M_n) \equiv \frac{A_E(\triangle M_n)}{4\pi} - 2M_E^2(\triangle M_n)\log\left(\frac{A_E(\triangle M_n)}{8\pi}\right).
$$

$$(3.21)$$

Considering eq. (3.19), one can show that

$$
\Delta A_E(\triangle M_n) = 16\pi M_E(\triangle M_n)\left[1 + \left(1 - \frac{J_E^2(\triangle M_n)}{M_E^4(\triangle M_n)}\right)^{-\frac{1}{2}}\right]\triangle M_n,
$$

$$(3.22)$$

and therefore the KBH's *effective area quanta number* is defined as

$$
N_E(\triangle M_n) \equiv \frac{A_E(\triangle M_n)}{|\Delta A_E(\triangle M_n)|} = \frac{M_E(\triangle M_n)}{2\triangle M_n}\sqrt{1 - \frac{J_E^2(\triangle M_n)}{M_E^4(\triangle M_n)}},
$$

$$(3.23)$$

which let us identify the KBH's *effective Bekenstein-Hawking entropy* as

$$
\begin{aligned}
(S_{\text{KBH}})_{\text{BH}}(\triangle M_n) &\equiv \frac{A_E(\triangle M_n)}{4} \\
&= 4\pi N_E(\triangle M_n) M_E(\triangle M_n) \left[1 + \left(1 - \frac{J_E^2(\triangle M_n)}{M_E^4(\triangle M_n)} \right)^{-\frac{1}{2}} \right] \cdot \triangle M_n.
\end{aligned}
\tag{3.24}
$$

As one can confirm, for $M_E^2 \gg J_E$, the mentioned equations reduce to those in [4, 5] for the SBH.

We recall that, to the second order approximation, the BH's entropy contains three parts: the usual Bekenstein-Hawking entropy, and two sub-leading corrections, the logarithmic term and the inverse area term [4, 5]

$$
S_{\text{total}} = S_{\text{BH}} - \ln S_{\text{BH}} + \frac{3}{2A}.
\tag{3.25}
$$

If one wants to satisfy the underlying quantum gravity theory, the logarithmic and inverse area terms are requested [4, 5]. In fact, for a better understanding of a BH's entropy in quantum gravity it is imperative to go beyond Bekenstein-Hawking entropy and identify the sub-leading corrections [4, 5]. Hence, the KBH's *total effective entropy* is [4, 5]

$$
S_{\text{total}}(\triangle M_n) \equiv (S_{\text{KBH}})_{\text{BH}}(\triangle M_n) - \ln(S_{\text{KBH}})_{\text{BH}}(\triangle M_n) + \frac{3}{2A_E(\triangle M_n)}.
\tag{3.26}
$$

At this point, we have successfully defined the KBH's effective state.

We note that one can start with eq. (3.19) and show that eq. (3.24) is valid for $J_E \ll M_E^2$. In fact, for $J_E \ll M_E^2$ eq. (3.19) implies

$$
A_E(\triangle M_n) = 16\pi M_E^2(\triangle M_n).
\tag{3.27}
$$

Using the area law with eq. (3.27), one obtains

$$
(S_{\text{KBH}})_{\text{BH}}(\triangle M_n) = 4\pi M_E^2(\triangle M_n).
\tag{3.28}
$$

Using eq. (3.23), which, for $J_E \ll M_E^2$, becomes

$$
N_E(\triangle M_n) \equiv \frac{A_E(\triangle M_n)}{|\Delta A_E(\triangle M_n)|} = \frac{M_E(\triangle M_n)}{2\triangle M_n},
\tag{3.29}
$$

one can replace one of $M_E(\triangle M_n)$ in eq. (3.27) in the following manner

$$
(S_{\text{KBH}})_{\text{BH}}(\triangle M_n) = 4\pi M_E(2N_E(\triangle M_n)\triangle M_n),
\tag{3.30}
$$

which is in agreement with eq. (3.24) when condition (2.2) is imposed on eq. (3.24).

4 Conclusion remarks

In the first section of this paper, we briefly explained the important issue that the non-strictly continuous character of the Hawking radiation spectrum generates a natural correspondence between Hawking radiation and BH QNMs [4, 5, 10]. In doing so, we found that

exemplifying the discrete character of the BH energy spectrum, QNM transition process, and radiation spectrum [4, 5, 10] is essential to BH physics because it authorizes us to encode this information as quantized structures with well-defined effectives for mass, horizon, and temperature that are fundamental to recognizing the underlying unitary quantum gravity theory.

Next, we took into due account the non-strictly thermal character of the spectrum [4, 5, 10], which is also necessary to BH physics because it permits us to use the effective quantities in [4, 5] to generalize the SBH results in [4, 5, 10] to the KBH framework. In particular, we demonstrated in section 2 that QNMs can be naturally interpreted in terms of KBH quantum levels, where the obtained KBH results are in full agreement with the SBH results in [4, 5, 10]. Therefore, these findings are meaningful because the effective quantities in [4, 5, 10] have been achieved for the stable four dimensional SBH and KBH solutions in Einstein's general relativity.

In section 3, we used the effective quantities in [4, 5, 10] as the foundation on which to construct the "effective state" of a KBH by generalizing the non-strictly thermal case results in [18]. It is imperative to express the KBH's effective state because we need additional features and knowledge to consider in future experiments and observations.

Acknowledgments

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