An Elegant Proof that the Catalan's Constant is Irrational

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Who is as the wise man? And who knoweth the interpretation of a thing?

Ecclesiastes 8:1a

ABSTRACT. We use the contradiction method for prove that the Catalan's constant is irrational.

1. INTRODUCTION

In Mathematics, the Catalan's constant [1] is defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}. \tag{1.1}$$

It is not known whenever G is irrational, let alone transcendental. The first open problem is the subject of my paper.

The Catalan's constant was named after Eugène Charles Catalan (30 May 1814 – 14 February 1894), a French and Belgian mathematician.

2. THE PROOF

THEOREM. *The Catalan's constant is irrational.*

Proof. We will use the *reductio ad absurdum*.

By hypothesis, we suppose that G is a rational number. Of course, there exist two positive integers a and b, such that $G = a/b$, where, clearly, $b > 1$. Firstly, we define the number

$$x := \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \cdot \left| G - \sum_{n=0}^{b} \frac{(-1)^n}{(2n+1)^2} \right|. \tag{2.1}$$

If G is rational, then x is an integer. We substitute $G = a/b$ into this definition to find

$$\begin{aligned} x &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \cdot \left| \frac{a}{b} - \sum_{n=0}^{b} \frac{(-1)^n}{(2n+1)^2} \right| \\ &= \left| \frac{a(2b+1)!^2}{4^b (b!)^2} - \sum_{n=0}^{b} \frac{(-1)^n (2b+1)!^2}{(2n+1)^2 4^b b((b-1)!)^2} \right|. \end{aligned} \tag{2.2}$$

It is obvious that the first term is an integer; because, for $b > 1$, then $4^b (b!)^2 < (2b+1)!^2$. The second term is an integer; because, for $b > 1$, then $(2n+1)^2 4^b b((b-1)!)^2 < (2b+1)!^2$. Hence x is an integer.

We, now, demonstrate that $0 < x < 1$.

First, we demonstrate that x is strictly positive, we insert the series representation of G into the definition of x and we find

$$(2.3)\ x = \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \sum_{n=0}^{b}\frac{(-1)^n}{(2n+1)^2}\right|$$
$$= \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right| = \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{\cos(\pi n)}{(2n+1)^2}\right|$$
$$> \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\int_{b+1}^{\infty}\frac{\cos(\pi x)}{(2x+1)^2}dx\right|$$
$$= \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|-\frac{1}{4}\pi\,\mathrm{Ci}\left(\left(b+\frac{3}{2}\right)\pi\right) - \frac{\cos(\pi b)}{4b+6}\right| > 0.$$

On the other hand, for all terms with $2n+1 \geq 2b+2$, i.e., $2n \geq 2b+1$, we have the upper estimate

$$(2.4)\qquad \frac{(2b+1)!}{(2n+1)!} \leq \frac{1}{(2b+2)^{2n-2b}}.$$

This inequality is strict for every $2n+1 \geq 2b+3$, i.e., $n \geq b+1$. Thereof, we substitute (1.1) and (2.4) in (2.1)

$$(2.5)\qquad x = \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \sum_{n=0}^{b}\frac{(-1)^n}{(2n+1)^2}\right|$$
$$= \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right| < \frac{(2b+1)!^2}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)!^2}\right|$$
$$= \frac{1}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n(2b+1)!^2}{(2n+1)!^2}\right| < \frac{1}{4^b b((b-1)!)^2}\left|\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2b+2)^{2n-2b}}\right|$$
$$= \frac{1}{4^b b((b-1)!)^2}\left|-\frac{(-1)^b}{4b^2+8b+5}\right| < 1.$$

Since there is no integer strictly between 0 and 1, we have get in a contradiction, and so G must be irrational. □

REFERENCES

[1] http://en.wikipedia.org/wiki/Catalan's_constant, available in July 12, 2013.