Smarandache Multi-Spaces

&

Mathematical Combinatorics

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Xi An, March   2007

ABSTRACT

In this speech, I will prescribe limitation on following topics and mainly concentrate on myself works for Smarandache multi-spaces by a combinatorial manner in the past three years.

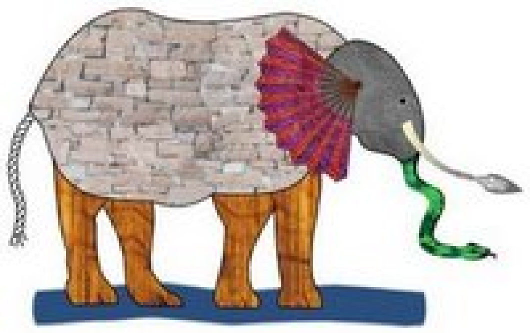
- *What is a Smarandache multi-space*?
- *Mixed non-euclidean geometries and Smarandache geometries, with Iseri's 2-dimensional model.*

- *Combinatorial maps with contribution to constructing general 2-dimensional Smarandache manifolds.*
- *Pseudo-manifolds for n-dimensional Smarandache manifolds endowed with a differential structure.*
- *Combinatorial manifolds endowed with a topological or differential structure and geometrical inclusions .*
- *Applications of multi-spaces to other sciences.*

1. What is a Smarandache multi-space?

- **Let's beginn from a famous proverb**

Six blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body.



- *the leg-toucher says it's like a pillar;*
- *the tail-toucher claims it's like a rope;*
- *the trunk-toucher compares it to a tree branch;*
- *the ear-toucher says it is like a hand fan;*
- *the belly-toucher asserts it's like a wall;*
- *the tusk-toucher insists it's like a solid pipe.*

Fig.1.1

All of you are right!

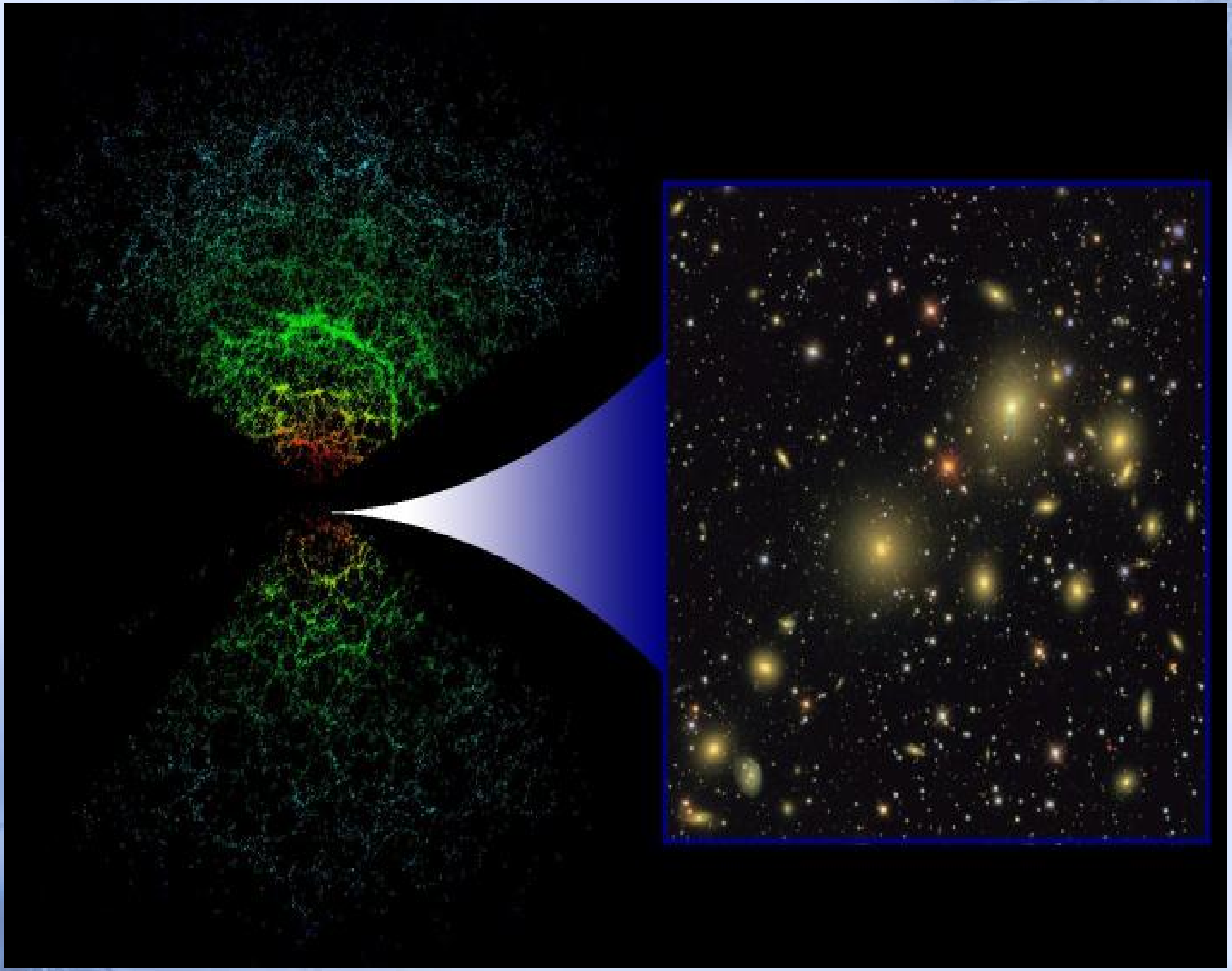
A wise man explains to them: *Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.*

- **A view of the sky by eyes of a man on the earth**

Fig. 1.2 A picture of the sky

- **A picture viewed by Hubble telescope(1995)**

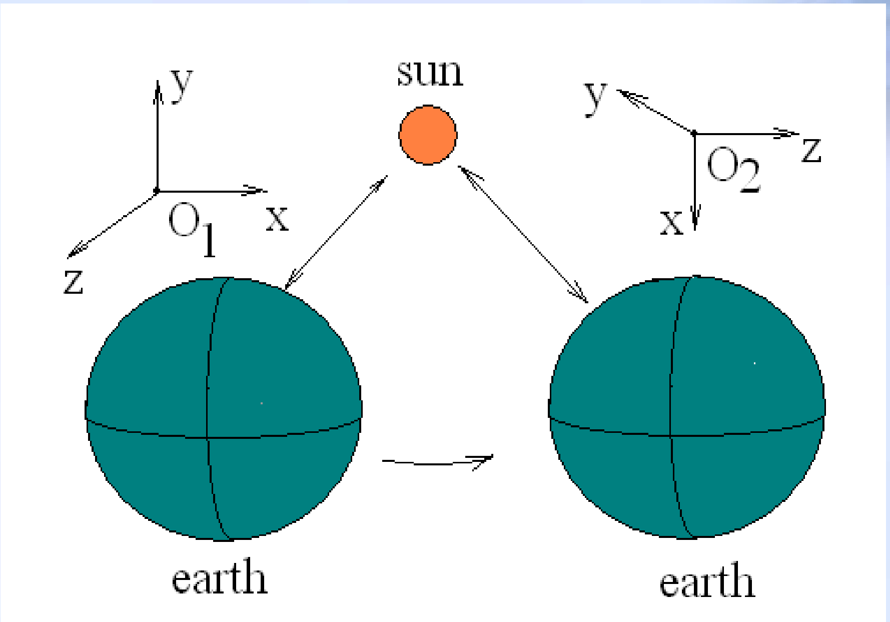
Fig.1.3



- **Einstein's general relativity**

  all laws of physics take the same form in any reference system.

Fig.1.4



- **Einstein's equivalence principle**

  there are no difference for physical effects of the inertial force and the gravitation in a field small enough.

- **Einstein's gravitational equation**

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

where $R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}_{\mu i\nu}$ and $R = g^{\nu\mu}R_{\nu\mu}$,

$$R^{\alpha}_{\mu i\nu} = \frac{\partial \Gamma^{i}_{\mu i}}{\partial x^{\nu}} - \frac{\partial \Gamma^{i}_{\mu\nu}}{\partial x^{i}} + \Gamma^{\alpha}_{\mu i}\Gamma^{i}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{i}_{\alpha i},$$

$$\Gamma^{g}_{mn} = \frac{1}{2}g^{pq}\left(\frac{\partial g_{mp}}{\partial u^{n}} + \frac{\partial g_{np}}{\partial u^{m}} - \frac{\partial g_{mn}}{\partial u^{p}}\right).$$

- **The big bang and evolution of the universe**

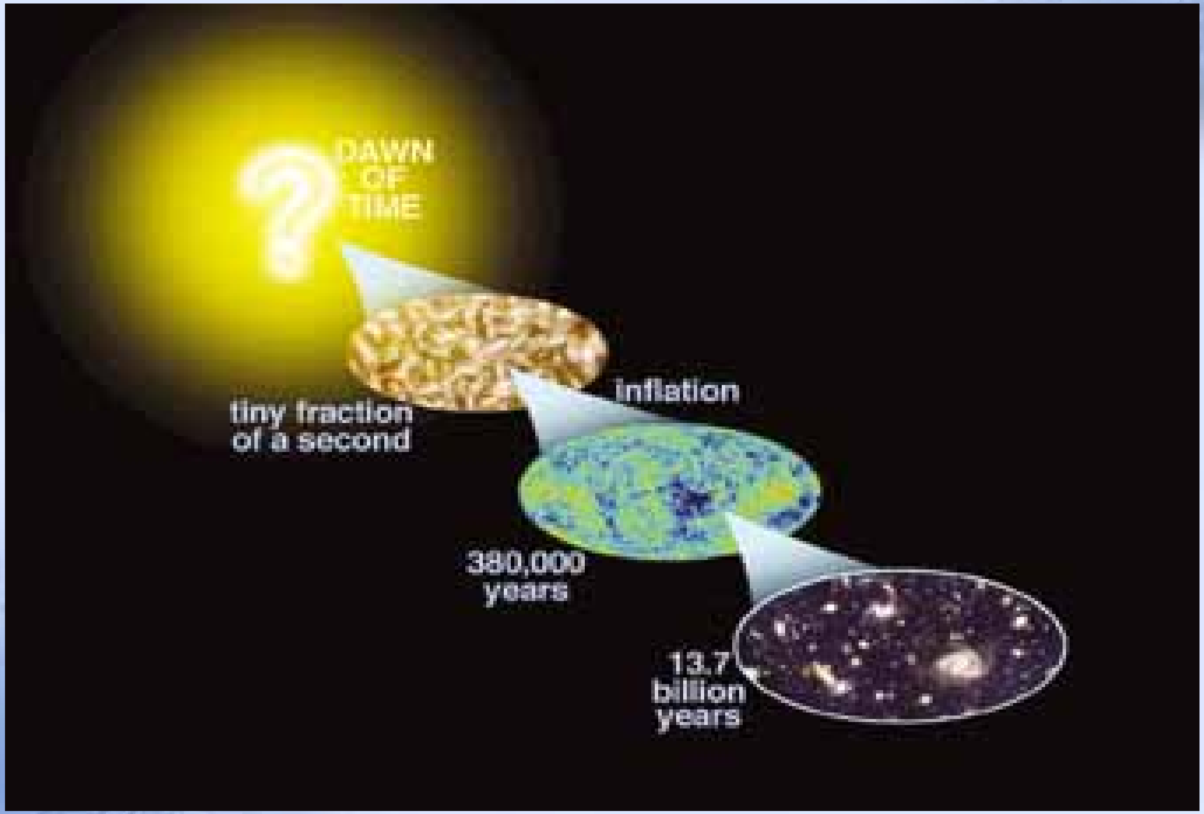
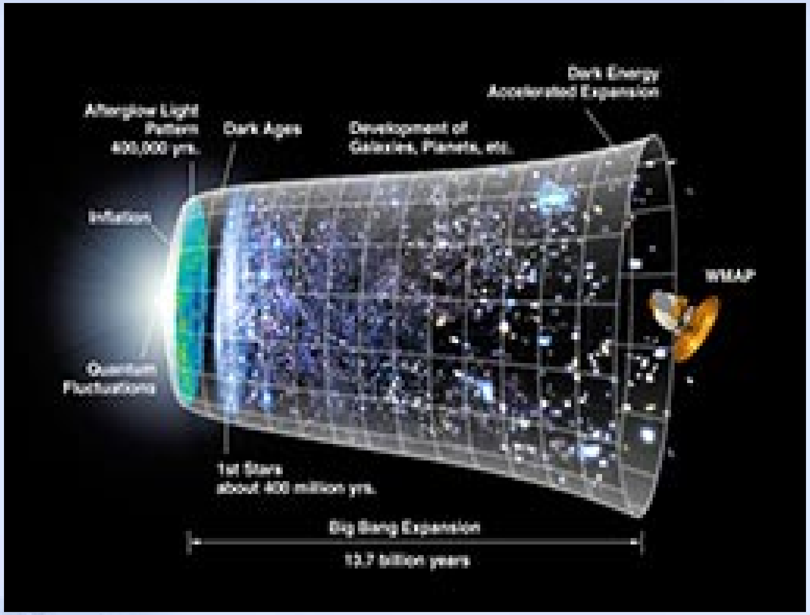
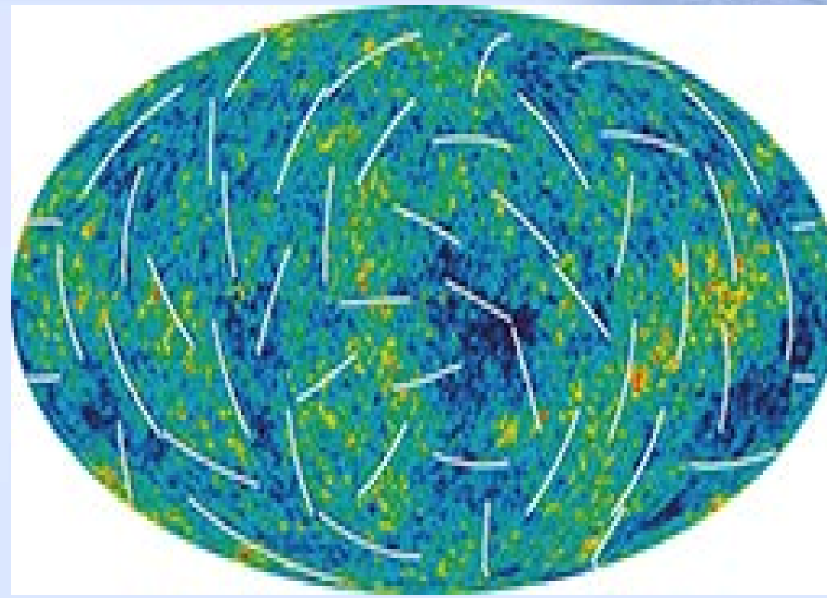


Fig.1.5 the big bang





CMBtimeline WMAP 2003

Fig.1.6

Peoples wined the Noble Prize for research on CMB:

A.Penzias and R.Wilson , *in physics in 1978*

G.F.Smoot and J.C.Mather, *in physics in 2006*

- **What is the right theory for the universe ∑?**

All matters are made of atoms and sub-atomic particles, held together by four fundamental forces: gravity, electromagnetism, strong nuclear force and weak force, partially explained by

Quantum Theory: *electromagnetism, strong nuclear force and weak force* (reduces forces to the exchange of discrete packet of quanta)

Relativity Theory: *gravity (*smooth deformation of the fabric spacetime)

- **Einstein's unification of fields**

find a unifying theory of fields to describe ***the*** *four fundamental forces, i.e., combine* **Quantum Theory** *and* **Relativity Theory.**

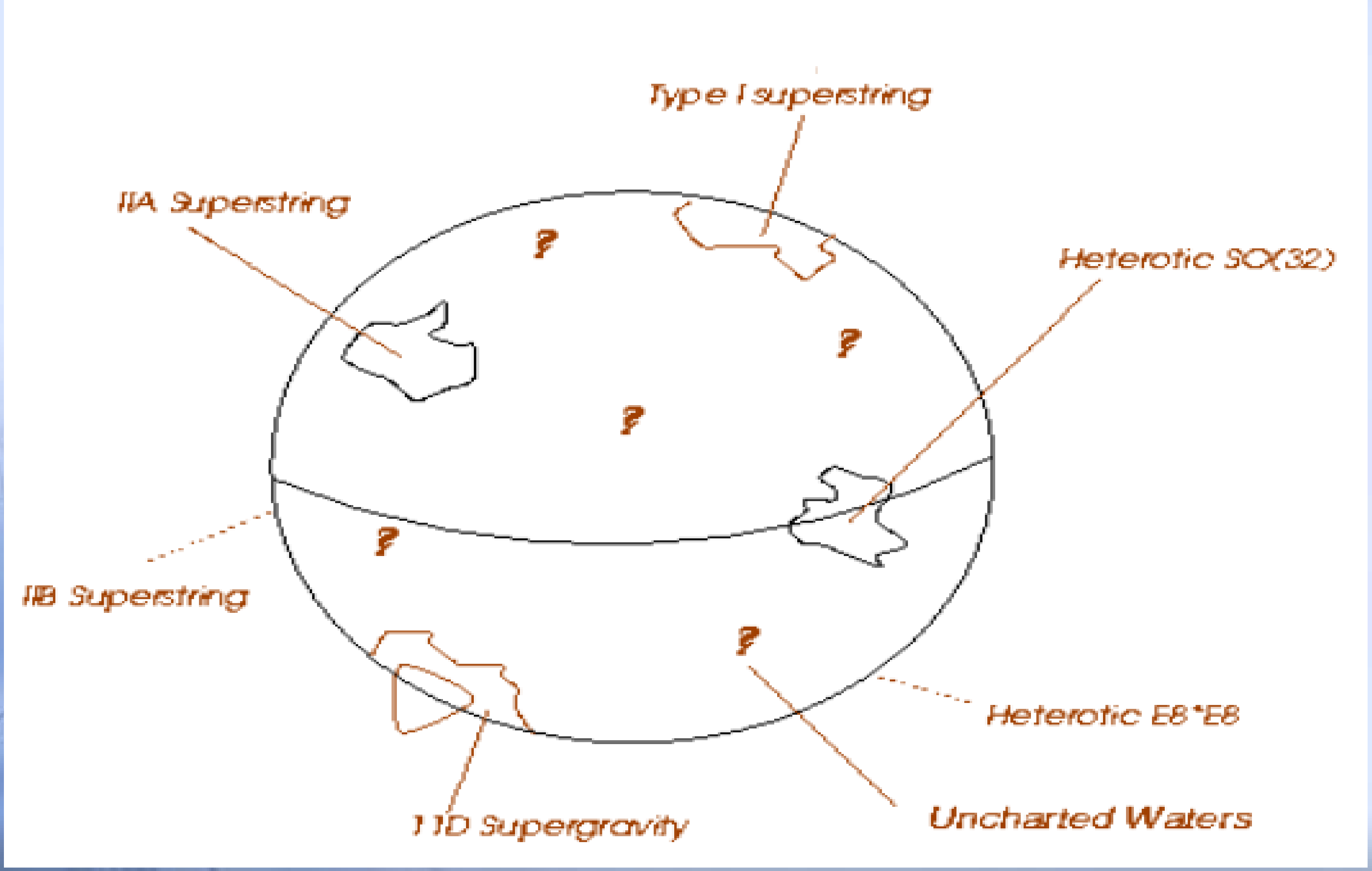
- **A description for the universe ∑ by *String theory***

The universe started as a perfect 10 dimensional space with nothing in it. However, this 10 dimensional space was unstable. The original 10 dimensional spacetime finally cracked into two pieces, a 4 and a 7 dimensional cosmos. The universe made the 7 of the 10 dimensions curled into a tiny ball, allowing the remaining 4 dimensional universe to inflate at enormous rates. It grows then at today's universe.

- **There are five string theory**

Types: $E_8 \times E_8$, SO(32), SO(32) I, IIA and IIB. Each of them is a extreme theory of M-theory.

Fig.1.7



- A right theory for the universe ∑

$$
\begin{aligned}
\sum \;=\; & \{E_8 \times E_8\ heterotic\ string\} \\
& \bigcup \{SO(32)\ heterotic\ string\} \\
& \bigcup \{SO(32)\ type\ I\ string\} \\
& \bigcup \{Type\ IIA\} \bigcup \{Type\ IIB\} \\
& \bigcup A \bigcup \cdots \bigcup B \cdots \bigcup C
\end{aligned}
$$

where A,…,B,…,C denote some unknown theories for the universe ∑.

- **What is a right theory for an objective Δ?**

The foundation of science is the measure and metric. Different characteristic A_i describes the different aspect Δ_i. A right theory for Δ should be

$$\triangle = \bigcup_{i \geq 1} \triangle_i = \bigcup_{i \geq 1} A_i$$

- **Definition of Smarandache multi-spaces**

A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer n≧2.

Whence Smarandache multi-spaces are mathematics for right theories of objectives.

- An example of Smarandache multi-spaces

Let n be an integer, $Z_1 = (\{0, 1, 2, \cdots, n-1\}, +)$ an additive group (mod$n$), $P = (0, 1, 2, \cdots, n-1)$ a permutation. For any integer $i, 0 \leq i \leq n-1$, define

$$Z_{i+1} = P^i(Z_1)$$

such that $P^i(k) +_i P^i(l) = P^i(m)$ in Z_{i+1} if $k + l = m$ in Z_1, where $+_i$ denotes the binary operation $+_i : (P^i(k), P^i(l)) \rightarrow P^i(m)$. Then

$$\bigcup_{i=1}^{n} Z_i$$

is a multi-space.

Mathematical Combinatorics

- Smarandache multi-spaces is really a combinatorial theory for mathematics, i.e., combine the same or different fields into a unifying one in logic.
- **Combinatorial Conjecture for mathematics** (Mao,2005)

  Every mathematical science can be reconstructed from or made by combinatorization.

This conjecture means that

(i) *One can selects finite combinatorial rulers to reconstruct or make generalization for classical mathematics.*

(ii) *One can combine different branches into a new theory and this process ended until it has been done for all mathematical sciences.*

2. Smaradache Geometries

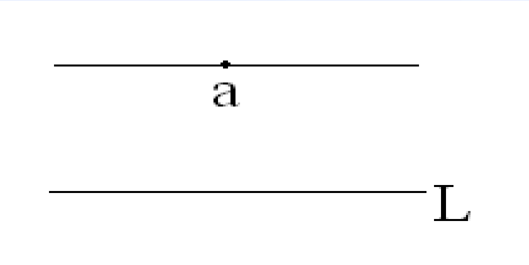
- Geometrical multi-space

A multi-metric space is a union $\cup_1^m M_i$ *such that each* M_i *is a space with metric* d_i *for any integer i,* $1 \leqq i \leqq m$.

- Mixed geometries

- **Axioms of Euclid geometry**

Fig.2.1



(A1) *there is a straight line between any two points.*

(A2) *a finite straight line can produce a infinite straight line continuously.*

(A3) *any point and a distance can describe a circle.*

(A4) *all right angles are equal to one another.*

(A5) *given a line and a point exterior this line, there is one line parallel to this line.*

- **Axioms of hyperbolic geometry**

(A1)-(A4) and (L5) following:

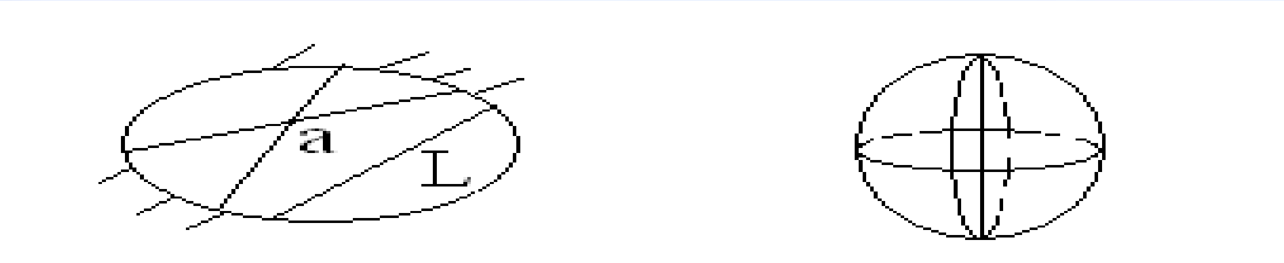
there are infinitely many lines parallel to a given line passing through an exterior point.

- **Axioms of Riemann geometry**

(A1)-(A4) and (R5) following:

there is no parallel to a given line passing through an exterior point.

Fig.2.2



(1) (2)

- **Question asked by Smarandache in 1969**

Are there other geometries by denying axioms in Euclid geometry not like the hyperbolic or Riemann geometry?

He also specified his question to the following questions:

Are there Paradoxist geometry, Non-Geometry, Counter-Projective Geometry, Anti-Geometry ?

- **Paradoxist geometry**

A geometry with axioms (A1)-(A4) and one of the following**:**

(1) *there are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.*

(2) *there are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.*

(3) *there are at least a straight line and a point exterior to it in this space for which only a finite number of lines* $L_1, L_2, \ldots, L_k$, $k \geqq 2$ *pass through the point and do not intersect the initial line.*

(4) *there are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.*

(5) *there are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.*

Non-geometry

A geometry by denial some axioms of (A1)-(A5).

(A1$^-$) *It is not always possible to draw a line from an arbitrary point to another arbitrary point.*

(A2$^-$) *It is not always possible to extend by continuity a finite line to an infinite line.*

(A3$^-$) *It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.*

(A4$^-$) *not all the right angles are congruent.*

(A5$^-$) *if a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.*

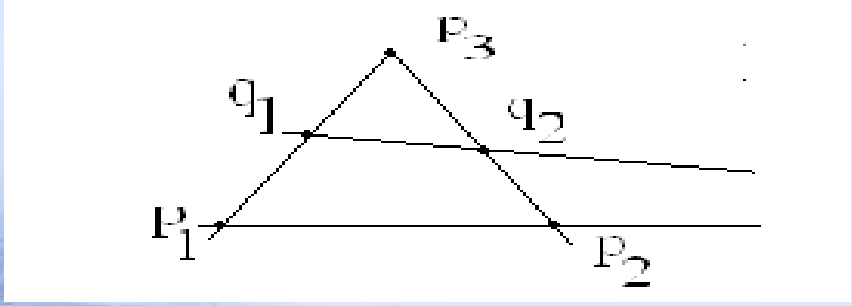
- **Counter-Projective geometry**

A geometry with these counter-axioms (C1)-(C3) following.

(C1) *there exist, either at least two lines, or no line, that contains two given distinct points.*

(C2) *let* p_1,p_2,p_3 *be three non-collinear points, and* q_1,q_2 *two distinct points. Suppose that* $\{p_1,q_1,p_3\}$ *and* $\{p_2,q_2,p_3\}$ *are collinear triples. Then the line containing* p_1, p_2 *and the line containing* q_1, q_2 *do not intersect.*

Fig.2.3



(C3) *every line contains at most two distinct points*.

- **Anti-geometry**

A geometry by denial some axioms of the Hilbert's 21 axioms of Euclidean geometry.

- **Definition of Smarandache geometries**

An axiom is said to be Smarandachely denied if it behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969).

- **Smarandache manifolds**

For any integer n, $n \geqq 1$, an n-manifold is a Hausdorff space M^n, i.e., a space that satisfies the T_2 separation axiom, such that for any $p \in M^n$, there is an open neighborhood U_p, $p \in U_p$ a subset of M^n and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$ or $\mathbf{C}^n$, respectively.

A Smarandache manifold is an n-dimensional manifold that support a Smarandache geometry.

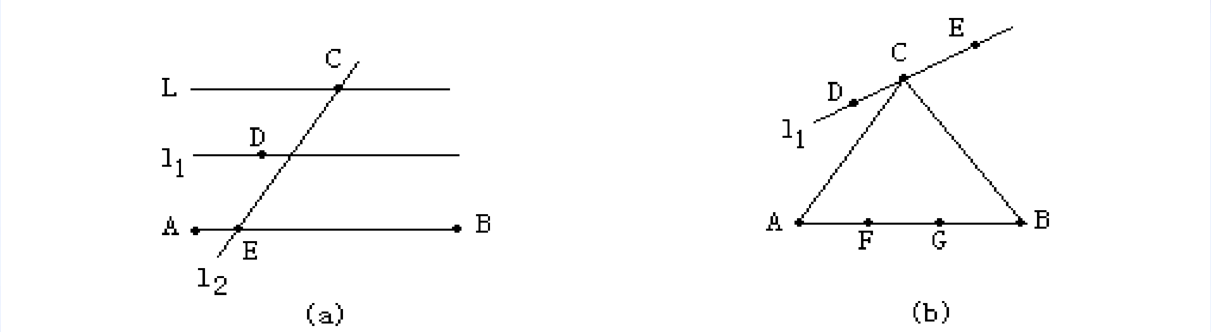
- **Question**

Can we construct Smarandache n-maniflods for any integer $n \geqq 2$?

An example of Smarandache geometries

Choose three non-collinear points A,B and C in an Euclid plane, points as all usual Euclidean point and lines any Euclidean line that passes through one and only one of points A,B and C.

Fig.2.4



Then the axiom that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel* and *no parallel*.

The axiom that through any two distinct points there exist one line passing through them is now replaced by; *one line* and *no line*.

Iseri's model for Smarandache manifolds

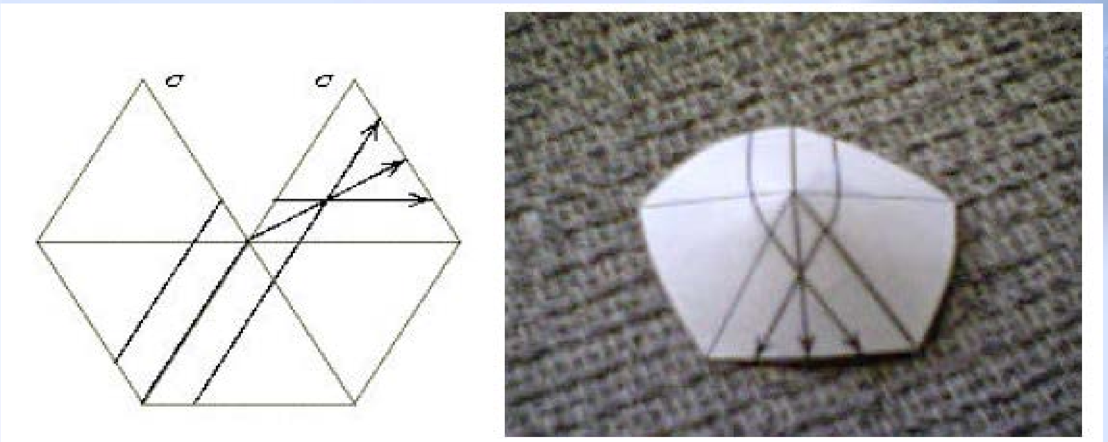
An s-manifold is any collection of these equilateral triangular disks T_i, $1 \leqq i \leqq n$ satisfying conditions following:

(i) each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks T_i, T_j, $1 \leqq i \leqq n$ and $i \neq j$;

(ii) each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks, called elliptic, euclidean or hyperbolic point.

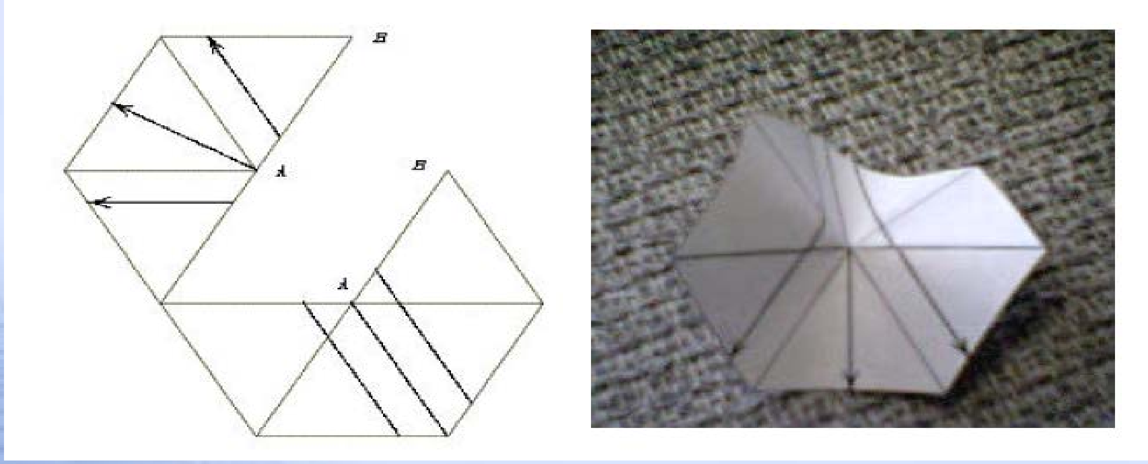
- **An elliptic point**

Fig.2.5



- **A hyperbolic point**

Fig.2.6



- **Iseri's result on s-manifolds**

There are Smarandache geometries, particularly, paradoxist geometries, non-geometries, counter-projective geometries and anti-geometries in s-manifolds.

See: Iseri's **Smarandache Manifolds,** American Research Press, 2002.

- **A classification for closed s-manifolds (Mao,2005)**

Let Δ_i, $1 \leqq i \leqq 7$ denote closed s-manifolds with valency 5, 6, 7, 5 and 6, 5 and 7, 6 and 7, 5,6 and 7 respectively.

- $|\Delta_i| = +\infty$ *for i = 2,3,4,6,7 and* $|\Delta_1| = 2$, $|\Delta_5| \geq 2$.

See Linfan Mao's **Automorphism groups of maps,surfaces and Smarandache geometries**, American Research Press, 2005.

3. Constructing Smarandache 2-manifolds

- **Maps and map geometries(Mao,2005)**

Iseri's model is Smarandache 2-manifolds. A generalization of his idea induced a general approach for constructing Smarandache 2-manifolds, i.e., map geometries on 2-manifolds.

- **What is a map?**

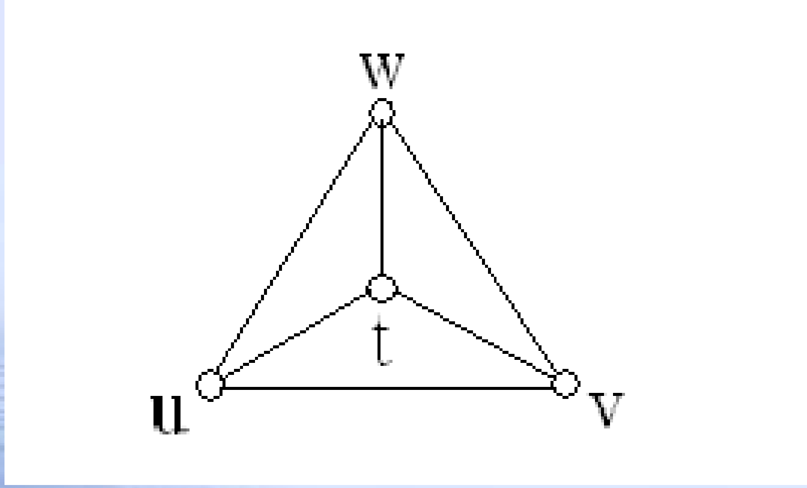
A map is a connected topological graph cellularly embedded in a 2-manifold (also called closed surface), which can be also seen as a cell-decomposition of closed surfaces.

- **What is a graph?**

A graph is a pair (V,E), where E is a subset of V×V.

A graph can be represented by a diagram on the plane, in which vertices are elements in V and two vertices u,v is connected by an edge e if and only if e=(u,v). As an example, the complete graph K^4 is shown in Fig.2.7.

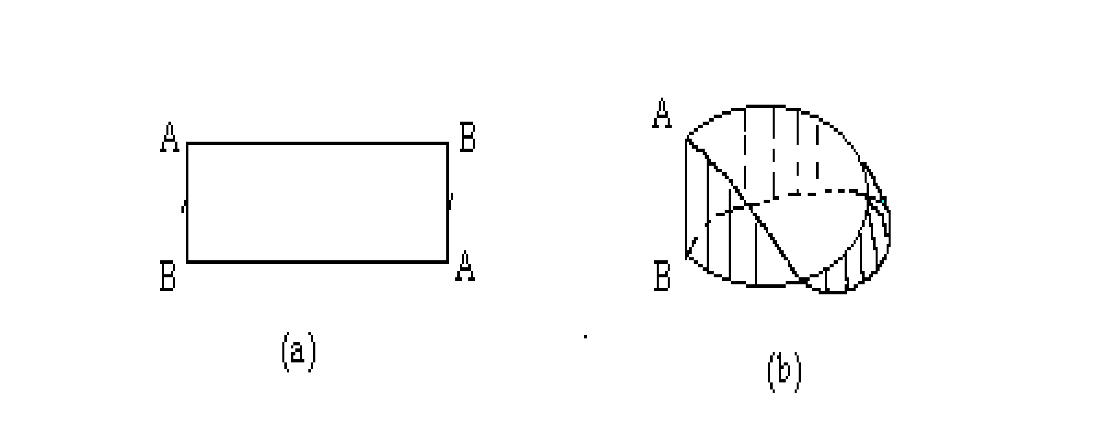
Fig.3.1



- **The classification theorem for 2-dimensional manifolds**

Each 2-manifold is homomorphic to the sphere P_0*, or to a 2-manifold* P_p *by adding p handles on* P_0*, or to a 2-manifold* N_q *by adding q crosscaps on* P_0*.*

Fig.3.2



The former is said an orientable 2-manifold of genus p and the later a non-orientable 2-manifold of genus q.

- **A polygon representation of 2-manifolds with even sides**

Any 2-manifold is homeomorphic to one of the standard 2=manifold following:

(P_0) *the sphere:* aa^{-1};

(P_n) *the connected sum of* $n, n \geqq 1$ *tori:*

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1};$$

(Q_n) *the connected sum of* $n, n \geqq 1$ *projective planes:*

$$a_1 a_1 a_2 a_2 \cdots a_n a_n.$$

- **Polygons of sphere, projective-plane, tours and Klein bottle**

Fig.3.3

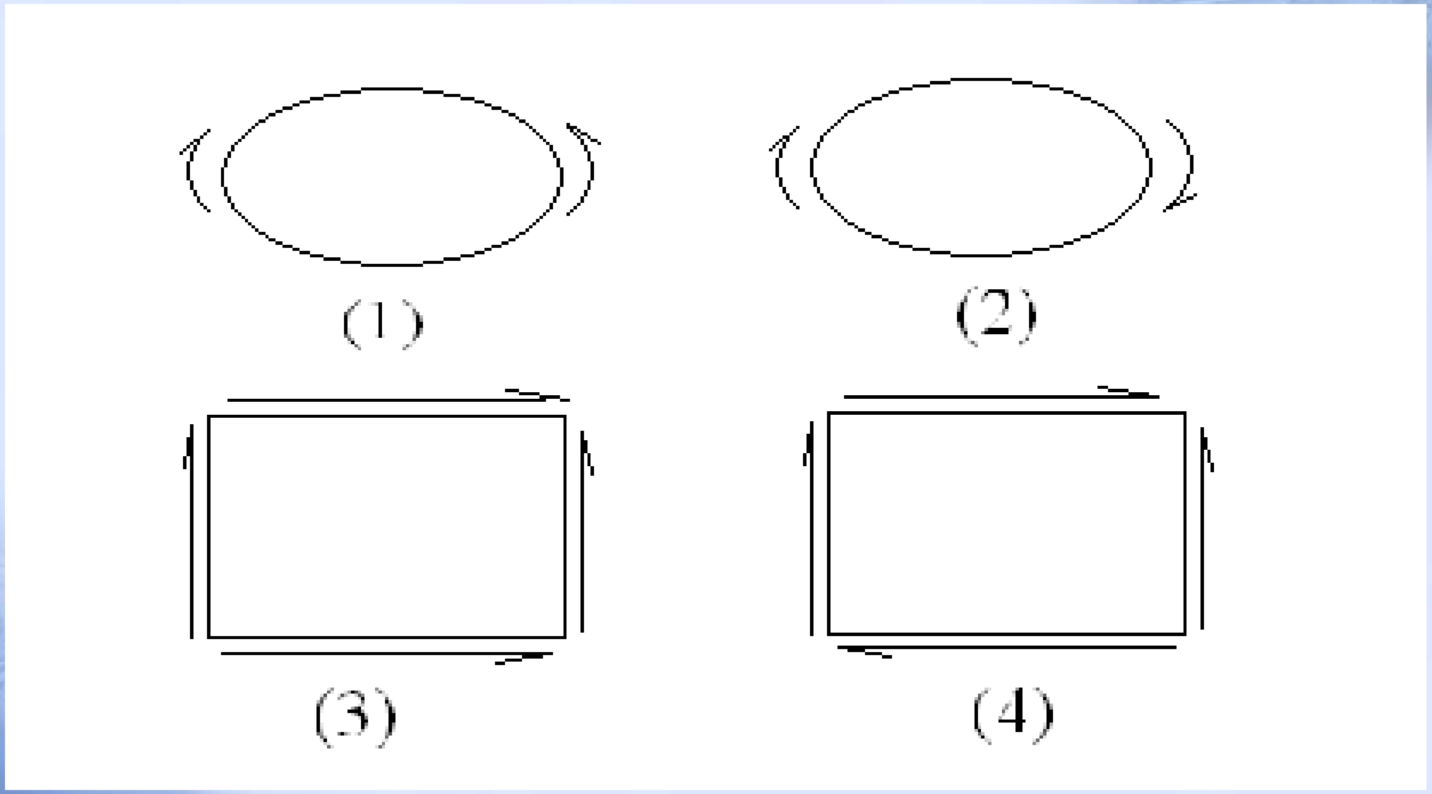


Fig.2.9

- **An example of K^4 on the Klein bottle**

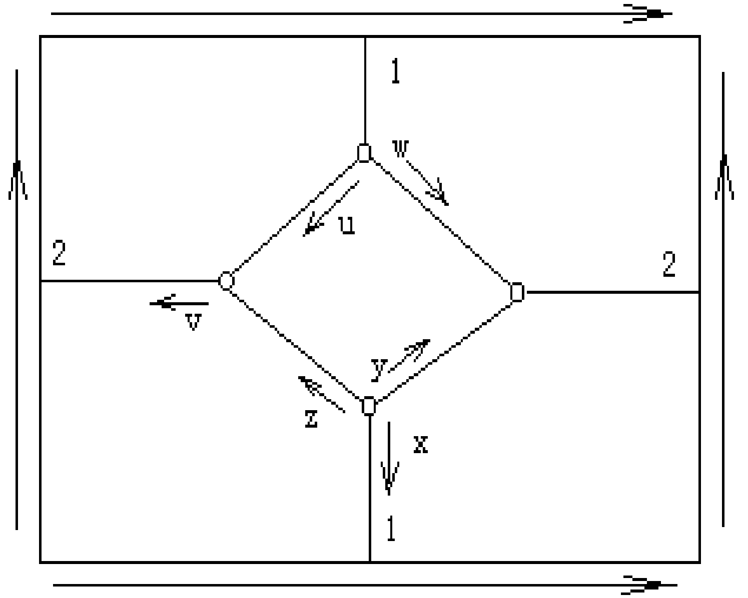
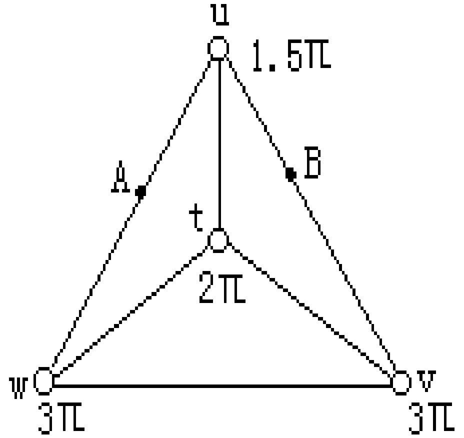


Fig.3.4 A graph K^4 on the Klein bottle

- **Definition of map geometries without boundary**

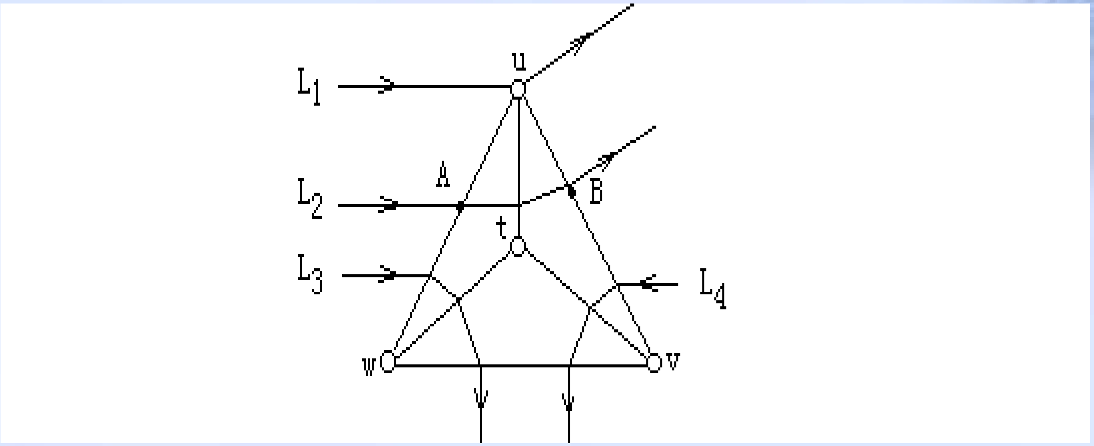
For a combinatorial map M with each vertex valency≧3, associates a real number $\mu(u)$, $0<\mu(u)< 4\pi/\mu(u)$ to each vertex u, $u\in V(M)$. Call (M,μ) a map geometry without boundary, $\mu(u)$ an angle factor of the vertex u and orientablle or non-orientable if M is orientable or not.

Fig.3.5



- **Lines behavior in the map geometry of Fig.3.5**

Fig.3.6

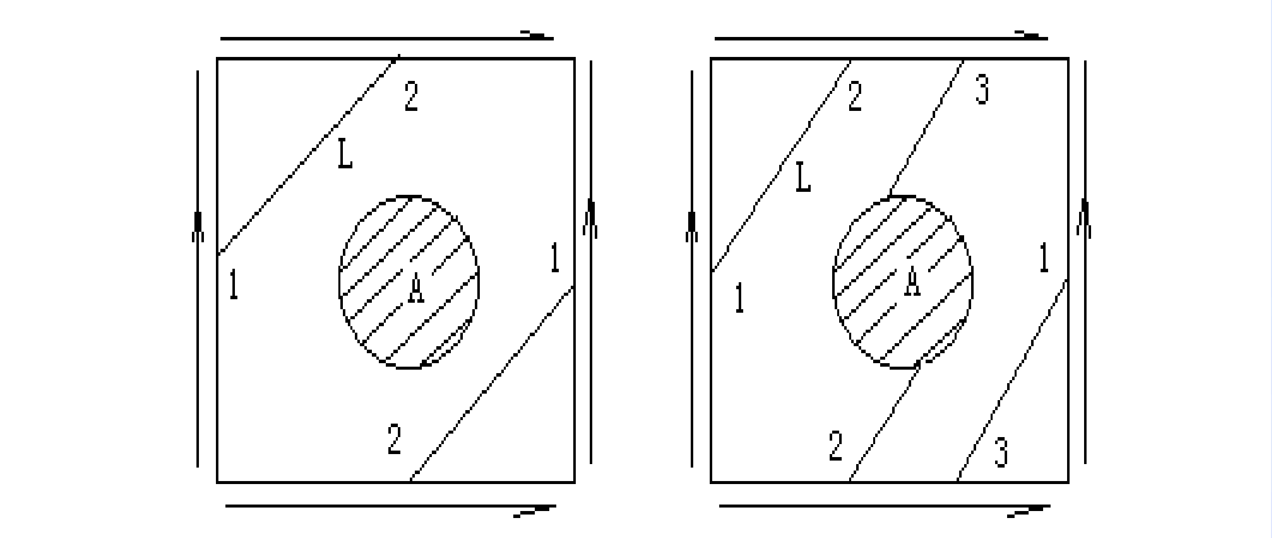


- **Definition of map geometries with boundary**

For a map geometry (M,μ) without boundary and faces $f_1, f_2,\ldots,f_k \in F(M), 1\leqq l\leqq \varphi(M)-1$, if $S(M)\setminus\{f_1,f_2,\ldots,f_k\}$ is connected, then call $(M,\mu)^{-k}=(S(M)\setminus\{f_1,f_2,\ldots,f_k\},\mu)$ a map geometry with boundary $f_1,f_2,\ldots,f_k$.

- **A representation of map geometries with boundary**

Fig.3.7



- **Theorem (Mao, 2005)**

(1)*For a map M on a 2-manifold with order*$\geqq 3$*, vertex valency*$\geqq 3$ *and a face* $f \in F(M)$*, there is an angle factor* μ *such that* (M,μ) *and* $(M,\mu)^{-1}$ *is a Smarandache geometry by denial the axiom (A5) with these axioms (A5),(L5) and (R5).*

(2)*There are non-geometries in map geometries with or without boundary.*

(3)*Unless axioms I-3, II-3, III-2, V-1 and V-2 in the Hilbert's axiom system for an Euclid geometry, an anti-geometry can be gotten from map geometries with or without boundary by denial other axioms in this axiom system.*

(4)*Unless the axiom (C3), a counter-projective geometry can be gotten from map geometries with or without boundary by denial axioms (C1) and (C2).*

4. Constructing Smarandache n-manifolds

- Pseudo-manifold geometries (Mao, 2006)

The idea applied in map geometries can be generalized to n-manifolds for Smarandache n-manifolds, which also enables us to affirm that Smarandache geometries include nearly all existent differential geometries, such as Finsler geometry and Riemannian geometry, etc..

Definition of differential n-manifolds

An *differential n-manifold* $(M^n, \mathcal{A})$ is an n-manifold M^n, $M^n = \bigcup_{i \in I} U_i$, endowed with a C^r differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n for an integer r with following conditions hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of M^n;

(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

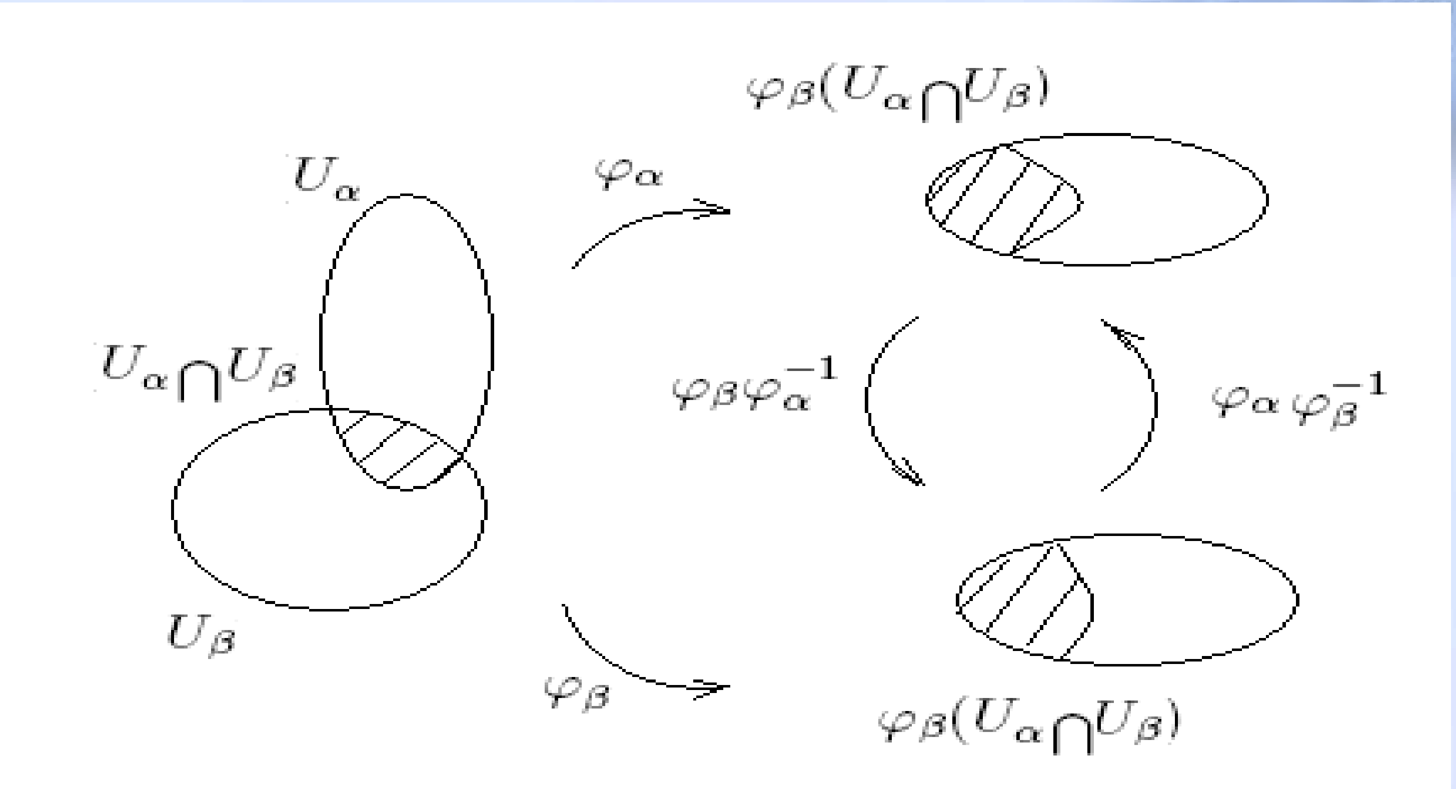
$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\beta) \ \text{ and } \ \varphi_\beta \varphi_\alpha^{-1} : \varphi_\beta(U_\alpha \bigcap U_\beta) \to \varphi_\alpha(U_\alpha)$$

are C^r;

(3) $\mathcal{A}$ is maximal, i.e., if (U, φ) is an atlas of M^n equivalent with one atlas in $\mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

An explanation for condition (2).

Fig.4.1



An n-manifold is *smooth* if it is endowed with a C^∞ differential structure.

Construction of pseudo-manifolds

Let M^n *be an* n*-manifold with an atlas* $\mathcal{A}=\{(U_p,\varphi_p)|p\in M^n\}$. *For* $\forall p\in M^n$ *with a local coordinates* $(x_1,x_2,\cdots,x_n)$,*define a spatially directional mapping* $\omega : p\rightarrow \mathbf{R}^n$ *action on* φ_p *by*

$$\omega : p \rightarrow \varphi_p^{\omega}(p)=\omega(\varphi_p(p))=(\omega_1,\omega_2,\cdots,\omega_n),$$

i.e., if a line L *passes through* $\varphi(p)$ *with direction angles* $\theta_1,\theta_2,\cdots,\theta_n$ *with axes* $\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n$ *in* $\mathbf{R}^n$, *then its direction becomes*

$$\theta_1-\frac{\vartheta_1}{2}+\sigma_1,\theta_2-\frac{\vartheta_2}{2}+\sigma_2,\cdots,\theta_n-\frac{\vartheta_n}{2}+\sigma_n$$

after passing through $\varphi_p(p)$, *where for any integer* $1\le i\le n$, $\omega_i\equiv\vartheta_i(mod 4\pi)$, $\vartheta_i\ge 0$ *and*

$$\sigma_i=\begin{cases}\pi, & if\ \ 0\le\omega_i<2\pi,\\ 0, & if\ \ 2\pi<\omega_i<4\pi.\end{cases}$$

A manifold M^n *endowed with such a spatially directional mapping* $\omega : M^n\rightarrow\mathbf{R}^n$ *is called an* n*-dimensional pseudo-manifold, denoted by* $(M^n,\mathcal{A}^{\omega})$.

Definition 4.1 *A spatially directional mapping* $\omega : M^n \to \mathbf{R}^n$ *is euclidean if for any point* $p \in M^n$ *with a local coordinates* $(x_1, x_2, \cdots, x_n)$, $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ *with* $k_i \equiv 1(mod2)$ *for* $1 \le i \le n$, *otherwise, non-euclidean.*

Definition 4.2 *Let* $\omega : M^n \to \mathbf{R}^n$ *be a spatially directional mapping and* $p \in (M^n, \mathcal{A}^\omega)$, $\omega(p)(mod4\pi) = (\omega_1, \omega_2, \cdots, \omega_n)$. *Call a point* p *elliptic, euclidean or hyperbolic in direction* $\mathbf{e}_i$, $1 \le i \le n$ *if* $o \le \omega_i < 2\pi$, $\omega_i = 2\pi$ *or* $2\pi < \omega_i < 4\pi$.

Corollary 4.1 *Let* $(M^n, \mathcal{A}^\omega)$ *be a pseudo-manifold. Then* $\varphi_p^\omega = \varphi_p$ *if and only if every point in* M^n *is euclidean.*

Theorem 4.2 *Let* $(M^n, \mathcal{A}^\omega)$ *be an n-dimensional pseudo-manifold and* $p \in M^n$. *If there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in* (U_p, φ_p), *then* $(M^n, \mathcal{A}^\omega)$ *is a Smarandache n-manifold.*

Basis of tangent space of a pseudo-manifold at a point

Theorem 4.3 *For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local alta (U_p, φ_p), $\varphi_p(p) = (x_1^0 x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of T_pM^n is*

$$\dim T_pM^n = 2n - s$$

with a basis

$$\{\frac{\partial}{\partial x^{i_j}}|_p \mid 1 \le j \le s\} \bigcup \{\frac{\partial^-}{\partial x^l}|_p, \frac{\partial^+}{\partial x^l}|_p \mid 1 \le l \le n \text{ and } l \ne i_j, 1 \le j \le s\}.$$

Basis of cotangent space of a pseudo-manifold at a point

Theorem 4.4 *For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local alta (U_p, φ_p), $\varphi_p(p) = (x_1^0 x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of $T_p^* M^n$ is*

$$\dim T_p^* M^n = 2n - s$$

with a basis

$$\{dx^{i_j}|_p \mid 1 \leq j \leq s\} \bigcup \{d^- x_p^l, d^+ x^l|_p \mid 1 \leq l \leq n \text{ and } l \neq i_j, 1 \leq j \leq s\},$$

where

$$dx^i|_p(\frac{\partial}{\partial x^j}|_p) = \delta_j^i \text{ and } d^{\epsilon_i} x^i|_p(\frac{\partial^{\epsilon_i}}{\partial x^j}|_p) = \delta_j^i$$

for $\epsilon_i \in \{+, -\}, 1 \leq i \leq n$.

Definition 4.3 *A Minkowski norm on a vector space* V *is a function* $F : V \to \mathbf{R}$ *such that*

(1) F *is smooth on* $V \setminus \{0\}$ *and* $F(v) \geq 0$ *for* $\forall v \in V$*;*

(2) F *is* 1*-homogenous, i.e.,* $F(\lambda v) = \lambda F(v)$ *for* $\forall \lambda > 0$*;*

(3) *for all* $y \in V \setminus \{0\}$*, the symmetric bilinear form* $g_y : V \times V \to \mathbf{R}$ *with*

$$g_y(u, v) = \sum_{i,j} \frac{\partial^2 F(y)}{\partial y^i \partial y^j}$$

is positive definite for $u, v \in V$*.*

Denote by $TM^n = \bigcup_{p \in (M^n, \mathcal{A}^\omega)} T_p M^n$.

Definition 4.4 *A pseudo-manifold geometry is a pseudo-manifold* $(M^n, \mathcal{A}^\omega)$ *endowed with a Minkowski norm* F *on* TM^n*.*

Theorem 4.5 *There are pseudo-manifold geometries.*

Definition 4.5 *A principal fiber bundle (PFB) consists of a pseudo-manifold* $(P, \mathcal{A}_1^{\omega})$, *a projection* $\pi : (P, \mathcal{A}_1^{\omega}) \to (M, \mathcal{A}_0^{\pi(\omega)})$, *a base pseudo-manifold* $(M, \mathcal{A}_0^{\pi(\omega)})$ *and a Lie group* G, *denoted by* (P, M, ω^{π}, G) *such that (1), (2) and (3) following hold.*

(1) *There is a right freely action of* G *on* $(P, \mathcal{A}_1^{\omega})$, *i.e., for* $\forall g \in G$, *there is a diffeomorphism* $R_g : (P, \mathcal{A}_1^{\omega}) \to (P, \mathcal{A}_1^{\omega})$ *with* $R_g(p^{\omega}) = p^{\omega} g$ *for* $\forall p \in (P, \mathcal{A}_1^{\omega})$ *such that* $p^{\omega}(g_1 g_2) = (p^{\omega} g_1) g_2$ *for* $\forall p \in (P, \mathcal{A}_1^{\omega})$, $\forall g_1, g_2 \in G$ *and* $p^{\omega} e = p^{\omega}$ *for some* $p \in (P^n, \mathcal{A}_1^{\omega})$, $e \in G$ *if and only if* e *is the identity element of* G.

(2) *The map* $\pi : (P, \mathcal{A}_1^{\omega}) \to (M, \mathcal{A}_0^{\pi(\omega)})$ *is onto with* $\pi^{-1}(\pi(p)) = \{pg | g \in G\}$, $\pi\omega_1 = \omega_0 \pi$, *and regular on spatial directions of* p, *i.e., if the spatial directions of* p *are* $(\omega_1, \omega_2, \cdots, \omega_n)$, *then* ω_i *and* $\pi(\omega_i)$ *are both elliptic, or euclidean, or hyperbolic and* $|\pi^{-1}(\pi(\omega_i))|$ *is a constant number independent of* p *for any integer* $i, 1 \leq i \leq n$.

(3) *For* $\forall x \in (M, \mathcal{A}_0^{\pi(\omega)})$ *there is an open set* U *with* $x \in U$ *and a diffeomorphism* $T_u^{\pi(\omega)} : (\pi)^{-1}(U^{\pi(\omega)}) \to U^{\pi(\omega)} \times G$ *of the form* $T_u(p) = (\pi(p^{\omega}), s_u(p^{\omega}))$, *where* $s_u : \pi^{-1}(U^{\pi(\omega)}) \to G$ *has the property* $s_u(p^{\omega} g) = s_u(p^{\omega}) g$ *for* $\forall g \in G, p \in \pi^{-1}(U)$.

Definition 4.6 *Let* (P, M, ω^{π}, G) *be a* PFB *with* $\dim G = r$. *A subspace family* $H = \{H_p | p \in (P, \mathcal{A}_1^{\omega}), \dim H_p = \dim T_{\pi(p)} M\}$ *of* TP *is called a connection if conditions (1) and (2) following hold.*

(1) *For* $\forall p \in (P, \mathcal{A}_1^{\omega})$, *there is a decomposition*

$$T_pP = H_p \bigoplus V_p$$

and the restriction $\pi_*|_{H_p} : H_p \to T_{\pi(p)}M$ *is a linear isomorphism.*

(2) H *is invariant under the right action of* G, *i.e., for* $p \in (P, \mathcal{A}_1^{\omega})$, $\forall g \in G$,

$$(R_g)_{*p}(H_p) = H_{pg}.$$

Theorem 4.6 (dimensional formula) *Let* (P, M, ω^{π}, G) *be a* PFB *with a connection* H. *For* $\forall p \in (P, \mathcal{A}_1^{\omega})$, *if the number of euclidean directions of* p *is* $\lambda_P(p)$, *then*

$$\dim V_p = \frac{(\dim P - \dim M)(2\dim P - \lambda_P(p))}{\dim P}.$$

Geometrical inclusions in Smarandache geometries

Theorem 4.7 *There are inclusions among Smarandache geometries, Finsler geometry, Riemann geometry and Weyl geometry:*

$$\{Smarandache\ geometries\} \supset \{pseudo-manifold\ geometries\}$$
$$\supset \{Finsler\ geometry\} \supset \{Riemann\ geometry\} \supset \{Weyl\ geometry\}.$$

Theorem 4.8 *There are inclusions among Smarandache geometries, pseudo-manifold geometry and Kähler geometry:*

$$\{Smarandache\ geometries\} \supset \{pseudo-manifold\ geometries\}$$
$$\supset \{Kähler\ geometry\}.$$

5. Geometry on Combinatorial manifolds (Mao,2006-2007)

The combinatorial speculation in Smarandache multi-spaces enables us to consider these geometrical objects consisted by manifolds with different dimensions, i.e., combinatorial manifolds. Certainly,each combinatorial manifold is a Smarandache manifold itself. Similar to the construction of Riemannian geometry, by introducing metrics on combinatorial manifolds, we can construct topological or differential structures on them and obtained an entirely new geometrical theory, which also convinces us those inclusions of geometries in Smarandache geometries established in Section 4 again.

- What is a combinatorial manifold?

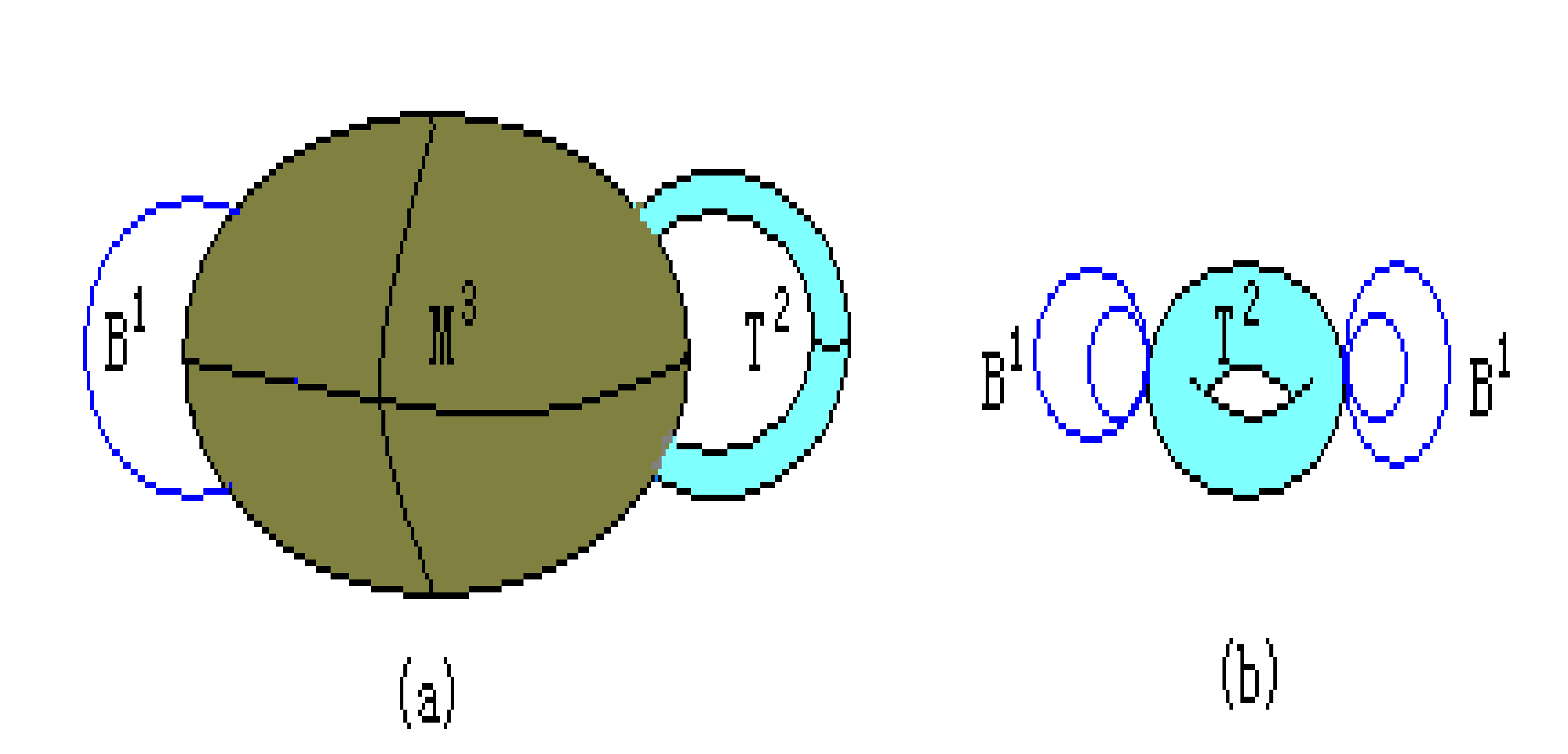


Fig.5.1

Definition 5.1 *For a given integer sequence* $n_1, n_2, \cdots, n_m, m \geq 1$ *with* $0 < n_1 < n_2 < \cdots < n_s$, *a combinatorial manifold* $\widetilde{M}$ *is a Hausdorff space such that for any point* $p \in \widetilde{M}$, *there is a local chart* (U_p, φ_p) *of* p, *i.e., an open neighborhood* U_p *of* p *in* $\widetilde{M}$ *and a homoeomorphism* $\varphi_p : U_p \to \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ *with* $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ *and* $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, *denoted by* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *or* $\widetilde{M}$ *on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. *The maximum value of* $s(p)$ *and the dimension* $\widehat{s}(p)$ *of* $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ *are called the dimension and the intersectional dimensional of* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *at the point* p, *respectively.*

- Topological structures

- **D-Connectedness**

Definition 5.2 *For two points p, q in a finitely combinatorial manifold $M(n_1, n_2, \cdots, n_m)$, if there is a sequence $B_1, B_2, \cdots, B_s$ of d-dimensional open balls with two conditions following hold.*

(1) $B_i \subset \widetilde{M}(n_1, n_2, \cdots, n_m)$ *for any integer* $i, 1 \leq i \leq s$ *and* $p \in B_1$, $q \in B_s$;

(2) *The dimensional number* $\dim(B_i \bigcap B_{i+1}) \geq d$ *for* $\forall i, 1 \leq i \leq s-1$.

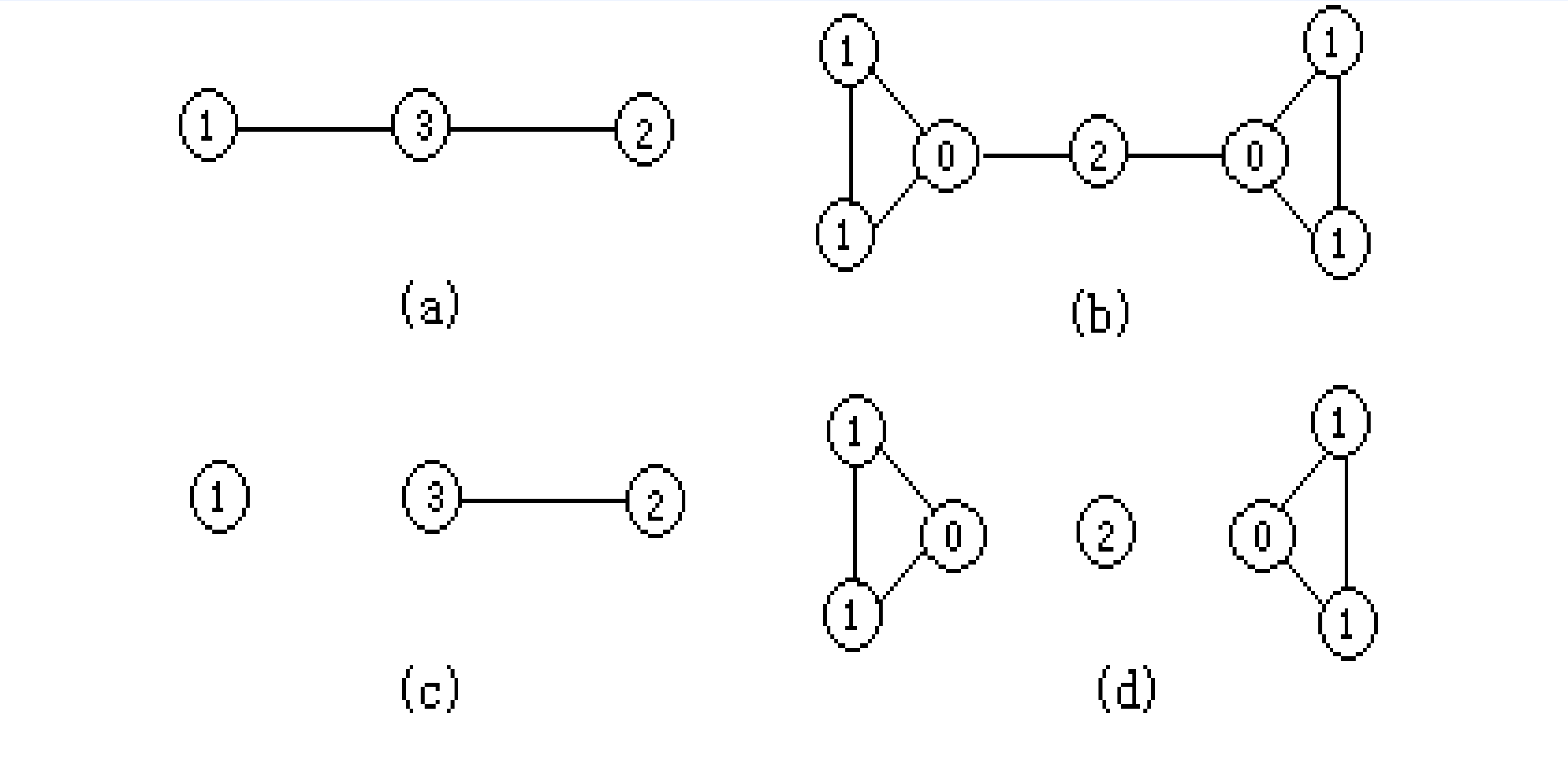
Then points p, q are called d-dimensional connected in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and the sequence $B_1, B_2, \cdots, B_e$ a d-dimensional path connecting p and q, denoted by $P^d(p, q)$.

If each pair p, q of points in the finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is d-dimensional connected, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called d-pathwise connected and say its connectivity $\geq d$.

- **Labeled graphs for finitely combinatorial manifolds**

Choose a graph with vertex set being manifolds labeled by its dimension and two manifold adjacent if there is a d-path in this combinatorial manifold. d=1 in (a) and (b), d=2 in (c) and (d) for Fig.5.1.

Fig.5.2



$\mathcal{H}^d(n_1, n_2, \cdots, n_m)$ — all these finitely combinatorial manifolds with connectivity $\geq d$.

$\mathcal{G}(n_1, n_2, \cdots, n_m)$ — all connected graphs with vertex labels $0, n_1, n_2, \cdots, n_m$ and induced subgraph by vertices labelled 1 in G is a union of complete; vertices labelled 0 only adjacent to vertices labelled 1.

Theorem 5.1 *Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}^d(n_1, n_2, \cdots, n_m)$ defines a labelled connected graph $G[n_1, n_2, \cdots, n_m] \in \mathcal{G}(n_1, n_2, \cdots, n_m)$. Conversely, every labelled connected graph $G[n_1, n_2, \cdots, n_m] \in \mathcal{G}(n_1, n_2, \cdots, n_m)$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}^d(n_1, n_2, \cdots, n_m)$ for any integer $1 \leq d \leq n_1$.*

Homotopy classes

Definition 5.3 *Two finitely combinatorial manifolds* $\widetilde{M}(k_1,k_2,\cdots,k_l)$ *and* $\widetilde{M}(n_1,n_2,\cdots,n_m)$ *are said to be homotopic if there exist continuous maps*

$$f:\widetilde{M}(k_1,k_2,\cdots,k_l)\to\widetilde{M}(n_1,n_2,\cdots,n_m),$$

$$g:\widetilde{M}(n_1,n_2,\cdots,n_m)\to\widetilde{M}(k_1,k_2,\cdots,k_l)$$

such that $gf\simeq$*identity*: $\widetilde{M}(k_1,k_2,\cdots,k_l)\to\widetilde{M}(k_1,k_2,\cdots,k_l)$ *and* $fg\simeq$*identity*: $\widetilde{M}(n_1,n_2,\cdots,n_m)\to\widetilde{M}(n_1,n_2,\cdots,n_m)$.

Theorem 5.2 *Let* $\widetilde{M}(n_1,n_2,\cdots,n_m)$ *and* $\widetilde{M}(k_1,k_2,\cdots,k_l)$ *be finitely combinatorial manifolds with an equivalence* $\varpi:G[\widetilde{M}(n_1,n_2,\cdots,n_m)]\to G[\widetilde{M}(k_1,k_2,\cdots,k_l)]$. *If for* $\forall M_1,M_2\in V(G[\widetilde{M}(n_1,n_2,\cdots,n_m)])$, M_i *is homotopic to* $\varpi(M_i)$ *with homotopic mappings* $f_{M_i}:M_i\to\varpi(M_i)$, $g_{M_i}:\varpi(M_i)\to M_i$ *such that* $f_{M_i}|_{M_i\bigcap M_j}=f_{M_j}|_{M_i\bigcap M_j}$, $g_{M_i}|_{M_i\bigcap M_j}=g_{M_j}|_{M_i\bigcap M_j}$ *providing* $(M_i,M_j)\in E(G[\widetilde{M}(n_1,n_2,\cdots,n_m)])$ *for* $1\le i,j\le m$, *then* $M(n_1,n_2,\cdots,n_m)$ *is homotopic to* $M(k_1,k_2,\cdots,k_l)$.

Fundamental d-groups

Definition 5.4 Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold. For an integer $d, 1 \leq d \leq n_1$ and $\forall x \in \widetilde{M}(n_1, n_2, \cdots, n_m)$, a fundamental d-group at the point x, denoted by $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x)$ is defined to be a group generated by all homotopic classes of closed d-pathes based at x.

Theorem 5.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a d-connected finitely combinatorial manifold with $1 \leq d \leq n_1$. Then*

(1) *for $\forall x \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,*

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong (\bigoplus_{M \in V(G^d)} \pi^d(M)) \bigoplus \pi(G^d),$$

where $G^d = G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$, $\pi^d(M), \pi(G^d)$ denote the fundamental d-groups of a manifold M and the graph G^d, respectively and

(2) *for $\forall x, y \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,*

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong \pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), y).$$

- **Euler-Poincare characteristic**

$$\chi(\widetilde{M}) = \sum_{K^k \in Cl(k), k \geq 2} \; \sum_{M_{i_j} \in V(K^k), 1 \leq j \leq s \leq k} (-1)^{s+1} \chi(M_{i_1} \bigcup \cdots \bigcup M_{i_s})$$

- Differential structures

- **Tangent vector spaces**

Definition 5.5 *Let* $(\widetilde{M}(n_1, n_2, \cdots, n_m), \widetilde{\mathcal{A}})$ *be a smoothly combinatorial manifold and* $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. *A tangent vector* v *at* p *is a mapping* $v : \mathscr{X}_p \to \mathrm{R}$ *with conditions following hold.*

(1) $\forall g, h \in \mathscr{X}_p, \forall \lambda \in \mathrm{R},\ v(h + \lambda h) = v(g) + \lambda v(h)$;

(2) $\forall g, h \in \mathscr{X}_p, v(gh) = v(g)h(p) + g(p)v(h)$.

Theorem 5.4 *For any point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is*

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$$

with a basis matrix

$$[\frac{\partial}{\partial \overline{x}}]_{s(p) \times n_{s(p)}} =$$

$$\begin{bmatrix}
\frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots \\
\frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)n_{s(p)}}}
\end{bmatrix}$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$.

Tensor fields

Definition 5.6 *Let* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *be a smoothly combinatorial manifold and* $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. *A tensor of type* (r, s) *at the point* p *on* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *is an* $(r+s)$*-multilinear function* τ,

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \to \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ *and* $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$.

Theorem 5.5 *Let* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *be a smoothly combinatorial manifold and* $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. *Then*

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ *and* $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$, *particularly,*

$$\dim T_s^r(p, \widetilde{M}) = (\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p)))^{r+s}.$$

Exterior differential on k-forms

Theorem 5.6 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then there is a unique exterior differentiation $\widetilde{d}:\Lambda(\widetilde{M})\to\Lambda(\widetilde{M})$ such that for any integer $k\geq 1$, $\widetilde{d}(\Lambda^k)\subset\Lambda^{k+1}(\widetilde{M})$ with conditions following hold.*

(1) *$\widetilde{d}$ is linear, i.e., for $\forall\varphi,\psi\in\Lambda(\widetilde{M})$, $\lambda\in\mathbf{R}$,*

$$\widetilde{d}(\varphi+\lambda\psi)=\widetilde{d}\varphi\wedge\psi+\lambda\widetilde{d}\psi$$

and for $\varphi\in\Lambda^k(\widetilde{M}),\psi\in\Lambda(\widetilde{M})$,

$$\widetilde{d}(\varphi\wedge\psi)=\widetilde{d}\varphi+(-1)^k\varphi\wedge\widetilde{d}\psi.$$

(2) *For $f\in\Lambda^0(\widetilde{M})$, $\widetilde{d}f$ is the differentiation of f.*

(3) *$\widetilde{d}^2=\widetilde{d}\cdot\widetilde{d}=0$.*

(4) *$\widetilde{d}$ is a local operator, i.e., if $U\subset V\subset\widetilde{M}$ are open sets and $\alpha\in\Lambda^k(V)$, then $\widetilde{d}(\alpha|_U)=(\widetilde{d}\alpha)|_U$.*

Connection on tensors

Definition 5.8 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. A connection on tensors of $\widetilde{M}$ is a mapping $\widetilde{D} : \mathscr{X}(\widetilde{M}) \times T^r_s\widetilde{M} \to T^r_s\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X,\tau)$ such that for $\forall X, Y \in \mathscr{X}\widetilde{M}$, $\tau, \pi \in T^r_s(\widetilde{M})$, $\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,*

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau + \lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;

(2) $\widetilde{D}_X(\tau \otimes \pi) = \widetilde{D}_X\tau \otimes \pi + \sigma \otimes \widetilde{D}_X\pi$;

(3) *for any contraction C on $T^r_s(\widetilde{M})$,*

$$\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau).$$

Geometry on combinatorial manifolds

Definition 5.9 *A combinatorially Finsler geometry is a smoothly combinatorial manifold $\widetilde{M}$ endowed with a Minkowski norm $\widetilde{F}$ on $T\widetilde{M}$, denoted by $(\widetilde{M};\widetilde{F})$.*

Theorem 5.6 *There are combinatorially Finsler geometries.*

Theorem 5.7 *A combinatorially Finsler geometry $(\widetilde{M}(n_1, n_2, \cdots, n_m);\widetilde{F})$ is a Smarandache geometry.*

Corollary 5.1 *There are inclusions among Smarandache geometries, Finsler geometry, Riemannian geometry and Weyl geometry:*

$$\{Smarandache\ geometries\} \supset \{combinatorially\ Finsler\ geometries\}$$

$$\supset \{Finsler\ geometry\}\ and\ \{combinatorially\ Riemannian\ geometries\}$$

$$\supset \{Riemannian\ geometry\} \supset \{Weyl\ geometry\}.$$

$$\supset \{Riemannian\ geometry\} \supset \{Weyl\ geometry\}.$$

Integration on combinatorial manifolds

$\mathscr{C} = \{(\widetilde{U}_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ — for $\forall \alpha \in \widetilde{I}$, $\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ is an constant $n_{\widetilde{U}_\alpha}$ for $\forall p \in \widetilde{U}_\alpha$.

$\mathscr{H}_{\widetilde{M}} = \{n_{\widetilde{U}} \mid \alpha \in \widetilde{I}\}$.

Definition 5.10 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with orientation $\mathscr{O}$ and $(\widetilde{U}; [\varphi])$ a positively oriented chart with a constant $n_{\widetilde{U}}$. Suppose $\omega \in \Lambda^{n_{\widetilde{U}}}(\widetilde{M}), \widetilde{U} \subset \widetilde{M}$ has compact support $\widetilde{C} \subset \widetilde{U}$. Then define*

$$\int_{\widetilde{C}} \omega = \int \varphi_*(\omega|_{\widetilde{U}}).$$

Now if $\mathscr{C}_{\widetilde{M}}$ is an atlas of positively oriented charts with an integer set $\mathscr{H}_{\widetilde{M}}$, let $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha)|\alpha \in \widetilde{I}\}$ be a partition of unity subordinate to $\mathscr{C}_{\widetilde{M}}$. For $\forall \omega \in \Lambda^n(\widetilde{M})$, $n \in \mathscr{H}_{\widetilde{M}}$, an integral of ω on $\widetilde{P}$ is defined by

$$\int_{\widetilde{P}} \omega = \sum_{\alpha \in \widetilde{I}} \int g_\alpha \omega.$$

- **A generalization of Stokes theorem**

Theorem 5.8 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $n \in \mathscr{H}_{\widetilde{M}}$ if $\omega \in \Lambda^n(\widetilde{M})$ has compact support, then*

$$\int_{\widetilde{D}} d\omega = \int_{\partial \widetilde{D}} \omega$$

with the convention $\int_{\partial \widetilde{D}} \omega = 0$ while $\partial \widetilde{D} = \emptyset$.

6.Application to other fields(Mao,2006)

- Applications to algebra

Definition 6.1 *Let* $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ *be a complete multi-algebra system with a binary operation set* $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. *If for any integer* $i, 1 \leq i \leq n$, $(G_i; \times_i)$ *is a group and for* $\forall x, y, z \in \widetilde{G}$ *and any two binary operations "$\times$" and "$\circ$", $\times \neq \circ$, there is one operation, for example the operation $\times$ satisfying the distribution law to the operation "$\circ$" provided their operation results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ *is called a multi-group.*

Definition 6.2 *Let* $\widetilde{R} = \bigcup_{i=1}^{m} R_i$ *be a complete multi-algebra system with double binary operation set* $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. *If for any integers* $i, j,\ i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ *is a ring and for* $\forall x, y, z \in \widetilde{R}$,

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

provided all their operation results exist, then $\widetilde{R}$ *is called a multi-ring. If for any integer* $1 \leq i \leq m$, $(R; +_i, \times_i)$ *is a filed, then* $\widetilde{R}$ *is called a multi-filed.*

Definition 6.3 *Let* $\tilde{V} = \bigcup_{i=1}^{k} V_i$ *be a complete multi-algebra system with binary operation set* $O(\tilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \le i \le m\}$ *and* $\tilde{F} = \bigcup_{i=1}^{k} F_i$ *a multi-filed with double binary operation set* $O(\tilde{F}) = \{(+_i, \times_i) \mid 1 \le i \le k\}$. *If for any integers* $i, j,\ 1 \le i, j \le k$ *and* $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \tilde{V}$, $k_1, k_2 \in \tilde{F}$,

(*i*) $(V_i; \dot{+}_i, \cdot_i)$ *is a vector space on* F_i *with vector additive* $\dot{+}_i$ *and scalar multiplication* $\cdot_i$;

(*ii*) $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c} = \mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;

(*iii*) $(k_1 +_i k_2) \cdot_j \mathbf{a} = k_1 +_i (k_2 \cdot_j \mathbf{a})$;

provided all those operation results exist, then $\tilde{V}$ *is called a multi-vector space on the multi-filed* $\tilde{F}$ *with a binary operation set* $O(\tilde{V})$, *denoted by* $(\tilde{V}; \tilde{F})$.

- Applications to theoretical physics

Applications of Smarandache multi-spaces to the universe by solving Einstein's equation…,etc..

More materials can be found in

Linfan Mao: *Aotomorphism Groups of Maps, Surfaces and Smarandache geometries*, American Research Press,2005.

Linfan Mao: ***Smarandache Multi-Space Theory***, HEXIS, Phoenix, AZ, 2006.

Linfan Mao: ***Selected Papers on Mathematical Combinatorics***, World Academic Press,2006

Linfan Mao: ***Smarandache Geometries & Map Theory with Applications***, Chinese Branch Xiquan House, 2006.

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Thanks！