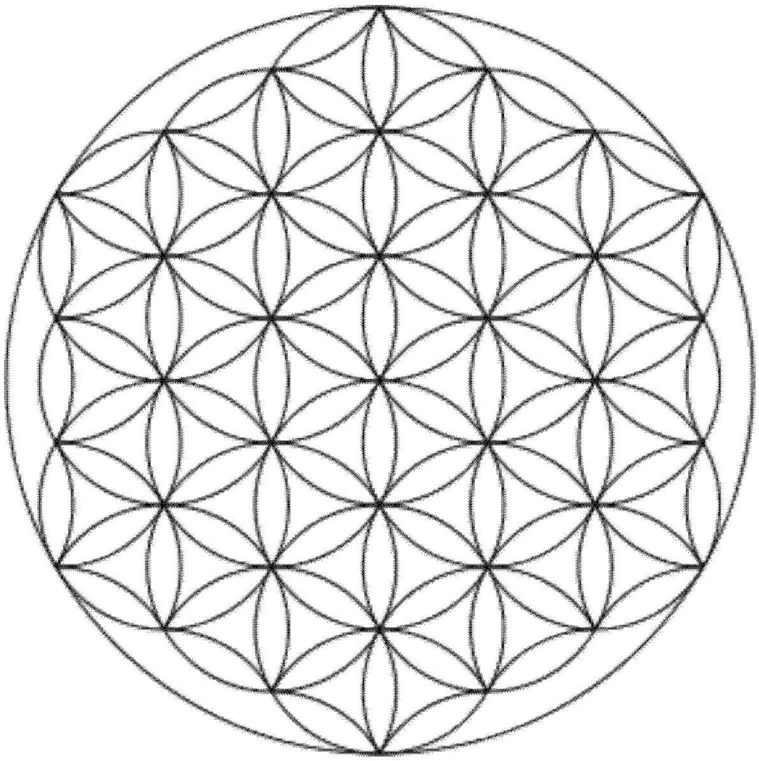
G2 Root System and 28 Nakshastra



By John Sweeney

Abstract

The G2 Root system is the only root system in which the angle of pi / 6 appears between two roots. This is made necessary in nature by the demand for circular objects which can be divided by six or twelve, according to Vedic literature concerning the 28 Nakshastra or astrological houses. In addition, G2 provides the key linkage between the Sedenions, which curiously contain properties related to the number 29 and the Octonions. G2 is key to the transformation from Binary to Trinary, Fano Plane to Tetrahedron and the Sedenions, mixing the 8 x 8 to 9 x 9 aspects of matter, or the Satwa and Raja aspects. Finally, from the Sedenions this paper develops toward the Hopf Fibration and the Boerdijk-Coxeter Helix, which is composed of Sedenions in the form of tetrahedra. Along the way we travel all the way back to the Osiris Temple of Abydos, Egypt, where G2 appears.

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Introduction

In mathematics, a **root** system is a configuration of vectors in a Euclidean space satisfying certain geometrical properties. (Wikipedia).

In this paper we examine the root system of G2, which holds the key to the transition from the Octonions to the Sedenions, and from binary to ternary math, and involving three - forms. The author further notes relationships between the Hopf Fibration and the Poincare Dodecahedral Space, as the momentum of recent papers appears to indicate close connections between these constructions.

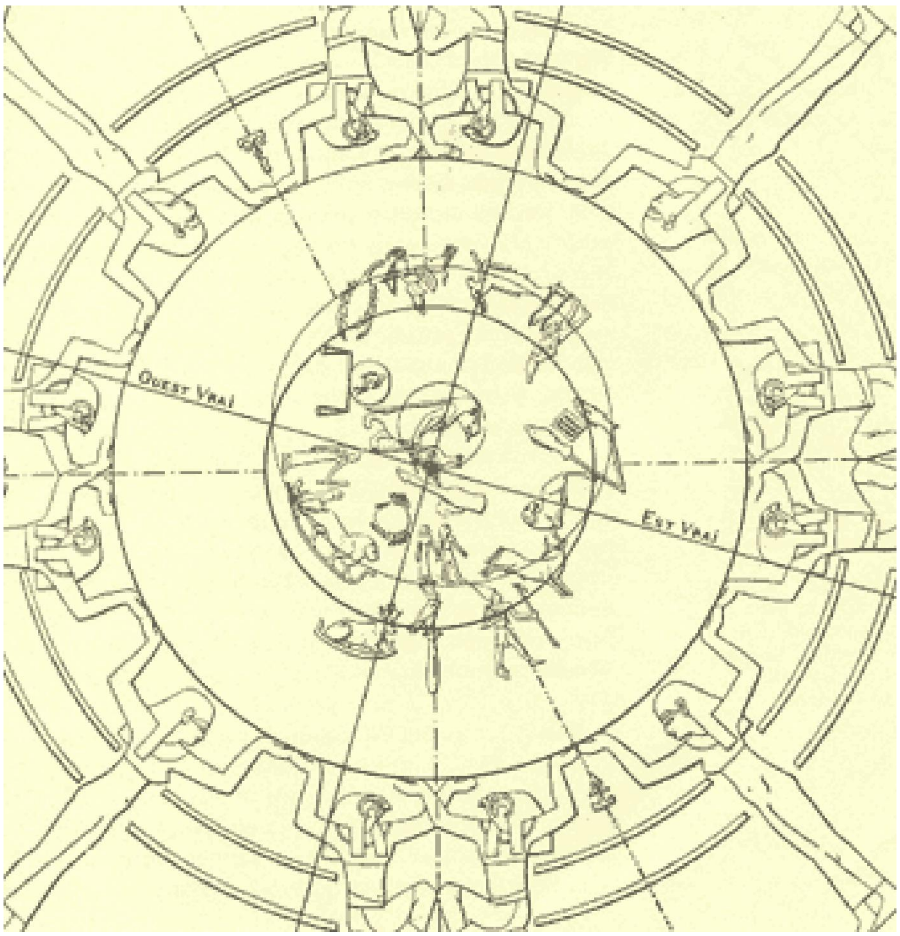
With regard to 12, 28, G2 and the 42 Assessors of the Sedenions, the ancient Egyptians knew of these relationships, since they depicted the 12 astrological houses, 42 Assessors in the scenes showing the weighing of the soul of Osiris. In addition, an Osiris Temple in Egypt holds a Flower of Life painted on its subterranean wall, and the Flower of Life is merely a depiction of the root system of G2. The Flower of Life is well - known to students of sacred geometry and has appeared in crop circles.

From the G2 Root System the paper transitions to Sedenions, which are closely related to the number 28. Tony Smith's essay explores many of these relationships, and so has been included in its entirety. Then, the author adds another piece of evidence regarding the importance of 28 in the An Lie Algebra. In passing we note the availability of G2 to three - forms or triplets, which again will prove important in the connection between Octonions and Sedenions.

Tony Smith suggests the importance of the Leech Lattice, and for this reason the author includes a section from Wolfram which describes the

Leech Lattice. The momentum of recent papers suggest that many of the forms described here enjoy close inter-relationships, with the Leech Lattice as key among these forms, as well as the Barnes - Wall Lattice.

As Tony Smith mentions the relationship between the Sedenions and the tetrahedra, the author includes a description of tetrahedra from Wolfram.



In Vedic Nuclear Physics, the number 28 weighs heavily, yet it must be met with the number 12, or twelve divisions. In this way, the ancients divided the sky into 28 houses, or Nakshastra, and then into 12 astrological houses, as we know today. Yet these 28 Nakshastra represent qualities which are part of Vedic Nuclear Physics.

The 28 differentiable structures of the 7-sphere enjoy an isometric relationship with the 28 Nakshastra of the Atharaveda of Hindu science, including the Hopf Fibration, which appears to stand out among the 28 7 - spheres. This paper explores the Hopf Fibration in search of special connections or reasons why this structure proves so important in physics, which are its relations to the icosahedron and the dodecahedron. These all prove essential to the Qi Men Dun Jia Model.

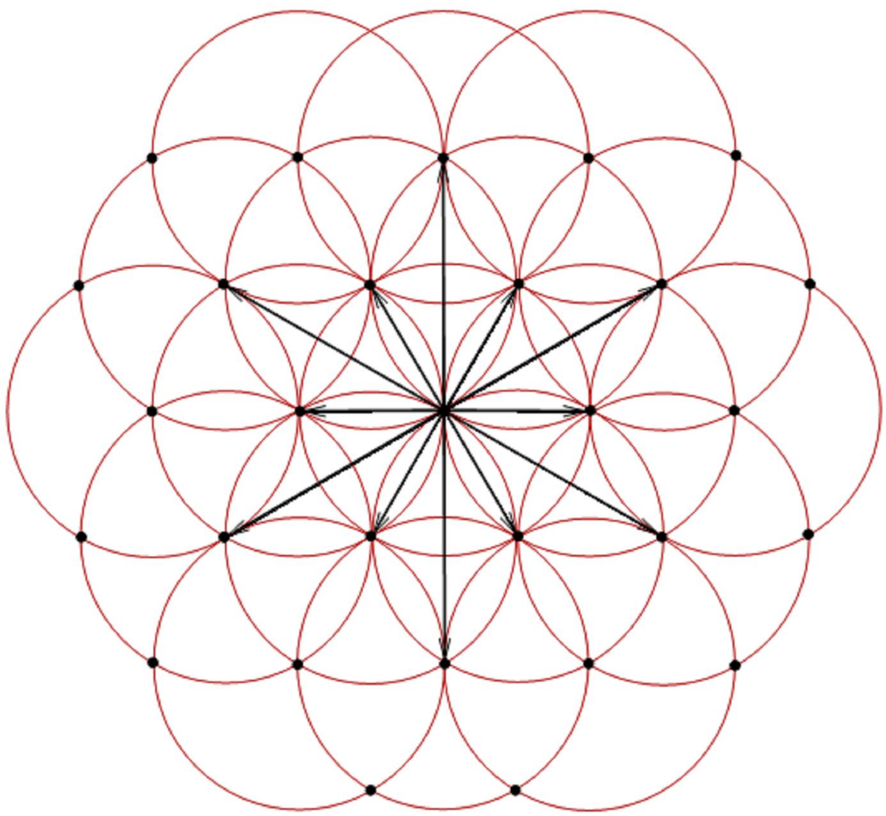
G2 and 14 Dimensions / Tony Smith

Reese Harvey (Spinors and Calibrations (Academic Press 1990))
shows (in exercise 6-17) that the 14-dim exceptional Lie Group G2,
the automorphism group of the octonions O,
can be represented in the graded exterior algebra of forms
on octonion imaginary space ImO = S7 /\(ImO)*
whose structure is dual to that of Cl(7)
2^7 = 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1

In particular, the representation space
/\^2(ImO)* decomposes as the direct sum /\^2_{7} + /\^2_{14}
where /\^2_{7} is Im(O) and /\^2_{14} is the Lie algebra G2
characterized by the kernel of the derivation map
from 49-dim GL(ImO) to 35-dim /\^3(ImO)*
(It is the antisymmetric part of GL(ImO), less S7>)
and
/\^3(ImO)* decomposes as /\^2_{1} + /\^2_{7} + /\^2_{27}
where /\^2_{7} is Im(O) and
/\^2_{1} + /\^2_{27} is the symmetric part of GL(ImO)
(It is the 1-dim scalar plus the 27-dim Jordan algebra
J3(O).)

The 14 "dimensions" of G2 described here reflect an essential structure in nature of 12 + 2 = 14, which include the 12 meridians which flow through the human body, plus the two central vessels.

Flower of Life



(only root system in which the angle $\pi/6$ appears between two roots)

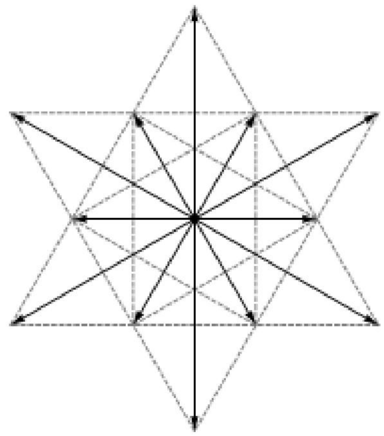


Figure 7: The root system G_2.

The G2 root system forms a Star of David, which originated in Egypt.

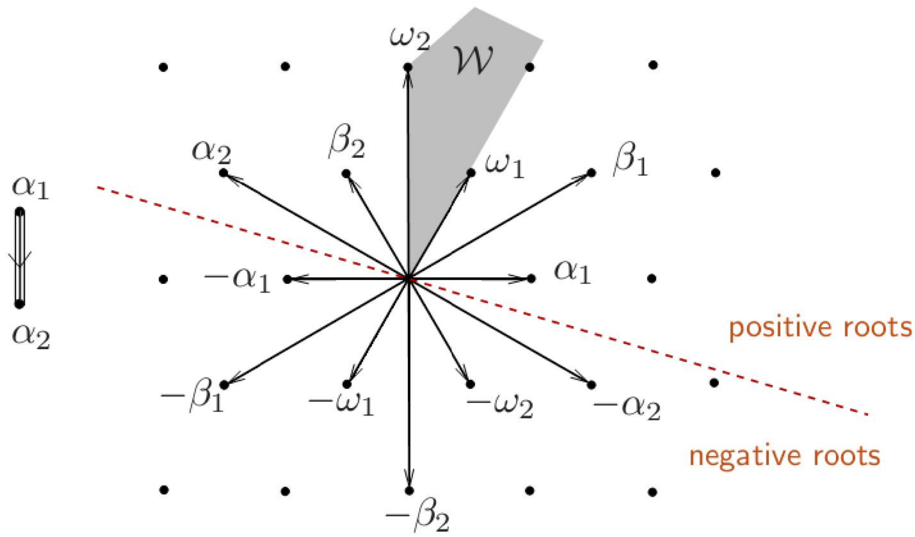
Finally, if $\theta = \pi/6$, the root system consists of 12 vectors. They correspond to the vertices of two regular hexagons that have different sizes and are rotated away from each other by an angle $\pi/6$ (see Fig. 7). The ratio of lengths of these vectors is $\sqrt{3}$. This is an "exceptional" root system and is called G_2.

There are no other root systems of rank 2, because in two dimensions the angle θ determines the root system completely.

The Flower of Life appears in the root system of G2 and by no accident: a Flower of Life was painted on the wall of an Osiris temple in Egypt, and Osiris is associated with the 42 Assessors as his soul is weighed in the underworld.

Root system of $\mathfrak{g}_2$ (II)

For a modern interpretation of the Cartan/Engel result, we need:



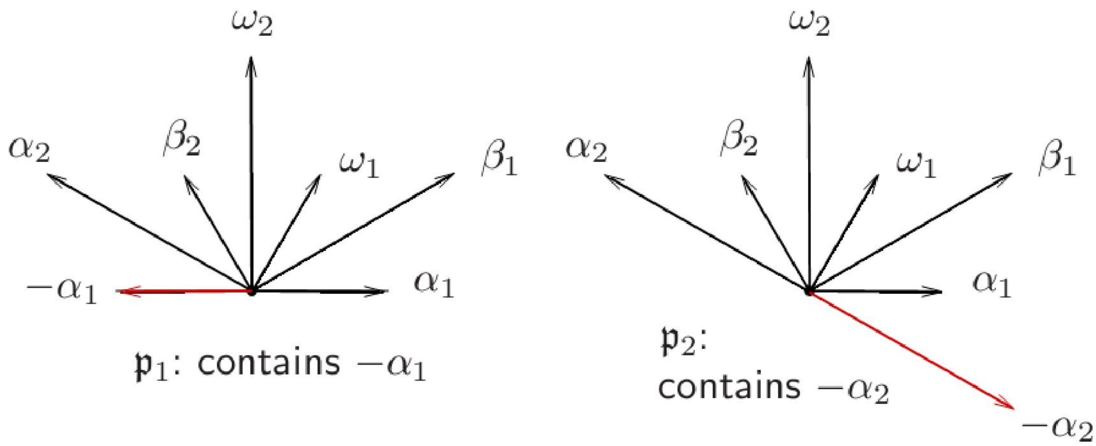
$\alpha_{1,2}$: simple roots

$\omega_{1,2}$: fundamental weights (ω_1: 7-dim. rep., ω_2: adjoint rep.)

$\mathcal{W}$: Weyl chamber = cone spanned by ω_1, ω_2

This view of the G2 Root system shows positive and negative roots and so implies binary or Yin and Yang roots. Notice the w1: as the seven "dimension" representation.

Parabolic subalgebras of $\mathfrak{g}_2$



$\mathfrak{p}_1 = \mathfrak{h} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1}$ [9-dimensional]

$\mathfrak{p}_2 = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}$ [9-dimensional]

$\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1}$ [8-dim. Borel alg.]

Modern interpretation

The complex Lie group G_2 has two maximal parabolic subgroups P_1 and P_2 (with Lie algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$)

$\Rightarrow G_2$ acts on the two 5-dimensional compact homogeneous spaces

- $M_1^5 := G_2/P_1 = \overline{G \cdot [v_{\omega_1}]} \subset \mathbb{P}(\mathbb{C}^7) = \mathbb{CP}^6$: a quadric
- $M_2^5 := G_2/P_2 = \overline{G \cdot [v_{\omega_2}]} \subset \mathbb{P}(\mathbb{C}^{14}) = \mathbb{CP}^{13}$ 'adjoint homogeneous variety'

where v_{ω_1}, v_{ω_2} are h. w. vectors of the reps. with highest weight ω_1, ω_2.

This diagram illustrates the 6 + 1 = 7 vectors of G2 roots.

Q: Is there a geometry based on 3-forms ?

• Generic 3-forms (i. e. with dense $\mathrm{GL}(n, \mathbb{C})$ orbit inside $\Lambda^3\mathbb{C}^n$) exist only for $n \leq 8$.

• To do geometry, we need existence of a compatible inner product, i. e. we want for generic $\omega \in \Lambda^3\mathbb{C}^n$

$$G_\omega := \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \omega = g^*\omega\} \subset \mathrm{SO}(n, \mathbb{C}).$$

This implies (dimension count!) $n = 7, 8$.

And indeed: for $n = 7$: $G_\omega = G_2$, for $n = 8$: $G_\omega = \mathrm{SL}(3, \mathbb{C})$.

The author will show in a future paper the relationship of 3 - forms to G2 and the importance of G2's ability to contain 3 - forms in a future paper.

Tony Smith - Why Not Sedenions?

In *q-alg/9710013 Guillermo Moreno describes The *Zero Divisors of the Cayley-Dickson Algebras over the Real Numbers. He shows, among other things, that the set of zero divisors (with entries of norm one) in the sedenions is homeomorphic to the *Lie group G2.

If the sedenions are regarded as the Cayley-Dickson product of two octonion spaces, then:

if you take one 7-sphere S7 in each octonion space,
and
if you take G2 as the space of zero divisors,
then

YOU CAN CONSTRUCT FROM THE SEDENIONS

the Lie group Spin(0,8)
as the *twisted fibration product S7 x S7 x G2.
Such a structure is represented
by the *design of the Temple of Luxor.

The late Robert de Marrais has constructed detailed models for Zero Divisors of algebras of dimension 2^N, with explicit results through N = 8. He (see *his paper *math.GM/0011260) visualizes the *zero divisors of sedenions and higher-dimensional algebras in terms of *Singularity / *Catastrophe Theory, saying "... *168 elements provide the exact count of "primitive" unit zero-divisors in the Sedenions. ([there are] Composite zero-divisors, comprising all points of certain hyperplanes of up to 4 dimensions ...)

The 168 are arranged in point-set quartets along the "42 Assessors."

In the Egyptian Book of the Dead, the soul of the deceased passes through a hall lined by 21 pairs of Assessors] (pairs of diagonals in planes spanned by pairs of pure imaginaries, each of which zero-divides only one such diagonal of any partner Assessor).

Wave-interference dynamics, depictable in the simplest case by Lissajous figures, ... derive from mutually zero-dividing trios of Assessors, a *D4-suggestive 28 in number ... ".Marrais constructs "box-kites" that are similar to *Onarhedra/heptavertons.

Although Marriais does not prove it in his paper, he conjectures that his box-kites are related to *the H3 Coxeter Group Singularity and, that it, in turn, is related to the *16-dimensional Barnes-Wall lattice.

The Barnes-Wall lattice is a d-dimensional lattice that exists when d is a power of 2. It is implemented in *Mathematica* as LatticeData[{"BarnesWall", n}].
Special cases are summarized in the following table.

d	lattice
2	square lattice $\mathbb{Z}^2$
4	root lattice D_4
8	root lattice E_8
16	laminated lattice $L_{16,1}$

The automorphism group has order

$$\begin{cases} 696\,729\,600 & \text{for } d=3 \\ 2^{d^2+d+1}\,(2^d-1)\prod_{i=1}^{d-1}(2^{2i}-1) & \text{otherwise,} \end{cases} \tag{1}$$

giving the first few terms as 2, 8, 1152, 696729600, 89181388800, 48126558103142400, ... (Sloane's A014116).
The 16-dimensional Barnes-Wall lattice can be constructed from the Leech lattice Λ_{24}.
The generating function of the theta series for the lattice for various orders d are summarized in the following table.

d	theta series generating function
2	$\vartheta_3^2(q)$
4	$\frac{1}{2}\left[\vartheta_3^4(q)+\vartheta_4^4(q)\right]$
8	$\frac{1}{2}\left[\vartheta_2^8(q)+\vartheta_3^8(q)+\vartheta_4^8(q)\right]$
16	$\frac{1}{2}\left[\vartheta_2^{16}(q)+\vartheta_3^{16}(q)+\vartheta_4^{16}(q)+30\,\vartheta_2^8(q)\,\vartheta_3^8(q)\right]$

The following table gives the theta series.

d	Sloane	theta series
2	A004018	$1+4q+4q^2+4q^4+8q^5+4q^8+4q^9+8q^{10}+\ldots$
4	A004011	$1+24q^2+24q^4+96q^6+24q^8+144q^{10}+\ldots$
8	A004009	$1+240q^2+2160q^4+6720q^6+17520q^8+30240q^{10}+\ldots$
16	A008409	$1+4320q^2+61440q^3+522720q^4+2211840q^5+\ldots$

I (Tony Smith) speculate that *the H4 Coxeter Group Singularity, based on the symmetries of *the 600-cell, which in turn can be constructed from a 24-cell, may be related to the *24-dimensional Leech lattice.

Leech Lattice

A 24-dimensional Euclidean lattice. An automorphism of the Leech lattice modulo a center of two leads to the Conway group Co_1. Stabilization of the one- and two-dimensional sublattices leads to the Conway groups Co_2 and Co_3, the Higman-Sims group *HS* and the McLaughlin group *McL*.
The Leech lattice appears to be the densest hypersphere packing in 24 dimensions, and results in each hypersphere touching $196\,560$ others. The number of vectors with norm n in the Leech lattice is given by

$$\theta(n) = \frac{65\,520}{691}\left[\sigma_{11}(n) - \tau(n)\right], \qquad (1)$$

where σ_{11} is the divisor function giving the sum of the 11th powers of the divisors of n and $\tau(n)$ is the tau function (Conway and Sloane 1993, p. 135). The first few values for $n = 1$, 2, ... are 0, 196560, 16773120, 398034000, ... (Sloane's A008408). This is an immediate consequence of the theta function for Leech's lattice being a weight 12 modular form and having no vectors of norm two.

$\theta(n)$ has the theta series

$$\begin{aligned} f(q) &= [E_4(q)]^3 - 720\, q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^{24} \\ &= [E_4(q)]^3 - 720\, q^2 (q^2, q^2)_{\infty}^{24} \\ &= \left[1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)\, q^{2m}\right]^3 - 720\, q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^{24} \\ &= 1 + 196\,560\, q^4 + 16\,773\,120\, q^6 + 3\,980\,034\,000\, q^8 + \ldots, \end{aligned}$$

where $E_4(q)$ is the Eisenstein series, which is the theta series of the E_8 lattice (Sloane's A004009), $(a, q)_\infty$ is a q-Pochhammer symbol, and $f(q)$ can be written in closed form in terms of Jacobi elliptic functions as

$$f(q) = \frac{1}{8}\left[\vartheta_2^8(q) + \vartheta_3^8(q) + \vartheta_4^8(q)\right] - \frac{45}{16}\vartheta_2^8(q)\vartheta_3^8(q)\vartheta_4^8(q).$$

Notice that the last entry mentions the E8 Lattice. Thus we move from 28 Nakshastra to the E8 Lattice.

Tony Smith (Continued)

The *28 new associative triple cycles of the sedenions
are related to the 28-dimensional Lie algebra Spin(0,8),
and to the 28 different differentiable structures on
the 7-sphere S7 that are used to construct
exotic structures on differentiable manifolds.

For sedenions,
*28 new associative triple cycles have appeared.
They correspond to the Lie algebra Spin(0,8).
The other 455 - 35 = 420 triples are not cycles.

The octonions correspond to a triangle,
a 2-dimensional simplex,
3 vertices corresponding to I,J,K ,
3 edges corresponding to i,j,k ,
and 1 face corresponding to E .
There are 3+3+1 = 7 things.

There are 7 (projective) lines each with 3 things.
An octonion multiplication table,
one of the *480 that exist,
is then determined by the 7 associative triple cycles

ijk -iJK -IjK -IJk EIi EJj EKk

where -iJK means that -iJK = -1.

For the octonions,
*6 new associative triple cycles have appeared.
They correspond to the Lie algebra Spin(4).
The other 35 - 7 = 28 triples are not cycles.

The sedenions correspond to a tetrahedron,
a 3-dimensional simplex,
4 vertices v of the tetrahedron corresponding to EIJK ;
6 edges e of the tetrahedron corresponding to ijkTUV ;
4 faces f of the tetrahedron corresponding to WXYZ ;
and the 1 entire tetrahedron T corresponding to S .
There are 4+6+4+1 = 15 things.

There are 35 (projective) lines each with 3 things.

Geometrically, they are of the form:

4 like eee (where eee are all on the same face);
6 like vev (these are the edges);
12 like vfe (where v is opposite e on face f);
3 like eTe (where e is opposite e on the whole tetrahedron T);
4 like vTf (where v is opposite f on T); and
6 like fef (where the edge of e is not on f or f,
that is, f and f are opposite to e).

Tetrahedron / Wolfram

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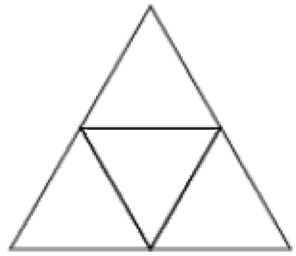
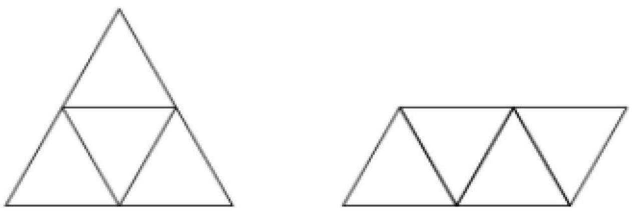


The regular tetrahedron, often simply called "the" tetrahedron, is the Platonic solid P_5 with four polyhedron vertices, six polyhedron edges, and four equivalent equilateral triangular faces, $4\{3\}$. It is also uniform polyhedron U_1 and Wenninger model W_1. It is described by the Schläfli symbol $\{3, 3\}$ and the Wythoff symbol is $3 \mid 2\,3$. It is an isohedron, and a special case of the general tetrahedron and the isosceles tetrahedron.
The tetrahedron has 7 axes of symmetry: $4C_3$ (axes connecting vertices with the centers of the opposite faces) and $3C_2$ (the axes connecting the midpoints of opposite sides).
There are no other convex polyhedra other than the tetrahedron having four faces.



The tetrahedron has two distinct nets (Buekenhout and Parker 1998).
Questions of polyhedron coloring of the tetrahedron can be addressed using the Pólya enumeration theorem.

The surface area of the tetrahedron is simply four times the area of a single equilateral triangle face

$$A = \tfrac{1}{4}\sqrt{3}\,a^2, \tag{1}$$

so

$$S = 4A = \sqrt{3}\,a^2. \tag{2}$$

The height of the regular tetrahedron is

$$h = \tfrac{1}{3}\sqrt{6} \tag{3}$$

and the inradius and circumradius are

$$r = \tfrac{1}{12}\sqrt{6} \tag{4}$$

$$R = \tfrac{1}{4}\sqrt{6}, \tag{5}$$

where $h = r + R$ as it must.

Since a tetrahedron is a pyramid with a triangular base, $V = \frac{1}{3} A_b h$, giving

$$V = \tfrac{1}{12}\sqrt{2}\,a^3. \tag{6}$$

The dihedral angle is

$$\alpha = \tan^{-1}\left(2\sqrt{2}\right) = 2\sin^{-1}\left(\tfrac{1}{3}\sqrt{3}\right) = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 70.53\,°. \tag{7}$$

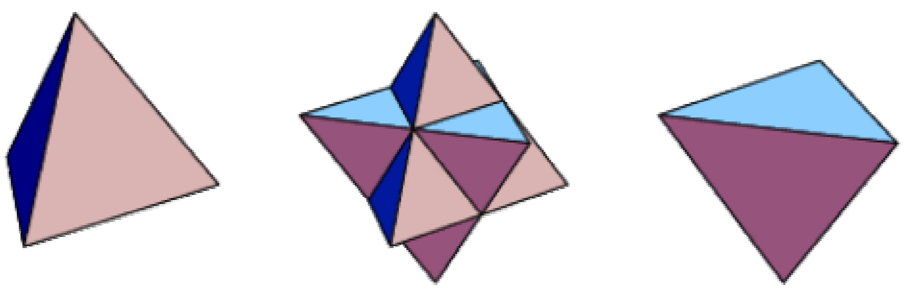
The midradius of the tetrahedron is

$$\rho = \sqrt{r^2 + d^2} = \sqrt{\tfrac{1}{8}}\,a = \tfrac{1}{4}\sqrt{2}\,a \tag{8}$$

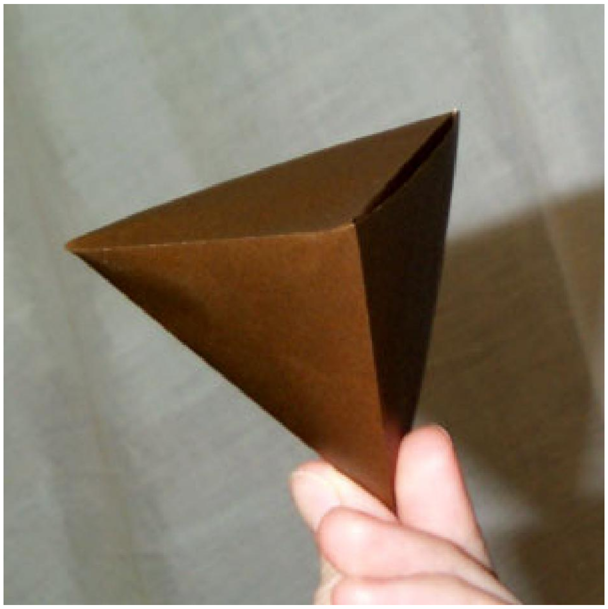
$$\approx 0.35355\,a. \tag{9}$$

Plugging in for the polyhedron vertices gives

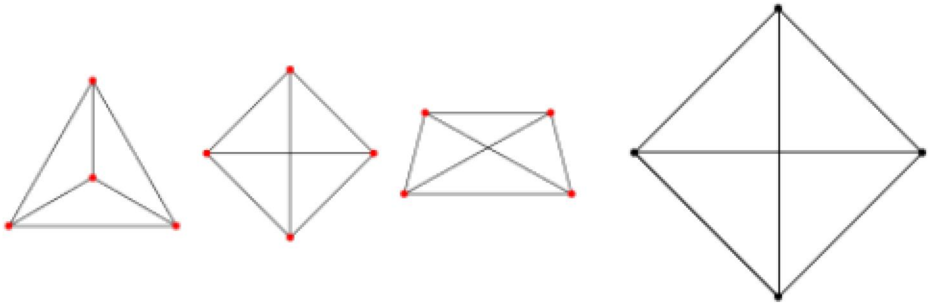
$$\left(\tfrac{1}{3}\sqrt{3}\,a, 0, 0\right), \left(-\tfrac{1}{6}\sqrt{3}\,a, \pm\tfrac{1}{2}\,a, 0\right), \text{ and } \left(0, 0, \tfrac{1}{3}\sqrt{6}\,a\right). \tag{10}$$



The dual polyhedron of an tetrahedron with unit edge lengths is another oppositely oriented tetrahedron with unit edge lengths.



The figure above shows an origami tetrahedron constructed from a single sheet of paper (Kasahara and Takahama 1987, pp. 56-57).



It is the prototype of the tetrahedral group T_d. The connectivity of the vertices is given by the tetrahedral graph, equivalent to the circulant graph $Ci_{1,2,3}(4)$ and the complete graph K_4.

The tetrahedron is its own dual polyhedron, and therefore the centers of the faces of a tetrahedron form another tetrahedron (Steinhaus 1999, p. 201). The tetrahedron is the only simple polyhedron with no polyhedron diagonals, and it cannot be stellated. If a regular tetrahedron is cut by six planes, each passing through an edge and bisecting the opposite edge, it is sliced into 24 pieces (Gardner 1984, pp. 190 and 192; Langman 1951).
Alexander Graham Bell was a proponent of use of the tetrahedron in framework structures, including kites (Bell 1903; Lesage 1956, Gardner 1984, pp. 184-185). The opposite edges of a regular tetrahedron are perpendicular, and so can form a universal coupling if hinged appropriately.

Eight regular tetrahedra can be placed in a ring which rotates freely, and the number can be reduced to six for squashed irregular tetrahedra (Wells 1975, 1991).

A sedenion multiplication table
(there are even more than 480 of them)
is then determined by the 35 associative triple cycles
used by *Onar Aam:

ijk -iJK -IjK -IJk EIi EJj EKk

-ijk iJK IjK IJk -EIi -EJj -EKk

-ijk iJK IjK IJk -EIi -EJj -EKk

-ijk iJK IjK IJk -EIi -EJj -EKk

eEE eII eJJ eKK eii ejj ekk

Exotic 7-Spheres by Tony Smith

**The 28 differentiable structures of the 7-sphere
enabled John Milnor (Ann. Math. 64 (1956) 399)
to construct exotic 7-spheres, denoted here by S7#.
There are 27 exotic S7# spheres, plus one (the 28th) normal S7.**

**A 7-sphere, whether exotic S7# or normal S7, can be "factored"
by a Hopf fibration into a 3-sphere S3 and a 4-sphere S4.
Each point of the S4 can be thought of as having one S3 attached.
The Hopf fibration can be denoted by S3 - S7 - S4.**

***Hopf fibrations can only be done for spheres of dimension 1,3,7,15:**

S0	**-**	**S1**	**-**	**S1**	**(S0 = point)**	**based on real numbers**
S1	**-**	**S3**	**-**	**S2**		**based on complex numbers**
S3	**-**	**S7**	**-**	**S4**		**based on quaternions**

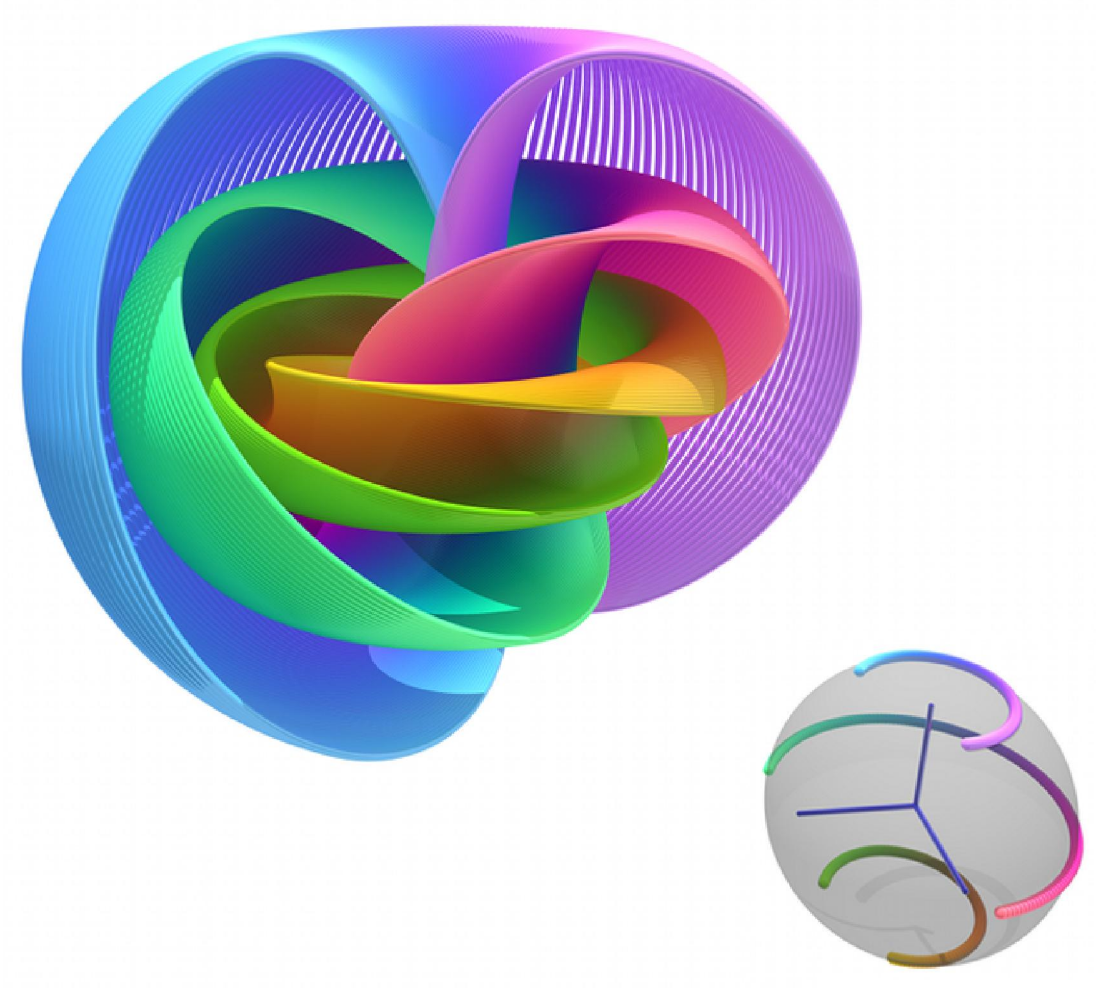
J. Francis notes how this is done, and then the homeomorphism to S7.

(2) We constructed 7-manifolds by fibering S^3 over S^4. We used that fact that $\pi_4 BSO_f = \mathbb{Z}\times\mathbb{Z}$. (This is a special n—$\pi_4 BSO(n) = \pi_4 BSO(5) = \mathbb{Z}$ for all $n \geq 5$; we're just out of stable range with $n = 4$.) Choosing a class (i, j) such that $i - j = 1$,

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d] & S^3 \arrow[d] \\
M_{i,j} \arrow[r] \arrow[d] & BSO(3) \arrow[d] \\
S^4 \arrow[r, "f_{ij}"'] & BSO(4)
\end{tikzcd}$$

from the long exact sequence on π_*, we see that M_{ij} is 6-connected. By the h-cobordism theorem, M^7 is hence homeomorphic to S^7.

Hopf Fibration / Wikipedia



In the mathematical field of topology, the **Hopf fibration** (also known as the **Hopf bundle** or **Hopf map**) describes a 3-sphere (a hypersphere in four-dimensional space) in terms of circles and an ordinary sphere. Discovered by Heinz Hopf in 1931, it is an influential early example of a fiber bundle. Technically, Hopf found a many-to-one continuous function (or "map") from the 3-sphere onto the 2-sphere such that each distinct *point* of the 2-sphere comes from a distinct *circle* of the 3-sphere (Hopf 1931). Thus the 3-sphere is composed of fibers, where each fiber is a circle — one for each point of the 2-sphere.

This fiber bundle structure is denoted

$$S^1 \hookrightarrow S^3 \xrightarrow{p} S^2,$$

meaning that the fiber space S^1 (a circle) is embedded in the total space S^3 (the 3-sphere), and $p : S^3 \to S^2$ (Hopf's map) projects S^3 onto the base space S^2 (the ordinary 2-sphere). The Hopf fibration, like any fiber bundle, has the important property that it is locally a product space. However it is not a *trivial* fiber bundle, i.e., S^3 is not *globally* a product of S^2 and S^1 although locally it is indistinguishable from it.

This has many implications: for example the existence of this bundle shows that the higher homotopy groups of spheres are not trivial in general. It also provides a basic example of a principal bundle, by identifying the fiber with the circle group.

Stereographic projection of the Hopf fibration induces a remarkable structure on $\mathbf{R}^3$, in which space is filled with nested tori made of linking Villarceau circles. Here each fiber projects to a circle in space (one of which is a line, thought of as a "circle through infinity"). Each torus is the stereographic projection of the inverse image of a circle of latitude of the 2-sphere. (Topologically, a torus is the product of two circles.) These tori are illustrated in the images at right. When $\mathbf{R}^3$ is compressed to a ball, some geometric structure is lost although the topological structure is retained (see Topology and geometry). The loops are homeomorphic to circles, although they are not geometric circles.

There are numerous generalizations of the Hopf fibration. The unit sphere in $\mathbf{C}^{n+1}$ fibers naturally over $\mathbf{CP}^n$ with circles as fibers, and there are also real, quaternionic, and octonionic versions of these fibrations. In particular, the Hopf fibration belongs to a family of four fiber bundles in which the total space, base space, and fiber space are all spheres:

$$\begin{aligned} S^0 &\hookrightarrow S^1 \to S^1, \\ S^1 &\hookrightarrow S^3 \to S^2, \\ S^3 &\hookrightarrow S^7 \to S^4, \\ S^7 &\hookrightarrow S^{15} \to S^8. \end{aligned}$$

By Adams' theorem such fibrations can occur only in these dimensions.

The Hopf fibration is important in twistor theory.

Why the Hopf Fibration?

Three of the six regular 4-polytopes – 8-cell (tesseract), 24-cell, and 120-cell – can each be partitioned into disjoint great circle (regular polygon) rings of cells forming discrete Hopf fibrations of these polytopes. The tesseract partitions into two interlocking rings of four cubes each. The 24-cell partitions into four rings of six octahedrons each. The 120-cell partitions[4] into twelve rings of ten dodecahedra each. The 24-cell also contains a fibration of six rings of four octahedrons each stacked end to end at their vertices.

The 600-cell partitions into 20 rings of 30 tetrahedra each in a very interesting, quasi-periodic chain called the Boerdijk–Coxeter helix. When superimposed onto the 3-sphere curvature it becomes periodic with a period of 10 vertices, encompassing all 30 cells. In addition, the 16-cell partitions into two 8-tetrahedron chains, four edges long, and the 5-cell partitions into a single degenerate 5-tetrahedron chain. The above fibrations all map to the following specific tilings of the 2-sphere[4].

S3	S2	# of rings	# of cells per ring	Cell Stacking
600-cell{3,3,5}	Icosahedron{3,5}	20	30	Boerdijk–Coxeter helix
120-cell{5,3,3}	Dodecahedron{5, 3}	12	10	face stacking
24-cell{3,4,3}	Tetrahedron{3,3}	4	6	face stacking
	Cube{4,3}	6	4	vertex stacking
16-cell{3,3,4}	Dihedron{n,2}	2	8	Boerdijk–Coxeter helix
8-cell{4,3,3}	Dihedron{n,2}	2	4	face stacking
5-cell{3,3,3}	Whole 2-sphere	1	5	Boerdijk–Coxeter helix

Given the Qi Men Dun Jia Model's close relationship to the icosahedron and the dodecahedron, these linkages prove essential, but especially the link to the **Boerdijk–Coxeter Helix.**

The 0-grade 1-dimensional scalar space of Cl(0,15)
represents the sedenion real axis.

There is an 8 to 1 correspondence between the
1x128 minimal ideal SSPINOR on which SL acts by Clifford action,
and
the 1x16 sedenion column vectors on which
SL acts by matrix-vector action.

This leads
to failure of the division algebra property of sedenions,
because the map SL from SSPINOR to S is 8 to 1
and invertible.

And there is probably a good reason why this is true. For details see the author's forthcoming essay on "Forgetting" in Mathematical Physics.

Consider SLx of sedenion left-multiplication by x
as being represented by the 16x16 real matrix

OLx	-*ORx
*OLx	ORx

and
consider the 16x16 real matrices
forming the 256-dim matrix algebra R(16),
which is the *Clifford algebra Cl(0,8) of Spin(0,8):

0	2	2	2	2	2	2	2	7	5	5	5	5	5	5	5
4	4	2	2	2	2	2	2	5	7	5	5	5	5	5	5
4	4	4	2	2	2	2	2	5	5	7	5	5	5	5	5
4	4	4	4	2	2	2	2	5	5	5	7	5	5	5	5
4	4	4	4	4	2	2	2	5	5	5	5	7	5	5	5
4	4	4	4	4	4	2	2	5	5	5	5	5	7	5	5
4	4	4	4	4	4	4	2	5	5	5	5	5	5	7	5
4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	7

1	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4
3	1	3	3	3	3	3	3	6	4	4	4	4	4	4	4
3	3	1	3	3	3	3	3	6	6	4	4	4	4	4	4
3	3	3	1	3	3	3	3	6	6	6	4	4	4	4	4
3	3	3	3	1	3	3	3	6	6	6	6	4	4	4	4

3	3	3	3	3	1	3	3	6	6	6	6	6	4	4	4
3	3	3	3	3	3	1	3	6	6	6	6	6	6	4	4
3	3	3	3	3	3	3	1	6	6	6	6	6	6	6	8

The numbers refer to the grade in Cl(0,8) of the matrix entry:

grade	0	1	2	3	4	5	6	7	8
dimension	1	8	28	56	70	56	28	8	1

*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	-0	3	-2	5	-4	-7	6	9	-8	-11	10	-13	12	15	-14
2	2	-3	-0	1	6	7	-4	-5	10	11	-8	-9	-14	-15	12	13
3	3	2	-1	-0	7	-6	5	-4	11	-10	9	-8	-15	14	-13	12
4	4	-5	-6	-7	-0	1	2	3	12	13	14	15	-8	-9	-10	-11
5	5	4	-7	6	-1	-0	-3	2	13	-12	15	-14	9	-8	11	-10
6	6	7	4	-5	-2	3	-0	-1	14	-15	-12	13	10	-11	-8	9
7	7	-6	5	4	-3	-2	1	-0	15	14	-13	-12	11	10	-9	-8
8	8	-9	-10	-11	-12	-13	-14	-15	-0	1	2	3	4	5	6	7
9	9	8	-11	10	-13	12	15	-14	-1	-0	-3	2	-5	4	7	-6
10	10	11	8	-9	-14	-15	12	13	-2	3	-0	-1	-6	-7	4	5
11	11	-10	9	8	-15	14	-13	12	-3	-2	1	-0	-7	6	-5	4
12	12	13	14	15	8	-9	-10	-11	-4	5	6	7	-0	-1	-2	-3
13	13	-12	15	-14	9	8	11	-10	-5	-4	7	-6	1	-0	3	-2
14	14	-15	-12	13	10	-11	8	9	-6	-7	-4	5	2	-3	-0	1
15	15	14	-13	-12	11	10	-9	8	-7	6	-5	-4	3	2	-1	-0

Table 2. Main portion of the Cayley table of the standard sedenion loop $S_L(\#2) = \pm\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ of order $n = 32$.

*	0	1	2	3	12	13	14	15	20	21	22	23	24	25	26	27
0	0	1	2	3	12	13	14	15	20	21	22	23	24	25	26	27
1	1	-0	3	-2	-13	12	15	-14	-21	20	23	-22	-25	24	27	-26
2	2	-3	-0	1	-14	-15	12	13	-22	-23	20	21	-26	-27	24	25
3	3	2	-1	-0	-15	14	-13	12	-23	22	-21	20	-27	26	-25	24
12	12	13	14	15	-0	-1	-2	-3	-24	25	26	27	20	-21	-22	-23
13	13	-12	15	-14	1	-0	3	-2	-25	-24	-27	26	21	20	-23	22
14	14	-15	-12	13	2	-3	-0	1	-26	27	-24	-25	22	23	20	-21
15	15	14	-13	-12	3	2	-1	-0	-27	-26	25	-24	23	-22	21	20
20	20	21	22	23	24	25	26	27	-0	-1	-2	-3	-12	-13	-14	-15
21	21	-20	23	-22	-25	24	-27	26	1	-0	3	-2	-13	12	-15	14
22	22	-23	-20	21	-26	27	24	-25	2	-3	-0	1	-14	15	12	-13
23	23	22	-21	-20	-27	-26	25	24	3	2	-1	-0	-15	-14	13	12
24	24	25	26	27	-20	-21	-22	-23	12	13	14	15	-0	-1	-2	-3
25	25	-24	27	-26	21	-20	-23	22	13	-12	-15	14	1	-0	3	-2
26	26	-27	-24	25	22	23	-20	-21	14	15	-12	-13	2	-3	-0	1
27	27	26	-25	-24	23	-22	21	-20	15	-14	13	-12	3	2	-1	-0

Table 3. Main portion of the Cayley table of the α-sedenion loop $S_L^{\alpha}(\#7) = \pm\{0,1,2,3,8,9,10,11,20,21,22,23,28,29,30,31\}$ of order $n = 32$.

*	0	1	2	3	8	9	10	11	20	21	22	23	28	29	30	31
0	0	1	2	3	8	9	10	11	20	21	22	23	28	29	30	31
1	1	-0	3	-2	9	-8	-11	10	-21	20	23	-22	29	-28	-31	30
2	2	-3	-0	1	10	11	-8	-9	-22	-23	20	21	30	31	-28	-29
3	3	2	-1	-0	11	-10	9	-8	-23	22	-21	20	31	-30	29	-28
8	8	-9	-10	-11	-0	1	2	3	28	29	30	31	-20	-21	-22	-23
9	9	8	-11	10	-1	-0	-3	2	29	-28	-31	30	21	-20	-23	22
10	10	11	8	-9	-2	3	-0	-1	30	31	-28	-29	22	23	-20	-21
11	11	-10	9	8	-3	-2	1	-0	31	-30	29	-28	23	-22	21	-20
20	20	21	22	23	-28	-29	-30	-31	-0	-1	-2	-3	8	9	10	11
21	21	-20	23	-22	-29	28	-31	30	1	-0	3	-2	-9	8	-11	10
22	22	-23	-20	21	-30	31	28	-29	2	-3	-0	1	-10	11	8	-9
23	23	22	-21	-20	-31	-30	29	28	3	2	-1	-0	-11	-10	9	8
28	28	-29	-30	-31	20	-21	-22	-23	-8	9	10	11	-0	1	2	3
29	29	28	-31	30	21	20	-23	22	-9	-8	-11	10	-1	-0	-3	2
30	30	31	28	-29	22	23	20	-21	-10	11	-8	-9	-2	3	-0	-1
31	31	-30	29	28	23	-22	21	20	-11	-10	9	-8	-3	-2	1	-0

Table 4. Main portion of the Cayley table of the β-sedenion loop $S_L^{\beta}(\#10) = \pm\{0,1,2,3,12,13,14,15,20,21,22,23,24,25,26,27\}$ of order $n = 32$.

*	0	1	2	3	4	5	6	7	24	25	26	27	28	29	30	31
0	0	1	2	3	4	5	6	7	24	25	26	27	28	29	30	31
1	1	-0	3	-2	5	-4	-7	6	-25	24	27	-26	29	-28	-31	30
2	2	-3	-0	1	6	7	-4	-5	-26	-27	24	25	30	31	-28	-29
3	3	2	-1	-0	7	-6	5	-4	-27	26	-25	24	31	-30	29	-28
4	4	-5	-6	-7	-0	1	2	3	-28	-29	-30	-31	24	25	26	27
5	5	4	-7	6	-1	-0	-3	2	-29	28	-31	30	-25	24	-27	26
6	6	7	4	-5	-2	3	-0	-1	-30	31	28	-29	-26	27	24	-25
7	7	-6	5	4	-3	-2	1	-0	-31	-30	29	28	-27	-26	25	24
24	24	25	26	27	28	29	30	31	-0	-1	-2	-3	-4	-5	-6	-7
25	25	-24	27	-26	29	-28	-31	30	1	-0	3	-2	5	-4	-7	6
26	26	-27	-24	25	30	31	-28	-29	2	-3	-0	1	6	7	-4	-5
27	27	26	-25	-24	31	-30	29	-28	3	2	-1	-0	7	-6	5	-4
28	28	-29	-30	-31	-24	25	26	27	4	-5	-6	-7	-0	1	2	3
29	29	28	-31	30	-25	-24	-27	26	5	4	-7	6	-1	-0	-3	2
30	30	31	28	-29	-26	27	-24	-25	6	7	4	-5	-2	3	-0	-1
31	31	-30	29	28	-27	-26	25	-24	7	-6	5	4	-3	-2	1	-0

Table 5. Main portion of the Cayley table of the γ-sedenion loop $S_L^\gamma(\#4) = \pm\{0,1,2,3,4,5,6,7,24,25,26,27,28,29,30,31\}$ of order $n = 32$.

Remark 1. *The Cayley table of the loop T_L (Table 1) of order 64 has 64 rows and 64 columns consisting of four portions (partitions) each with 32 rows and 32 columns. Such a table is somewhat large and we only show its main portion: the multiplication table of the basis $\mathbb{T}_E$ of $\mathbb{T}$. Similarly, Tables 2, 3, 4, and 5 only show their main portions.*

Important Notation Notes:

The two blocks of the form

0	2	2	2	2	2	2	2
4	4	2	2	2	2	2	2
4	4	4	2	2	2	2	2
4	4	4	4	2	2	2	2
4	4	4	4	4	2	2	2
4	4	4	4	4	4	2	2
4	4	4	4	4	4	4	2
4	4	4	4	4	4	4	4

are more symbolic than literal. They mean that:
the 28 entries labeled 2 correspond
to the antisymmetric part of an 8x8 matrix;
the 35 entries labeled 4 correspond
to the traceless symmetric part of an 8x8 matrix; and
the 1 entry labeled 0 corresponds
to the trace of an 8x8 matrix.

A more literal, but more complicated, representation
of the graded structure of those two blocks is:

0	2,4	2,4	2,4	2,4	2,4	2,4	2,4
2,4	4	2,4	2,4	2,4	2,4	2,4	2,4
2,4	2,4	4	2,4	2,4	2,4	2,4	2,4
2,4	2,4	2,4	4	2,4	2,4	2,4	2,4
2,4	2,4	2,4	2,4	4	2,4	2,4	2,4
2,4	2,4	2,4	2,4	2,4	4	2,4	2,4
2,4	2,4	2,4	2,4	2,4	2,4	4	2,4
2,4	2,4	2,4	2,4	2,4	2,4	2,4	4

However, in the more literal representation,
the entries are not all independent. The more symbolic
representation is a more accurate reflection of the
number of independent entries of each grade.

The two blocks of the form

1	3	3	3	3	3	3	3
3	1	3	3	3	3	3	3
3	3	1	3	3	3	3	3
3	3	3	1	3	3	3	3
3	3	3	3	1	3	3	3
3	3	3	3	3	1	3	3
3	3	3	3	3	3	1	3
3	3	3	3	3	3	3	1

can be taken more literally, as they mean that:
the 8 entries labelled 1 correspond
to the diagonal part of an 8x8 matrix; and
the 56 entries labelled 3 correspond
to the off-diagonal part of an 8x8 matrix.

The even subalgebra Cle(0,8) of Cl(0,8) is then
the block diagonal

0	2	2	2	2	2	2	2
4	4	2	2	2	2	2	2
4	4	4	2	2	2	2	2
4	4	4	4	2	2	2	2
4	4	4	4	4	2	2	2
4	4	4	4	4	4	2	2
4	4	4	4	4	4	4	2
4	4	4	4	4	4	4	4

4	4	4	4	4	4	4	4
6	4	4	4	4	4	4	4
6	6	4	4	4	4	4	4
6	6	6	4	4	4	4	4
6	6	6	6	4	4	4	4
6	6	6	6	6	4	4	4
6	6	6	6	6	6	4	4
6	6	6	6	6	6	6	8

The SLx matrix action of sedenion left-multiplication by x
restricted to the block diagonal
of the even subalgebra Cle(0,8) is then

OLx
ORx

and
the block diagonal part of the SL matrices
is just the direct sum OL + OR
each of which is an 8x8 real matrix
acts on 8-dimensional vector space
isomorphically
to its action of 8-dimensional spinor space OSPINOR.

Denote the OL spinor space by OSPINOR+
and the OR spinor space by OSPINOR-.

Then, the direct sum OSPINOR+ + OSPINOR-
represent
the +half-spinor space and the -half-spinor space
of the Clifford algebra Cl(0,8) of Spin(0,8)

The +half-spinor space OSPINOR+ is acted on
by the OL elements of Cle(0,8) of

grade	0	2	4
dimension	1	28	35

while

the -half-spinor space OSPINOR- is acted on
by the OR elements of Cle(0,8) of

grade	4	6	8
dimension	35	28	1

we have the useful result that
the block diagonal part of the
adjoint left action SL of sedenions
represents
the 16-dimensional full spinor representation of
the *Clifford algebra Cl(0,8) of the Lie algebra Spin(0,8).

Now (at last) we can see what all this has
to do with sedenions.

The sedenion multiplication table is 16x16
so it has 256 = 2^8 entries and can be written
as a 16x16 matrix:

```
   r i j k E I J K S T U V W X Y Z
 r x x x x x x x x x x x x x x x x
-i x x
-j x   x q
-k x     x
-E x       x o o o
-I x         x o o
-J x           x o
-K x             x
-S x               x s s s s s s s
-T x                 x s s s s s s
-U x                   x s s s s s
-V x                     x s s s s
-W x                       x s s s
-X x                         x s s
-Y x                           x s
-Z x                             x
```

The 16+15+15 = 46 x entries denote the "real" products that
cannot belong to an associative triple cycle of the type ijk.

For the ri part of the table, the complex numbers,
there are no associative triple cycles.

For the rijk part of the table, the quaternions,
there is only one associative triple cycle,
the ijk triple itself, denoted by the q entry.
It occupies the upper triangular part
of the lower right quadrant, and so is
only 1/2 (from picking the upper triangular part)
of 1/3 (from picking the lower right quadrant
out of the 3 new quadrants that come from
extending the complex numbers to the quaternions).
That is equivalent to considering the overlapping
arising from the 6 permutations of the
one associative triple cycle ijk, for which the
first two elements would be ij,jk,ki,ji,kj,ik.

The 6 o entries represent the 6 new
associative triple cycles that come with the octonions.

The *28 s entries represent the 28 new
*associative triple cycles that come with the sedenions.

R*ecall that
the block diagonal part of SL sedenion left-multiplication
corresponds to the full spinor representation of the
16x16=256-dim Clifford algebra Cl(0,8):

2^8 = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1

The 28-dimensional bivector space represents
the 28-dimensional Lie algebra Spin(0,8),
using the commutator product.
It is called the 28-dimensional adjoint
representation of Spin(0,8).

The 8-dimensional vector space represents
the vector space on which the Lie group Spin(0,8)
acts as a rotation group.
It is called the 8-dimensional vector
representation of Spin(0,8).

Each bivector, or Lie algebra element,
can be represented by a "new" sedenion
associative triple.

THIS IS A NEW WAY TO LOOK AT *LIE ALGEBRAS.

Now look at the even subalgebra Cle(0,8) of Cl(0,8).
Since Cl(0,8) is given by

2^8 = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1

Cle(0,8) is given by

1 + 28 + 70 + 28 + 1

Now pick a 7-dimensional subspace (imaginary octonions)
of the 8-dimensional vector space (octonions), and
look at the rotations (Spin(0,8) bivectors) whose
axes are those 7 vectors. That lets us pick out
7 of the 28 Spin(0,8) bivectors and regard them as
vectors in a 7-dimensional space. The other 21
bivectors then represent rotations in that 7-dimensional
space, that is, they represent elements of Spin(0,7),
the Lie algebra of the Cl(0,7) Clifford algebra.
Therefore, Cle(0,8) then can be written as:

1 + 7 + 21 + 70 + 21 + 7 + 1

We have already seen that the Spin(0,7) Clifford algebra
can be written as:

2^7 = 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1

So we can see, by splitting the 70-dimensional

4-vector space of Cl(0,8) into two spaces,
one 35-dimensional associative 3-vector space of Cl(0,7),
and
one 35-dimensional coassociative 4-vector space of Cl(0,7),
that

le(0,8), the even subalgebra of Cl(0,8),
is just Cl(0,7), the real 7-dimensional Clifford algebra.

Going to lower and lower dimensions,
we can see that the 6 o entries
representing the 6 new associative triple cycles that
come with the octonions
represent the 6-dimensional Lie algebra Spin(0,4),
or rotations in 4-dimensional spacetime,
which looks like
3-dimensional spatial rotations
plus 3-dimensional Lorentz boosts.

Also, the one q entry
representing the associative quaternion triple
represents the 1-dimensional Lie algebra Spin(0,2) = U(1),
the rotations in the complex plane.

WHAT ABOUT GOING UP TO HIGHER DIMENSIONS?

For 32-ons, we get 120 new associative triple cycles,
and they represent the Lie algebra Spin(0,16) of
the Clifford algebra Cl(0,16).

HOWEVER, *NOTHING REALLY NEW HAPPENS BECAUSE OF
*THE PERIODICITY PROPERTY OF REAL CLIFFORD ALGEBRAS.

The periodicity theorem says that
Cl(0,N+8) = Cl(0,8) x Cl(0,N) (here x = tensor product)

That means that the Clifford algebra Cl(0,16) of the 32-ons
is just the tensor product of two copies
of the Clifford algebra Cl(0,8).

So, everything that happens in the 32-on Clifford algebra
is just a product of what happens with Cl(0,8).

That point is emphasized by the fact
(see Lohmus, Paal, and Sorgsepp,
Nonassociative Algebras in Physics (Hadronic Press 1994))
that the derivation algebra of ALL Cayley-Dickson algebras
at the level of octonions or larger,
that is, of dimension 2^N where N = 3 or greater,
is the exceptional Lie algebra G2,
the Lie algebra of the automorphism group of the octonions.

The exceptional Lie algebra G2 is 14-dimensional,
larger than the 8-dimensional octonions,
but smaller than the 16-dimensional sedenions.

Each 8-dimensional half-spinor space of Cl(0,8)
has a 7-dimensional subspace of "pure" spinors
(see Penrose and Rindler, Spinors and space-time,
vol. 2, Cambridge 1986)
that correspond by triality to the
7-dimensional null light-cone of the 8-dim vector space.

The 7-dimensional spaces are 7-dimensional representations
of the 14-dimensional Lie algebra G2.
With respect to the *D4-D5-E6-E7 physics model,
the block diagonal part of
the SL sedenion left-multiplication matrix
represents the 16-dimensional full spinors of Cl(0,8),
whose bivector Lie algebra is 28-dim D4 Spin(0,8).

You can then make the conformal group, 45-dim D5 Spin(1,9),
extending 28-dim D4 Spin(0,8),
by forming the 2x2 matrices over Cl(0,8),
which are the Clifford algeba Cl(1,9),
as described by Ian Porteous
in chapters 18 and 23 of
Clifford Algebras and the Classical Groups (Cambridge 1995).
Note that there is a Spin(0,2) = U(1) symmetry
due to the 2x2 matrices that can be used

to give the vector extension to D5 from D4
a complex structure.

The conformal group, 78-dim E6,
extending 45-dim D5 Spin(1,9),
may be formed in an exceptional way,
by constructing E6 commutation relations
WITHIN the Cl(1,9) Clifford algebra,
by going from the Spin(1,9) group of Cle(1,9)
to the Pin(1,9) group of Cl(1,9).
The effect is to add to 45-dim Spin(1,9)
the 32-dim full spinors of Cle(1,9)
plus a Spin(0,2) = U(1) symmetry arising from
the 2-fold covering of Cle(1,9) by Cl(1,9).
The Spin(0,2) = U(1) symmetry gives the spinors
a complex structure.
(Such a construction cannot be done in all dimensions.
It is exceptional, and is due to underlying octonion structures.)

Then, as discussed by Gilbert and Murray
in Clifford algebras and Dirac operators
in harmonic analysis (Cambridge 1991),
Dirac operators and related structures
can be constructed and used to build
the *D4-D4-E6 physics model.

Sedenions may also be a good way to express
the structure of the last Hopf fibration
S7 --> S15 --> S8.
Since there are no Hopf fibrations other than
S0 --> S1 --> RP1 = Spin(2)/Spin(1)
S1 --> S3 --> S2 = CP1 = SU(2)/U(1) = Sp(1)/U(1) =
Spin(3)/Spin(2)
S3 --> S7 --> S4 = HP1 = Sp(2)/(Sp(1)xSp(1)) =
Spin(5)/Spin(4)
S7 --> S15 --> S8 = OP1 = Spin(9)/Spin(8)
sedenion structure may show why the sequence terminates.

The sedenions may show the structure

of the 16-dimensional octonionic projective plane OP2 = F4/Spin(9)
and may show
why there is no higher-dimensional octonionic projective space.

I think that the highest and best use of sedenions
is not for the full algebra structure
but
as a very useful expression
of "global" organization of the structure of octonions.

Other useful 16-dimensional structures are
the *16-dim Barnes-Wall lattice /\16
and

to me, the useful part of the sedenion product is
not the full sedenion product,
but only the block diagonal part

OLx	
	ORx

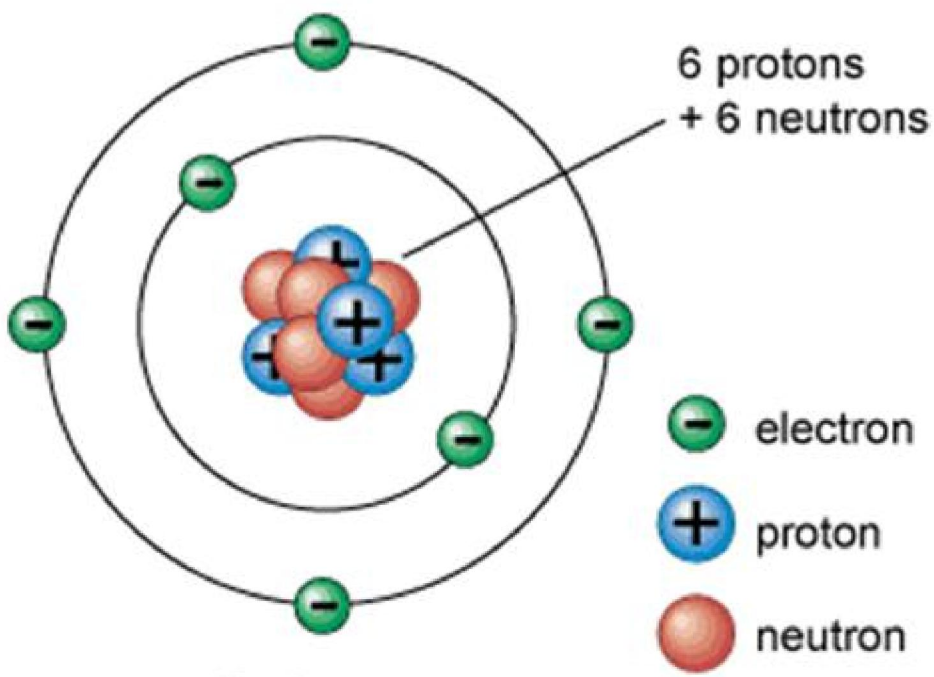
of the SL sedenion left-multiplication matrix,
because
it represents the 16-dimensional full spinors of Cl(0,8)
and
it is equivalent to the direct sum OL + OR
of
octonion left-multiplication
and
octonion right-multiplication.

28 Nakshatra as Key to Physics

A book on Vedic nuclear physics discusses the atom and the need for an angular division for a coherent state in this way:

The organic spectrum is governed by a flexible Bhava disposition that retains the Linga coherent state. The angular division for a coherent state must have a ratio of ½ or 30 degrees.

Axiomatically then, a coherent structure with six or twelve divisions in a cycle would be required. For this reason, carbon - based chemistry forms the base of organic states.



Carbon atom

For similar reasons, astral stress transmigrations that distort or twist this hexagonal form only need to be studied. Astrological and Ayurvedic divisions are based on 12 sectors, with 10 as the summation total following the Sankhya Guna principle of self -

similar interactions.

For this reason, 120 divisions empirically relate the twist due to stresses caused by planetary transits or configurations. The Vernier Effect of 10-1=9 gives the degree of overlap of adjacent angular dispositions.

Vernier Scale

The vernier scale is constructed so that it is spaced at a constant fraction of the fixed main scale. So for a decimal measuring device each mark on the vernier is spaced nine tenths of those on the main scale. If you put the two scales together with zero points aligned, the first mark on the vernier scale is one tenth short of the first main scale mark, the second two tenths short, and so on up to the ninth mark—which is misaligned by nine tenths. Only when a full ten marks are counted is there alignment, because the tenth mark is ten tenths—a whole main scale unit short, and therefore aligns with the ninth mark on the main scale.
Now if you move the vernier by a small amount, say, one tenth of its fixed main scale, the only pair of marks that come into alignment are the first pair, since these were the only ones originally misaligned by one tenth. If we move it two tenths, the second pair aligns, since these are the only ones originally misaligned by that amount. If we move it five tenths, the fifth pair aligns—and so on. For any movement, only one pair of marks aligns and that pair shows the value between the marks on the fixed scale.

The coherent divisions from the Andhathaamisra regions give 28 orders of counts that remain numerically stable . The 12 sector (30 deg.) give the reference level for characteristics that add or subtractact from the three guna states and the transitions by the linga/bhava and the abhiman/ahankar polarisation factor.

The 28 levels of change in descending order are characterized by the Nakshatra nomenclature in the Atharvaveda. The original classification of genetic characteristics affected by the stress transmigration patterns were defined in the 28 nakshatra qualities. Starting from the zero level ecliptic crossing called Punarvasu, when the organic form remains in its most flexible and balanced state, the list of 28 positions and their principle influence on the twelve sectors are identified in the Nakshatra devatyam passages in the Atharvaveda.

The Nakshatra devatyam passages lay the ratio of change by which the zero position of the ecliptic changed due to the precession of the equinoxes in about 25,760 years. The original classification of genetic characteristics affected by the stress transmigration patterns were defined in the 28 nakshatra qualities.

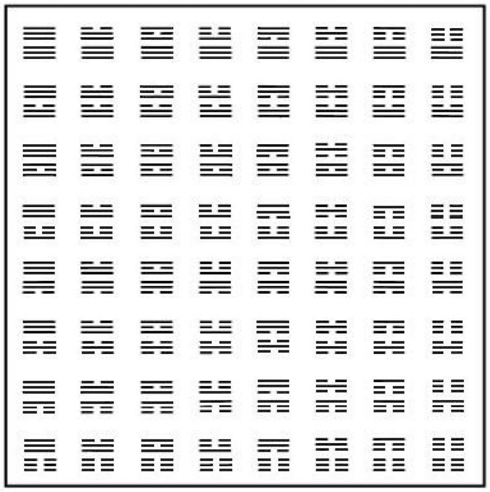


Figure 1. The **64** hexagrams of the I Ching table.

Trigrams of the eight hexagrams along the diagonal of the table symbolise the three cube faces whose intersection defines a corner of a cube. The pairing of each trigram with a different one symbolises an association between the three faces defining a corner of the central cube and the three faces of one of the cubes surrounding it that is contiguous with this corner. Each corner of the cube at the centre of the 3×3×3 array of cubes (the grey one in Fig. 2a) is the point of intersection of the corners of seven other cubes. Three of

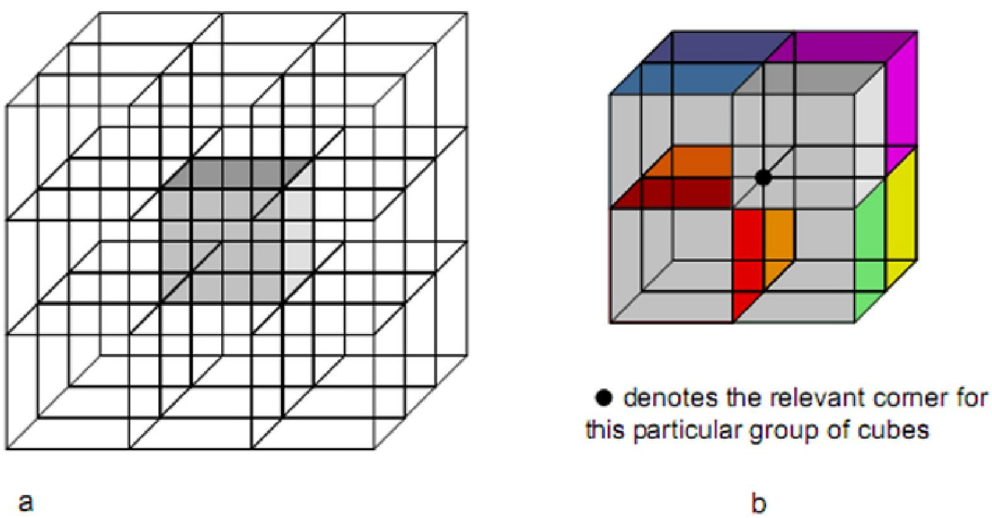


Figure 2. Each corner of the grey cube (a) at the centre of a 3×3×3 array of cubes touches three cubes (coloured blue, indigo & violet in (b)) at the same level as itself and four cubes (red, orange, yellow &

green) that are either above or below it.

These (the violet, indigo and blue cubes in Fig. 2b) are at the same height as the cube and the remaining four (the green, yellow, orange and red ones) are above or below it. Let us express the pattern of eight cubes centred on any corner of the central cube as

$$8 = (1 + 3) + 4, \quad (1)$$

where '1' always denotes the central cube, '3' denotes the three cubes contiguous with it at the same height and '4' denotes the four contiguous cubes

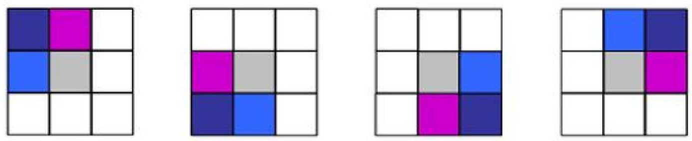


Figure 3. Three cubes on the same level are contiguous with each corner of the central (grey) cube.

above or below it. As we jump from corner to corner in the top face of the central cube, the L-shaped pattern of three coloured cubes in the same plane rotates (Fig. 3), as do the four cubes above them. Likewise, as we go from corner to corner in the bottom face of the central cube, the three cubes in the same plane change, as do the four cubes below them. Movement through all eight corners of the central cube involves every one of the **26**[1] cubes in the

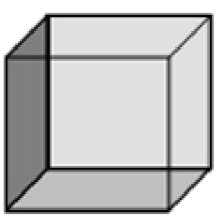


Figure 4. Each corner of a cube is the intersection of three faces of each of the seven cubes that surround it.

$3\times3\times3$ array because they are all contiguous with its corners.
Each corner of the central cube is the point of intersection of the three orthogonal faces (3) of each of the seven cubes that are contiguous with it (Fig. 4). The number of such faces generating the corners of the

central cube is

$$8\times\underline{3}\times 7 = \mathbf{168}. \tag{2}$$

Counting these corners in the same way as the pattern of cubes, that is, differentiating any corner from the three corners at the same height and the four corners below them, then

$$\begin{aligned}\mathbf{168} &= [(1+3) + 4]\times\underline{3}\times 7 \\ &= 24\times 7,\end{aligned} \tag{3}$$

Where

$$24 = [(1+3) + 4]\times\underline{3}. \tag{4}$$

The three orthogonal faces consist of the face perpendicular to one coordinate axis (for convenience, let us choose the X-axis) and the two faces perpendicular to the two other axes. So

$$\underline{3} = 1_X + 1_Y + 1_Z. \tag{5}$$

Therefore,

$$24 = [(1+3) + 4]\times(1_X + 1_Y + 1_Z)$$
$$3$$

$$= (1_X + 1_Y + 1_Z) + (3_X + 4_X) + (3_Y + 4_Y) + (3_Z + 4_Z), \tag{6}$$

where 3_X ($\equiv 3\times 1_X$) signifies the three faces of cubes that are orthogonal to the X-axis and associated with the three corners at the same height as the corner labelled ‘1’ (and similarly for the Y- and Z-axes), and 4_X ($\equiv 4\times 1_X$) denotes the four faces orthogonal to the X-axis associated with the four corners below corner ‘1’ (and similarly for 4_Y and 4_Z). Therefore, substituting Equation 6 in Equation 3,

$$\mathbf{168} = [(1_X + 1_Y + 1_Z) + (3_X + 4_X) + (3_Y + 4_Y) + (3_Z + 4_Z)]\times 7, \tag{7}$$

The **168** faces of the **26** cubes surrounding the central cube that touch its eight corners consist of seven sets of 24 faces.

Each set consists of the three orthogonal faces of the cube corresponding to corner '1', the seven faces perpendicular to the X-axis at the seven other corners, the seven faces perpendicular to the Y-axis at these corners and the seven faces perpendicular to the Z-axis at these corners.

Each group of seven faces comprises a face for each of the three corners of the central cube at the same height as corner '1' and a face for each of the four corners below it. The 24 faces naturally divide into two groups of twelve:

$$24 = (1_X + 1_Y + 1_Z) + (3_X + 4_X) + (3_Y + 4_Y) + (3_Z + 4_Z),$$
$$= (4_X + 4_Y + 4_Z) + [(1_X + 1_Y + 1_Z) + 3_X + 3_Y + 3_Z].$$
$$= 12 + 12',$$

where

$$12 \equiv 4_X + 4_Y + 4_Z \qquad (8)$$

and

$$12' \equiv (1_X + 1_Y + 1_Z) + 3_X + 3_Y + 3_Z. \qquad (9)$$

Hence,

$$\mathbf{168} = 24\times7 = (12 + 12')\times7 = 84 + 84',$$

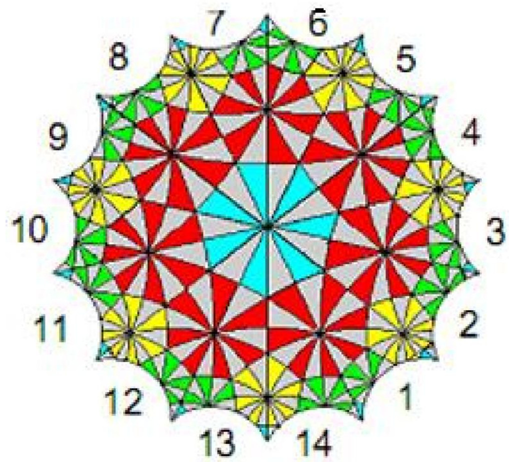
where

$$84 \equiv (4_X + 4_Y + 4Z)\times7 \qquad (10)$$

and

$$84' \equiv [(1_X + 1_Y + 1_Z) + 3_X + 3_Y + 3_Z]\times7. \qquad (11)$$

We see that the **168** faces are made up of two distinct sets of 84 faces. Each set comprises 28 faces orthogonal to the X-axis, 28 faces orthogonal to the Y-axis and 28 faces orthogonal to the Z-axis. One set is associated with the four upper corners and the other set is associated with the four lower corners.



14 slices
12 coloured triangles per slice
14×12 = **168** coloured triangles in 24 heptagons

Figure 7. The Klein Configuration of the **168** elements of the group PSL(2,7) of transformations of the Klein quartic.

The Klein Configuration can be embedded in a 3-torus because the latter is a compact manifold of genus 3. The 3-torus is topologically equivalent to a cube whose opposite faces are imagined to be glued together (Fig. 10), so that movement along each coordinate axis is periodic, like motion around a circle.

This means that all eight corners of a cube represent the same point of a space that has the topology of a 3-torus. However, suppose that they were tagged or labeled in some way that differentiated them.

As the cube has three 4-fold axes of symmetry, namely, rotations of 90°, 180° and 270° about the X-, Y- and Z-axes, there are nine possible configurations of labelled corners into which the original one can be rotated, that is, there are ten different configurations of corners connected by these rotations about the coordinate axes. They form the tetractys pattern shown in Fig. 11.

The cubes at the corners of the tetractys are the central cube after rotation through 90°, 180° and 270° about the X-axis. The two triads of cubes that are connected by dotted lines are the result of similar rotations about the Y- and Z-axes.

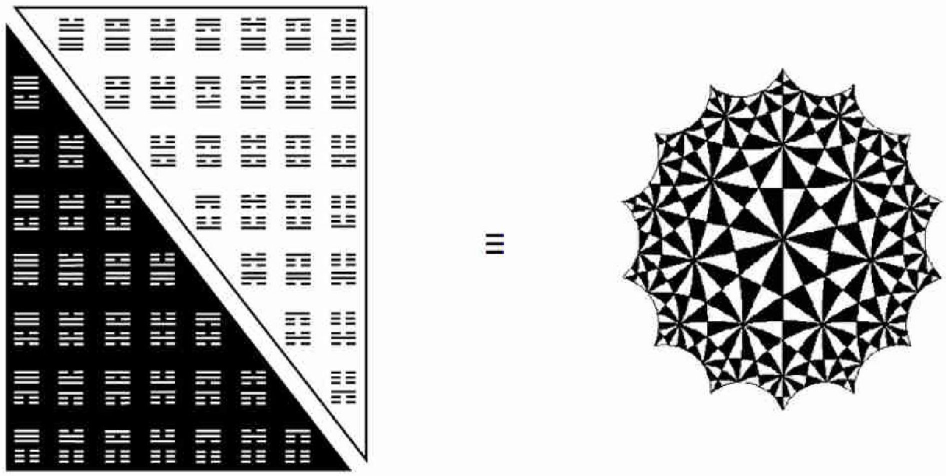
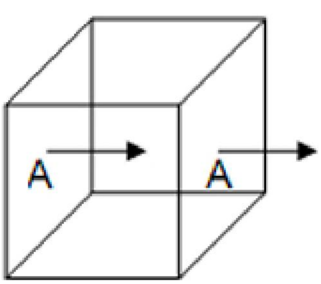
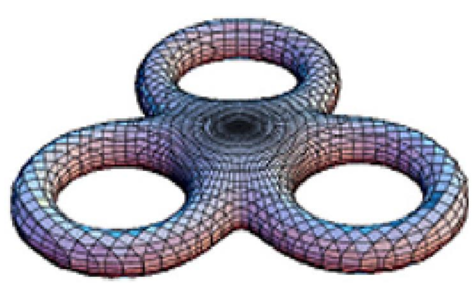


Figure 8. The isomorphism between the I Ching table and the Klein Configuration. The **168** (▲) correspond to the **168** yin and yang lines in the 28 hexagrams in the white triangle on one side of the diagonal of the table and the **168** (△) correspond to the **168** yin and yang lines in the 28 hexagrams in the black triangle on the other side. Each black line/triangle denotes an automorphism and a white line/triangle denotes an anti-automorphism of the Klein quartic.

$$\mathbf{168} = 7\times24 = 7\times[4_X + 4_Y + 4_Z + (1_X + 3_X) + (1_Y + 3_Y) + (1_Z + 3_Z)].$$

This shows that the **168** faces, likewise, comprise six different types of 28 faces. The splitting of each hexagram into two trigrams corresponds in the 3×3×3 array of cubes to the pair of opposite faces of a cube with four corners (the distinction between 4_X and $(1_X + 3_X)$, etc) and in the Klein Configuration to each of its seven sectors being divided into half. The I Ching table, the 3×3×3 array of cubes that it symbolises and the Klein





3-torus

Figure 10. The 3-torus, in which the Klein map can be embedded, is topologically identical to a cube whose opposite faces are imagined stuck together.

Configuration are isomorphic to one another. It was argued in Article 20 that the degree of correspondence between the first two is too high to be dismissed as coincidence. The same conclusion applies to the lattermost two.

The 1:3:4 pattern of cubes associated with the central cube corresponds in each sector of the Klein Configuration to the cyan hyperbolic triangle of the

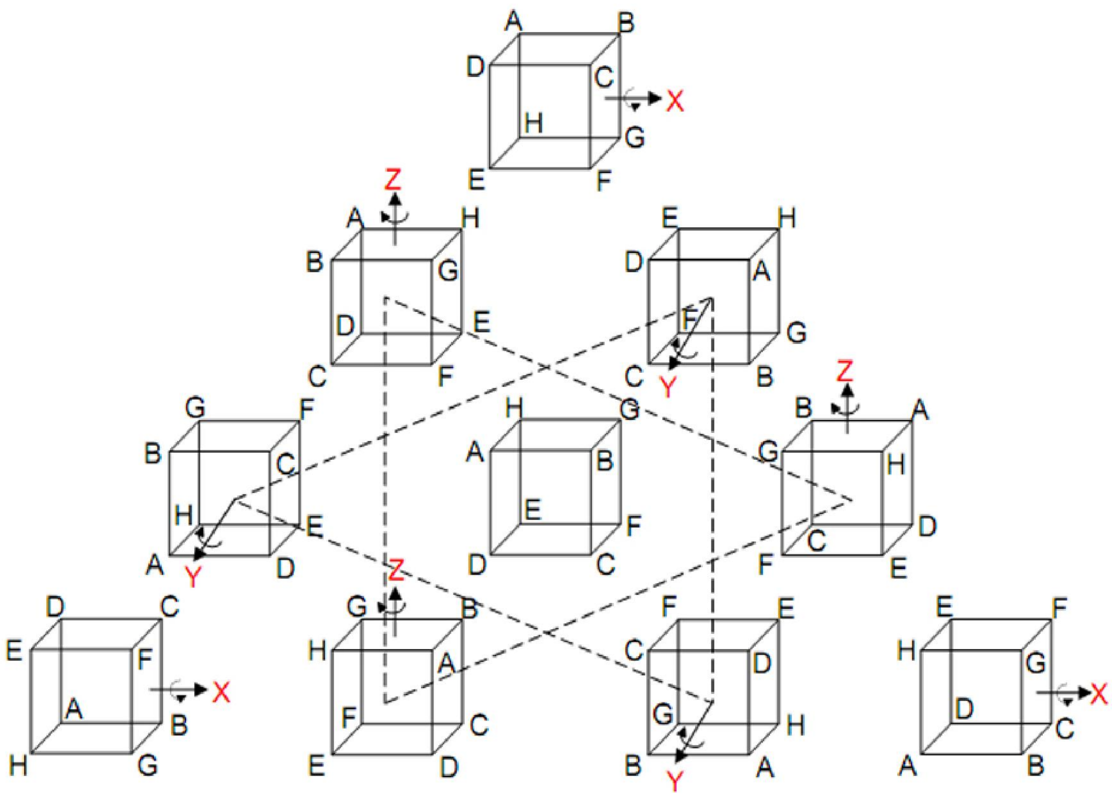


Figure 11. The three rotations of the central cube about the X-, Y- and Z-axes (labelled red) generate a tetractys array of ten orientations with the unrotated cube at its centre.

This is another example of the Tetrad Principle expressing properties of mathematical objects that are archetypal. Between the ends of its twelve sides are two corners, each defined by three

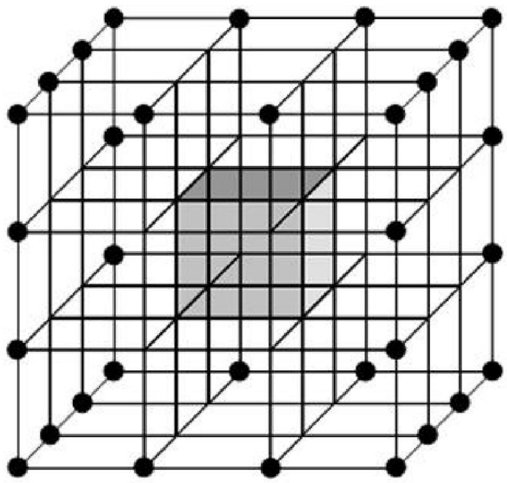
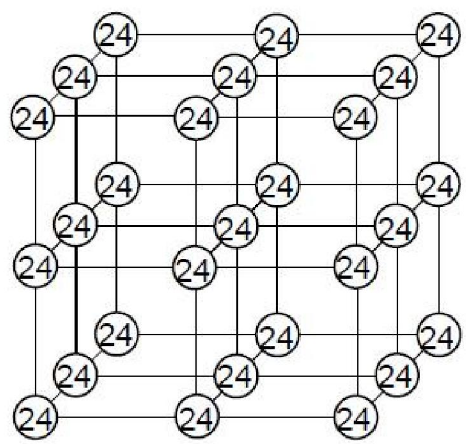


Figure 13. **168** faces of the **26** cubes in the 3×3×3 array surrounding the central (grey) cube define the 32 corners (●) along the 12 sides of the array.

faces of cubes. Three faces define each corner of the array itself. The number of cube faces defining corners along the sides of the array = 3[12×(2+2) + 8] = 3×56 = **168** . This is the same as the number of faces of the **26** cubes whose intersections define the corners of the central cube. They correspond to the **168** grey triangles of the Klein Configuration shown in Fig. 7 that represent the **168** anti-automorphisms of the Klein quartic. Including the (8×3=24) faces that define the eight corners of the central cube, the total number of faces defining the **64** corners of the array = **168** + 288 + **168** + 24 =

648 =



24 = 1×2×3×4

Figure 14. Four cubes with 1×2×3×4 assigned to their 27 corners generate the number of faces defining the **64** corners of the 3×3×3 array of cubes.

648. They consist of (648/3=**216**) sets of three perpendicular faces, where **216** = 6^3. There are (**168** +24=192) faces defining the eight corners inside the array and (288+**168** =456) faces defining the 56 corners on its surface. As 192 = 8×24 = 2^3×24 and 648 = 27×24 = 3^3×24, these two numbers can be represented by an array of four cubes (Fig. 14) with (24=1×2×3×4) assigned to each corner. This is another beautiful illustration of the Tetrad Principle at work in expressing archetypal patterns and the numbers that they embody because four joined cubes with the number 4! at each of their corners generate the number (648) of faces

defining the 4^3 corners of the 3×3×3 array of cubes, whilst one cube with 4! assigned to each of its 2^3 corners generates the number (192) of faces defining the corners of the central cube.

3. Superstring structural parameter 168

It was pointed out in Article 11[8] that the mathematical pattern through which Plato said in his *Timaeus* the Demiurge proportioned the cosmos is incomplete because it involved only the integers 1, 2 and 3, whereas Pythagoras taught that the integers 1, 2, 3 and 4 constituted the fundamental numbers expressing all aspects of the physical and spiritual cosmos.

The realisation that his famous 'Lambda' is but two sides of a tetractys of integers

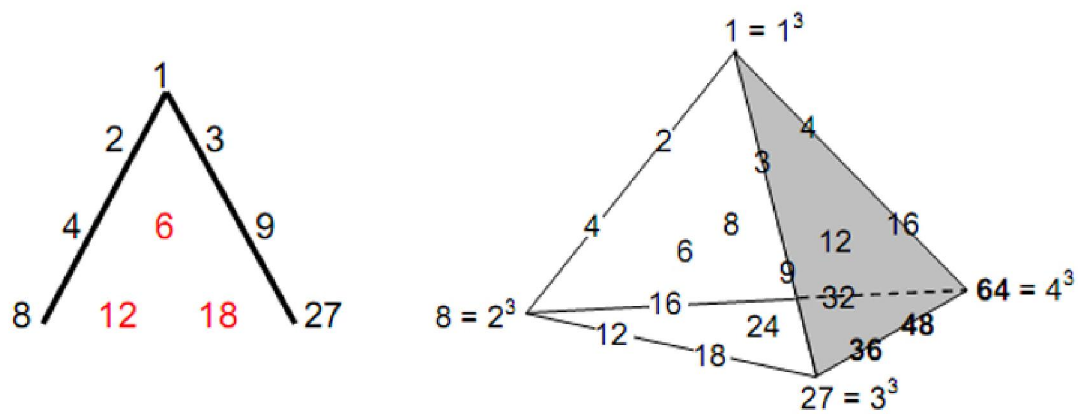


Figure 17. Plato's Lambda and its tetrahedral generalisation.

and that this tetractys is but one triangular face of a tetrahedral array of 20 integers (Fig. 17) has been demonstrated [9] to generate the tone ratios of the Pythagorean musical scale in a symmetric way that is absent from the account known to historians of music. Article 20[10] showed that the first four powers of 2, 3 and 4 spaced along the edges of the tetrahedron are the numbers of 0-cubes (points) in the sequence of generation from a point of a cube, a 2×2×2 array of cubes and a 3×3×3 array of cubes.

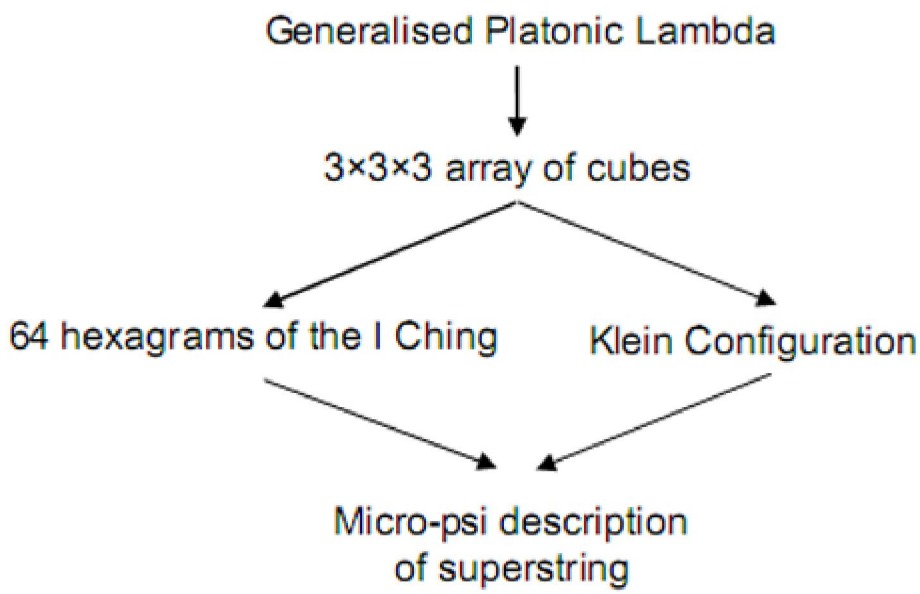


Figure 18

Ultimately, Philipps argues that the Exceptional Lie Algebra E8 squared leads to 168, and forms a key relationship to superstring theory.

Conclusion

In this paper we have seen that Sedenions may hold a key to understanding the number 28, through the extended essay by Frank "Tony" Smith. Through some of the work of S.M. Philipps, we have seen why the number 168 holds importance, and his website lists many more reasons, but chiefly the relationship to the Octonions and the Pythagorean music system hold vital importance.

A theme of this paper is that the Ancient Egyptians knew of the Sedenions, G2 and the importance of 28, the 42 Assessors of the Sedenions, if not of 168. Given the evidence presented in this paper, it would not be far-fetched to imagine a member of the Pythagorean school as having etched Greek graffiti into the Osiris Temple where many Flowers of Life have been carved into the walls. The idea that European civilization instantly materialized in the brains of Socrates, Plato and Aristotle grows increasingly more difficult to accept, while Plato's Timeas appears increasingly descriptive of the ancient world.

A fundamental tenet of this series of papers is that the ancient world contained a high level of mathematical physics, which our present civilization is nigh approaching. This concept is difficult for educated members of western nations to accept, since they consider themselves the epitome of the highest level of civilization the world has **ever** known. As Gandhi said, British civilization would have been a good idea, now that Churchill has destroyed the empire and the UK gradually sinks into a 1984 type of Orwellian nightmare.

S.M. Philipps next provides many reasons why the number 168 holds such importance in mathematical physics, and we repeat some of his arguments here. In the author's view, the most important of these isomorphic relations concern PS/ 7, the Octonions and the Fano Plane, as well as the Pythagorean music scale, which probably originated in ancient Egypt.

The number 28 holds great importance in Vedic Nuclear Physics as the author will describe in a future paper. Yet it is clear from Tony Smith's exposition, "Why Not Sedenions" that 28 holds a pre - eminent position in the relationship between Octonions and Sedenions, if not to the trigantiduonions, or the 32 - on's of Tony's original paper. Then, relations to the Hopf Fibration appear, which reflects back to the icosahedron and the dodecahedron of the Qi Men Dun Jia Model.
The main reason why the Hopf Fibration proves so important to the Qi Men Dun Jia Model is the following:

The 600-cell partitions into 20 rings of 30 tetrahedra, each a *Boerdijk–Coxeter helix*. When superimposed onto the 3-sphere curvature it becomes periodic, with a period of ten vertices, encompassing all 30 cells. The collective of such helices in the 600-cell represent a discrete Hopf fibration. While in 3 dimensions the edges are helices, in the imposed 3-sphere topology they are geodesics and have no torsion. They spiral around each other naturally due to the Hopf fibration.

Consider that the Boerdijk - Coxeter Helix consists of a string of tetrahedra, which are composed of Sedenions, which are composed of Octonions. The active ingredient of the Boerdijk - Coxeter Helix is the Hopf Fibration, which causes the edges to spiral around one another, that is that they comprise the collective of helices in the 600 - cell.

Finally, we note the following paper and its tracing from S3 to E4:

Sphere packing, helices and the polytope $\{3, 3, 5\}$

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Received 27 March 2001

Abstract. The packing of tetrahedra in face contact is well-known to be relevant to atomic clustering in many complex alloys. We briefly review some of the structures that can arise in this way, and introduce methods of dealing with the geometry of the polytope $\{3, 3, 5\}$, which is highly relevant to an understanding of these structures. Finally, we present a method of projection from S_3 to E_3 that enables coordinates for the key vertices of the collagen model of Sadoc and Rivier to be calculated.

In which is stated:

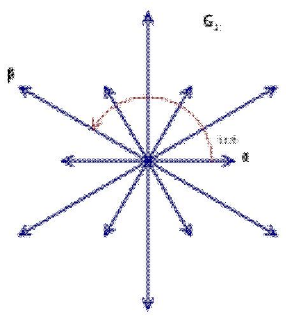
Table 1. Radii of successive shells around a vertex of $\{3,3,5\}$.

$2x_0$	n		d	$\cos\alpha$
2	1	"centre"	0	1
τ	12	icosahedron	$1/\tau$	τ
1	20	dodecahedron	1	$1/2$
$-\sigma$	12	icosahedron	$\sqrt{(3-\tau)}$	$1/2\tau$
0	30	icosidodecahedron	$\sqrt{2}$	0
σ	12	icosahedron	τ	$-1/\tau$
-1	20	dodecahedron	$\sqrt{3}$	$-1/2$
$-\tau$	12	icosahedron	$\sqrt{(2+\tau)}$	$-\tau/2$
-2	1	antipodal vertex	2	-1

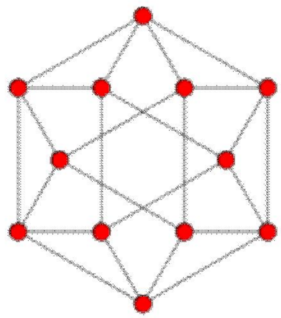
Which thus illustrate the importance of icosahedral and dodecahedral architecture to the 335 Polytope. Similar architecture proves key to the Qi Men Dun Jia Model. Finally, we note that the BC Helix stands composed of Sedenions and Octonions in tetrahedra form, which demonstrates the importance of linking Octonions and Sedenions, binary and ternary forms through the number 28.

Appendix I

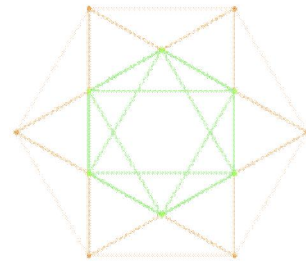
G_2 Roots of G_2



The 12 vector root system of G_2 in 2 dimensions.



The A_2 Coxeter plane projection of the 12 vertices of the cuboctahedron contain the same 2D vector arrangement.



Graph of G2 as a subgroup of F4 and E8 projected into the Coxeter plane

G2 is the name of three simple Lie groups (a complex form, a compact real form and a split real form), their Lie algebras $\mathfrak{g}_2$, as well as some algebraic groups. They are the smallest of the five exceptional simple Lie groups. G_2 has rank 2 and dimension 14. Its fundamental representation is 7-dimensional. The compact form of G_2 can be described as the automorphism group of the octonion algebra or, equivalently, as the subgroup of SO(7) that preserves any chosen particular vector in its 8-dimensional real spinor representation.
In older books and papers, G_2 is sometimes denoted by E_2.

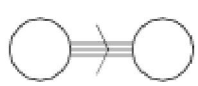
Real forms

There are 3 simple real Lie algebras associated with this root system:

- The underlying real Lie algebra of the complex Lie algebra G_2 has dimension 28. It has complex conjugation as an outer automorphism and is simply connected. The maximal compact subgroup of its associated group is the compact form of G_2.
- The Lie algebra of the compact form is 14 dimensional. The associated Lie group has no outer automorphisms, no center, and is simply connected and compact.
- The Lie algebra of the non-compact (split) form has dimension 14. The associated simple Lie group has fundamental group of order 2 and its outer automorphism group is the trivial group. Its maximal compact subgroup is $SU(2) \times SU(2)/(-1,-1)$. It has a non-algebraic double cover that is simply connected.

Algebra

Dynkin diagram and Cartan matrix

The Dynkin diagram for G_2 is given by .
Its Cartan matrix is:

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Although they span a 2-dimensional space, as drawn, it's much more symmetric to consider them as vectors in a 2-dimensional subspace of a three dimensional space.

(1,−1,0),	(2,−1,−1),
(−1,1,0)	(−2,1,1)
(1,0,−1),	(1,−2,1),
(−1,0,1)	(−1,2,−1)
(0,1,−1),	(1,1,−2),
(0,−1,1)	(−1,−1,2)

One set of **simple roots**, for  is:

(0,1,−1), (1,−2,1)

Weyl/Coxeter group

Its Weyl/Coxeter group is the dihedral group, D_6 of order 12.

Special holonomy

G_2 is one of the possible special groups that can appear as the holonomy group of a Riemannian metric. The manifolds of G_2 holonomy are also called G_2-manifolds.

Polynomial Invariant

G_2 is the automorphism group of the following two polynomials in 7 non-commutative variables.

$$C_1 = t^2 + u^2 + v^2 + w^2 + x^2 + y^2 + z^2$$

$$C_2 = tuv + wtx + ywu + zyt + vzw + xvy + uxz \text{(± permutations)}$$

which comes from the octionion algebra. The variables must be non-commutative otherwise the second polynomial would be identically zero.

Generators

Adding a representation of the 14 generators with coefficients A..N gives the matrix:

$$A\lambda_1 + ... + N\lambda_{14} = \begin{bmatrix} 0 & C & -B & E & -D & -G & -F+M \\ -C & 0 & A & F & -G+N & D-K & E+L \\ B & -A & 0 & -N & M & L & K \\ -E & -F & N & 0 & -A+H & -B+I & -C+J \\ D & G-N & -M & A-H & 0 & J & -I \\ G & K-D & -L & B-I & -J & 0 & H \\ F-M & -E-L & -K & C-J & I & -H & 0 \end{bmatrix}$$

Representations

The characters of finite dimensional representations of the real and complex Lie algebras and Lie groups are all given by the Weyl character formula. The dimensions of the smallest irreducible representations are (sequence A104599 in OEIS):

1, 7, 14, 27, 64, 77 (twice), 182, 189, 273, 286, 378, 448, 714, 729, 748, 896, 924, 1254, 1547, 1728, 1729, 2079 (twice), 2261, 2926, 3003, 3289, 3542, 4096, 4914, 4928 (twice), 5005, 5103, 6630, 7293, 7371, 7722, 8372, 9177, 9660, 10206, 10556, 11571, 11648, 12096, 13090….

The 14-dimensional representation is the adjoint representation, and the 7-dimensional one is action of G_2 on the imaginary octonions.
There are two non-isomorphic irreducible representations of dimensions 77, 2079, 4928, 28652, etc. The fundamental representations are those with dimensions 14 and 7 (corresponding to the two nodes in the Dynkin diagram in the order such that the triple arrow points from the first to the second).
Vogan (1994) described the (infinite dimensional) unitary irreducible representations of the.split real form of G_2.

Finite groups

The group $G_2(q)$ is the points of the algebraic group G_2 over the finite field $\mathbf{F}_q$. These finite groups were first introduced by Leonard Eugene Dickson in Dickson (1901) for odd q and Dickson (1905) for even q. The order of $G_2(q)$ is $q^6(q^6-1)(q^2-1)$. When $q\neq 2$, the group is simple, and when $q = 2$, it has a simple subgroup of index 2 isomorphic to ${}^2A_2(3^2)$. The J_1 was first constructed as a subgroup of $G_2(11)$. Ree (1960) introduced twisted Ree groups ${}^2G_2(q)$ of order $q^3(q^3+1)(q-1)$ for $q=3^{2n+1}$ an odd power of 3.

See also

- Seven-dimensional cross product

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  See section 4.1: G_2; an online HTML version of which is available at http://math.ucr.edu/home/baez/octonions/node14.html.
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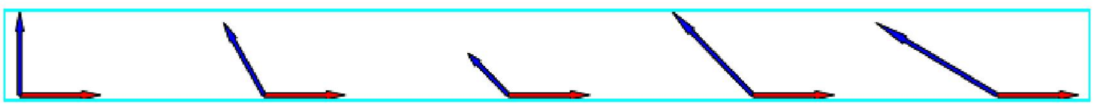
The Root System of G2 from its Dynkin Diagram

$D_2 = so(4)$ $A_2 = su(3)$ $B_2 = so(5)$ $C_2 = sp(2 \cdot 2)$ $G_2 = Aut(\mathbb{O})$

FIGURE 3. Rank 2 Dynkin Diagrams



$D_2 = so(4)$ $A_2 = su(3)$ $B_2 = so(5)$ $C_2 = sp(2 \cdot 2)$ $G_2 = Aut(\mathbb{O})$

FIGURE 4. Rank 2 Simple Roots

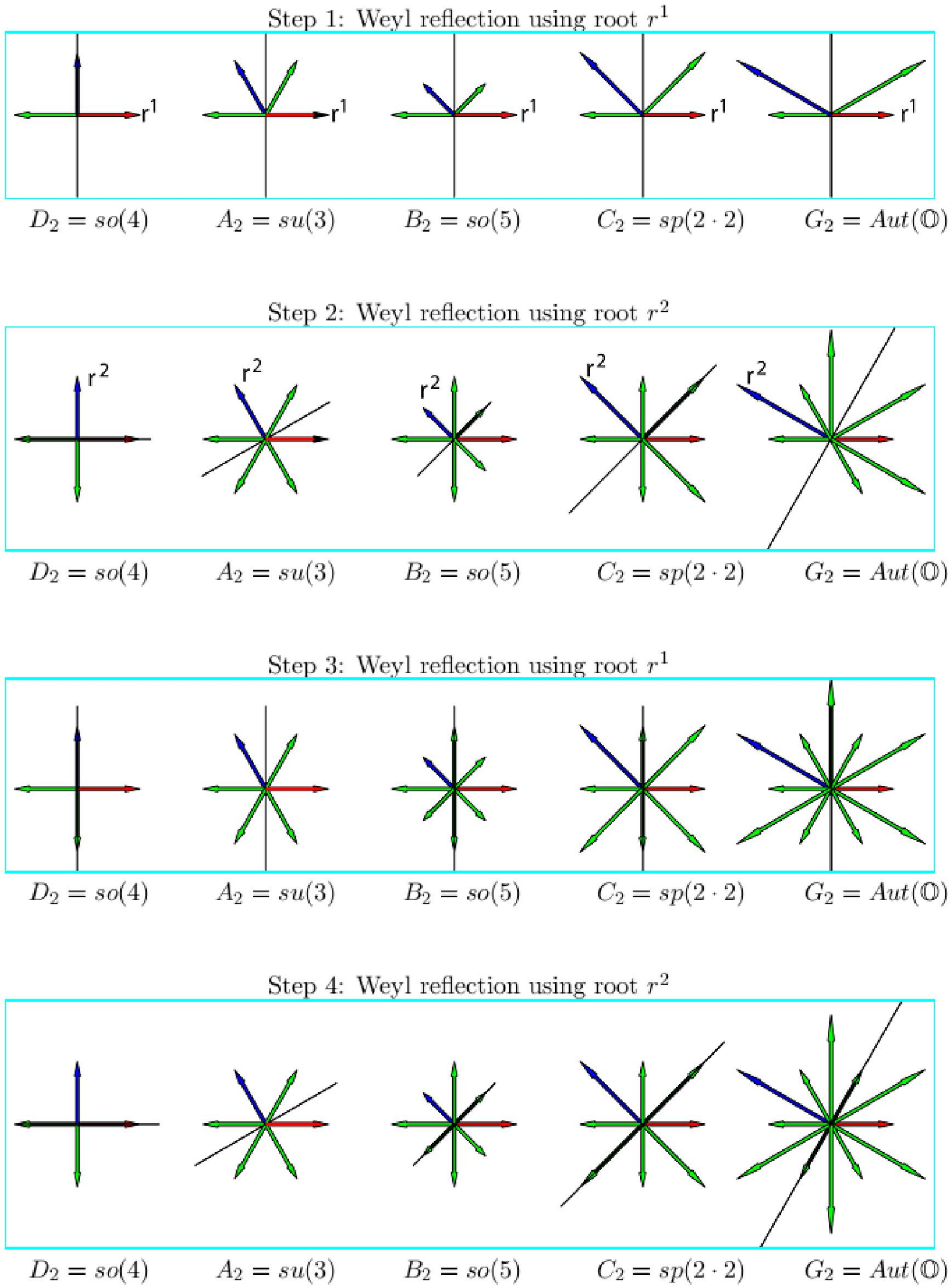


FIGURE 5. Generating an algebra's full root system using Weyl reflections

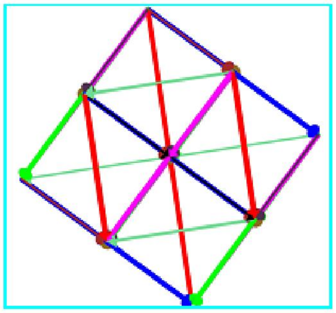


FIGURE 14. $B_2 = so(5) \subset B_3 = so(7)$

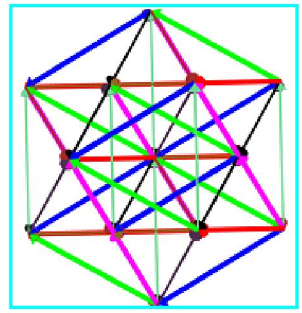


FIGURE 15. $G_2 \subset B_3 = so(7)$

Identify Subalgebras of $B_3 = so(7)$ using Projections of its Root Diagram

VISUALIZING LIE SUBALGEBRAS
USING ROOT AND WEIGHT DIAGRAMS
AARON WANGBERG AND TEVIAN DRAY
Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

Appendix II

27 Nakshatra and their 4 Pada

#	Name	Pada 1	Pada 2	Pada 3	Pada 4
1	Ashwini (अश्विनि)	चु Chu	चे Che	चो Cho	ला La
2	Bharani (भरणी)	ली Li	लू Lu	ले Le	लो Lo
3	Kritika (कृत्तिका)	अ A	ई I	उ U	ए E
4	Rohini(रोहिणी)	ओ O	वा Va/Ba	वी Vi/Bi	वु Vu/Bu
5	Mrigashīrsha (म्रृगशीर्षा)	वे Ve/Be	वो Vo/Bo	का Ka	की Ke
6	Ārdrā (आर्द्रा)	कु Ku	घ Gha	ङ Ng/Na	छ Chha
7	Punarvasu (पुनर्वसु)	के Ke	को Ko	हा Ha	ही Hi
8	Pushya (पुष्य)	हु Hu	हे He	हो Ho	ड Da
9	Āshleshā (आश्लेषा)	डी Di	डू Du	डे De	डो Do
10	Maghā (मघा)	मा Ma	मी Mi	मू Mu	मे Me
11	Pūrva or Pūrva Phalgunī (पूर्व फाल्गुनी)	नो Mo	टा Ta	टी Ti	टू Tu
12	Uttara or Uttara Phalgunī (उत्तर फाल्गुनी)	टे Te	टो To	पा Pa	पी Pi
13	Hasta (हस्त)	पू Pu	ष Sha	ण Na	ठ Tha
14	Chitra (चित्रा)	पे Pe	पो Po	रा Ra	री Ri
15	Svātī (स्वाति)	रू Ru	रे Re	रो Ro	ता Ta
16	Viśākhā (विशाखा)	ती Ti	तू Tu	ते Te	तो To
17	Anurādhā (अनुराधा)	ना Na	नी Ni	नू Nu	ने Ne
18	Jyeshtha (ज्येष्ठा)	नो No	या Ya	यी Yi	यू Yu
19	Mula (मूल)	ये Ye	यो Yo	भा Bha	भी Bhi
20	Pūrva Ashādhā (पूर्वाषाढ़ा)	भू Bhu	धा Dha	फा Bha/Pha	ढा Dha
21	Uttara Aṣāḍhā (उत्तराषाढ़ा)	भे Bhe	भो Bho	जा Ja	जी Ji
22	Śrāvaṇa (श्रावण)	खी Ju/Khi	खू Je/Khu	खे Jo/Khe	खो Gha/Kho
23	Śrāviṣṭha (श्रविष्ठा) or Dhanishta	गा Ga	गी Gi	गु Gu	गे Ge
24	Shatabhisha (शतभिषा)or Śatataraka	गो Go	सा Sa	सी Si	सू Su
25	Pūrva Bhādrapadā (पूर्वभाद्रपदा)	से Se	सो So	दा Da	दी Di
26	Uttara Bhādrapadā (उत्तरभाद्रपदा)	दू Du	थ Tha	झ Jha	ञ Da/Tra
27	Revati (रेवती)	दे De	दो Do	च Cha	ची Ch

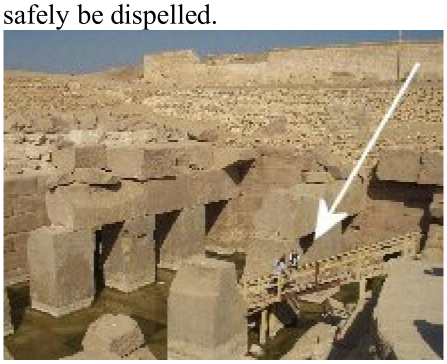
Appendix III

The Osirion and the Flower of Life

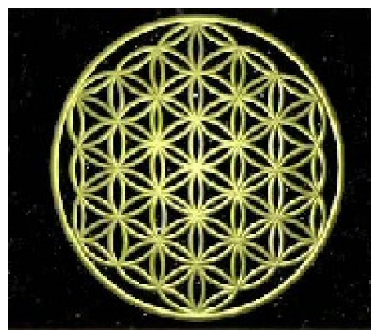
http://www.kch42.dial.pipex.com/egypttour_osirion.html

The Osirion

At the rear of Seti I temple at Abydos is a very strange structure called the Osirion that holds a number of mysteries.
The Osirion was originally meant to be entered from the Transverse passageway leading from the back of Seti's temple, but at the moment this passageway is not open to the public, and visitors must exit Seti's temple at the rear and approach the Osirion from above at modern ground level. The first thing of note is the enormous size of the red granite blocks used in its construction, which draws parallels with similar Old Kingdom megalithic structures, like the Valley Temple, and the Sphinx Temple at Giza. The similarities are inescapable - the stark and simple megalithic design, the lack of prolific inscriptions, and the fact that some of the larger stone blocks weigh up to 100 tons. This suggests that the Osirion pre-dates the temple, dated to around 1300 bc, by at least a thousand years.
Inscribed on some of these blocks and visible from the metal staircase can be seen a number of 'Flower of Life' patterns of interlocking circles, which were first drawn to my attention in the writings of Drunvalo Melchizedek. Many wild claims are made for the dating of this pattern, which can now safely be dispelled.



The Osirion at the rear of Seti I temple at Abydos. The position of the "Flower of Life" is marked



A diagram of the 'Flower of Life' pattern found on a number of the columns in the Osirion.

A closer inspection of these "Flower or Life" patterns reveals some very interesting information.

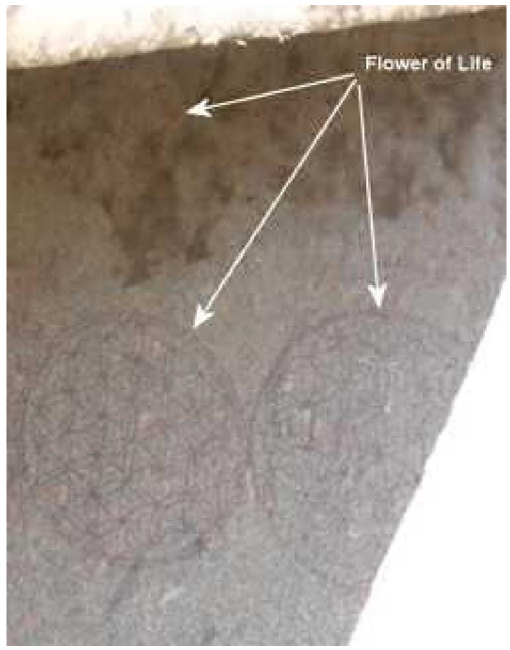
Origins

A possible clue to the origin of these designs can be found in some barely noticeable text that adjoins the patterns. Although I am no Greek scholar it is obvious that this is Greek writing for the following letters can be clearly made out:

θ = Theta
Є = Epsilon
λ = Lamda

Closer inspection shows that the two Greeks words that can be deciphered read: "Theos Nilos", which means the "God of the Nile." Another letter close to where the arrow is pointing looks like a large **F**. This appears to be the Greek letter "Digamma".
The Wikipedia Encyclopedia states:

The position of the "Flower of Life" patterns can be found principally on two supporting columns.The one that is indicated below and the face of the column opposite. On these blocks at least thirteen such patterns can be made out, although some are very faded.
The picture below shows three of these patterns, although only two are clearly visible. The symbol at the top is much discoloured and only apparent when the photo is viewed at high definition.
The two flowers designs that abut each other are about 6 inches in diameter and can be clearly made out. They are unlike anything else that I have ever seen in Egypt and therefore remain something of a mystery.



"Digamma is an archaic letter of the Greek alphabet, used primarily as aGreek numeral.
Its original name is unknown, but was probably (wau). It was later called **διγαμμα** (digamma — double gamma because of its shape). It is attested in archaic and dialectal ancient Greek inscriptions, and is occasionally used as a symbol in later Greek mathematical texts.
It is also used as the Greek numeral 6. In ancient usage, the numeral had the same form as the letter digamma."
Digamma, like Upsilon, derives from the Phoenician letter Waw, and in its turn gave rise to the Roman letter **F**."

It is clear from the above that the inscription is ancient Greek. The first thought is that this would have been inscribed after Alexander's invasion in 332 bc and before the beginning of the Roman period in 30 bc, when the Greek Ptolemies ruled Egypt. Against this view according to Herbert Weir Smyth in his book Greek Grammar the Digamma "disappeared when Athens adopted the Ionic alphabet in 403 bc. However it disappeared gradually and was still used in Boetia as late as 200 bc." This implies that it could already have vanished by Alexander's time.
So if the letter truly is "Digamma" it is quite possible that it was inscribed before the Ptolemaic period. On the present evidence this seems to be very unlikely but if so what other possibility might there be? We can be fairly sure that Pythagoras had a ten year stay in Egypt from around 535bc. Could this have been inscribed by the mighty sage or one of his followers?
Pythagoras
According to Porphyry (233-c.305) Pythagoras, after travelling extensively in Egypt, was eventually accepted for initiation into the temple mysteries by the priests at Diospolis. It is generally agreed that this referred to Thebes or Luxor as it is known today and that the priesthood would have been that of the temple of Amun at Karnak.
By road Thebes is a about 150 km south of Abydos and journeys between Thebes in Upper Egypt and the Delta would have passed by the Osirion site. Another possibility is the town of Diospolis Parva, which is only 47 km south of Abydos, which has a temple dedicated to the goddess Hathor. On this evidence we can be fairly sure that Pythagoras would at least have visited the temple shrine of Osiris.
We should not forget that when the Egyptologist Margaret Murray carried the intial survey and excavation work on the Abydos temple she found numerous examples of Greek text dating from the 3rd century BC to the 2nd century AD. Needless to say only the decipherment of the Greek text, alongside of the Flower of Life design will tell us more about the originator of these geometric designs.
2008 - Update
In January and December 2008 we had a chance to re-examine all of the Osirion graffiti and to take more detailed photos. It is clear that the Flower of Life images appear quite extensively on two columns that face in to each other. There are at least eight versions of the FOL on the column not readily visible to the public. Photos of these images will be released when they have been fully

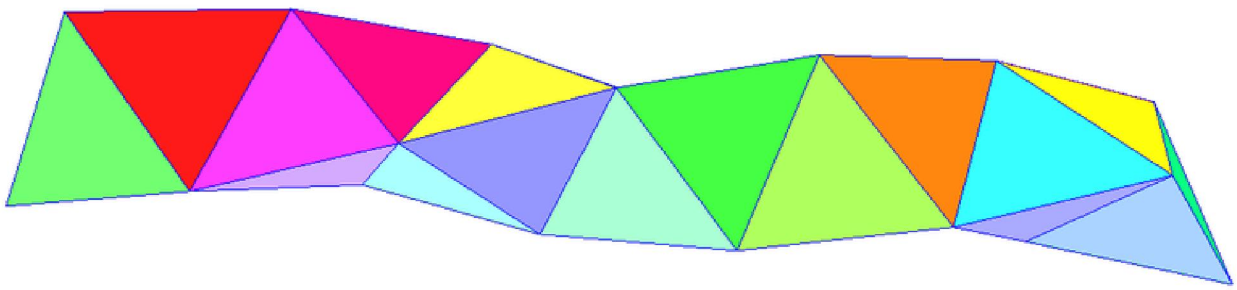
analysed. However it is now clear that all of the graffiti are close to the top of the columns, which are some 13.5 ft (4.1m) tall. To be easily drawn someone would have had to have been on a platform some 8ft from the present base floor level. This suggests that these graffiti were inscribed long after the pharonic temple had fallen into disuse and the Osirion had begun to fill up with sand. In addition the Greek text IXCX can be clearly made out close to the top of one of the columns. This would date the graffiti well into the Christian epoch perhaps as late as the fifth or sixth century CE.

http://www.kch42.dial.pipex.com/egypttour_osirion.html

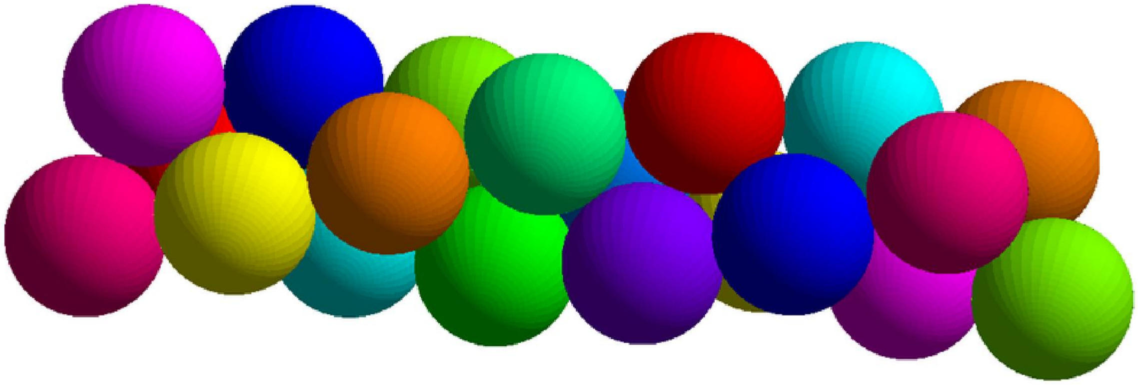
Appendix IV

Boerdijk–Coxeter Helix

From Wikipedia, the free encyclopedia



Coxeter regular tetrahedral helix



A Boerdijk helical sphere packing has each sphere centered at a vertex of the Coxeter helix. Each sphere is in contact with 6 neighboring spheres.

The **Boerdijk–Coxeter helix**, named after H. S. M. Coxeter and A. H. Boerdijk, is a linear stacking of regular tetrahedra, arranged so that the edges of the complex that belong to a single tetrahedron form three intertwined helices. There are two chiral forms, with either clockwise or counterclockwise windings. Contrary to any other stacking of Platonic solids, the Boerdijk–Coxeter helix is not rotationally repetitive. Even in an infinite string of stacked tetrahedra, no two tetrahedra will have the same orientation. This is because the helical pitch per cell is not a rational fraction of the circle.

Buckminster Fuller named it a *tetrahelix* and considered them with regular and irregular tetrahedral elements.[1]

Higher dimensional geometry

The 600-cell partitions into 20 rings of 30 tetrahedra, each a *Boerdijk–Coxeter helix*. When superimposed onto the 3-sphere curvature it becomes periodic, with a period of ten vertices, encompassing all 30 cells. The collective of such helices in the 600-cell represent a discrete Hopf fibration. While in 3 dimensions the edges are helices, in the imposed 3-sphere topology they are geodesics and have no torsion. They spiral around each other naturally due to the Hopf fibration.

Appendix V

Hopf Fibration / Map / Etc.

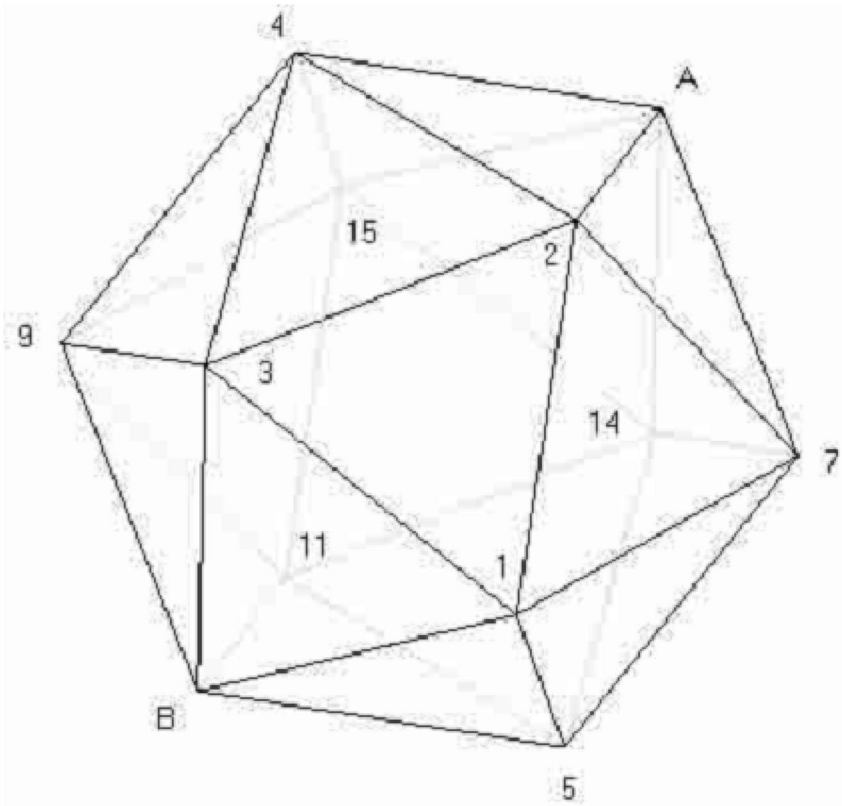


Fig. 13. Representation of the configuration of twelve fibres of a Hopf libration.

Hopf Map

The first example discovered of a map from a higher-dimensional sphere to a lower-dimensional sphere which is not null-homotopic. Its discovery was a shock to the mathematical community, since it was believed at the time that all such maps were null-homotopic, by analogy with homology groups.

The Hopf map $f: \mathbb{S}^3 \to \mathbb{S}^2$ arises in many contexts, and can be generalized to a map $\mathbb{S}^7 \to \mathbb{S}^4$. For any point p in the sphere, its preimage $f^{-1}(p)$ is a circle $\mathbb{S}^1$ in $\mathbb{S}^3$. There are several descriptions of the Hopf map, also called the Hopf fibration.

As a submanifold of $\mathbb{R}^4$, the 3-sphere is

$$\mathbb{S}^3 = \{(X_1, X_2, X_3, X_4) : X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1\}, \quad (1)$$

and the 2-sphere is a submanifold of $\mathbb{R}^3$,

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}. \quad (2)$$

The Hopf map takes points (X_1, X_2, X_3, X_4) on a 3-sphere to points on a 2-sphere (x_1, x_2, x_3)

$$x_1 = 2(X_1 X_2 + X_3 X_4)$$

$$x_2 = 2(X_1 X_4 - X_2 X_3)$$

$$x_3 = (X_1^2 + X_3^2) - (X_2^2 + X_4^2).$$

Every point on the 2-sphere corresponds to a circle called the Hopf circle on the 3-sphere.

By stereographic projection, the 3-sphere can be mapped to $\mathbb{R}^3$, where the point at infinity corresponds to the north pole. As a map, from $\mathbb{R}^3$, the Hopf map can be pretty complicated. The diagram above shows some of the preimages $f^{-1}(p)$, called Hopf circles. The straight red line is the circle through infinity.

By associating $\mathbb{R}^4$ with $\mathbb{C}^2$, the map is given by $f(z, w) = z/w$, which gives the map to the Riemann sphere.

The Hopf fibration is a fibration

$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2, \quad (6)$$

and is in fact a principal bundle. The associated vector bundle

$$L = \mathbb{S}^3 \times \mathbb{C} / U(1), \quad (7)$$

where

$$((z, w), v) \sim ((e^{it} z, e^{it} w), e^{-it} v) \quad (8)$$

is a complex line bundle on $\mathbb{S}^2$. In fact, the set of line bundles on the sphere forms a group under vector bundle tensor product, and the bundle L generates all of them. That is, every line bundle on the sphere is $L^{\otimes k}$ for some k.

The sphere $\mathbb{S}^3$ is the Lie group of unit quaternions, and can be identified with the special unitary group $SU(2)$, which is the simply connected double cover of $SO(3)$. The Hopf bundle is the quotient map $\mathbb{S}^2 \cong SU(2)/U(1)$.

SEE ALSO: Fibration, Fiber Bundle, Homogeneous Space, Principal Bundle, Stereographic Projection, Vector Bundle

Hopf fibration

A locally trivial fibration $f : S^{2n-1} \to S^n$ for . This is one of the earliest examples of locally trivial fibrations, introduced by H. Hopf in [1]. These mappings induce trivial mappings in homology and cohomology; however, they are not homotopic to the null mapping, which follows from the fact that their Hopf invariant is non-trivial. The creation of the mappings requires the so-called Hopf construction.

Let $X * Y$ be the join of two spaces X and Y, which has natural coordinates $\langle x, t, y \rangle$, where $x \in X$, $t \in [0, 1]$, $y \in Y$. Here, for example, $X * S^0 = SX$, where SX is the suspension of X. The Hopf construction $\mathfrak{H}$ associates with a mapping $f : X \times Y \to Z$ the mapping $\mathfrak{H}(f) : X * Y \to SZ$ given by $\mathfrak{H}(f)\langle x, t, y \rangle = \langle f(x, y), t \rangle$.

Suppose that mappings $\mu_n : S^{n-1} \times S^{n-1} \to S^{n-1}$ are defined for $n = 2, 4, 8$ by means of multiplications: in the complex numbers for $n = 2$, in the quaternions for $n = 4$, and in the Cayley numbers for $n = 8$. Then $S^{n-1} * S^{n-1} = S^{2n-1}$, and the Hopf mapping is defined as

$$\mathfrak{H}_n = \mathfrak{H}(\mu_n) : S^{2n-1} \to S^n .$$

The Hopf mapping $\mathfrak{H}_n$, $n = 2, 4, 8$, is a locally trivial fibration with fibre S^{n-1}. If $f : S^{n-1} \times S^{n-1} \to S^{n-1}$ is a mapping of bidegree (d_1, d_2), then the Hopf invariant of the mapping $\mathfrak{H}(f)$ is $d_1 d_2$. In particular, the Hopf invariant of the Hopf fibration is 1.

Sometimes the Hopf fibration is defined as the mapping $f : S^{2n+1} \to CP^n$ given by the formula $(z_0, \dots, z_n) \to [z_0 : \dots : z_n]$, $z_i \in \mathbf{C}$. This mapping is a locally trivial fibration with fibre S^1. For $n = 1$ one obtains the classical Hopf fibration $f : S^3 \to S^2$.

References

[1] H. Hopf, "Ueber die Abbildungen von Sphären niedriger Dimension" *Fund. Math.* , **25** (1935) pp. 427–440
[2] D. Husemoller, "Fibre bundles" , McGraw-Hill (1966)

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Hopf fibration. A.V. Shokurov (originator), *Encyclopedia of Mathematics.* URL: http://www.encyclopediaofmath.org/index.php?title=Hopf_fibration&oldid=11978

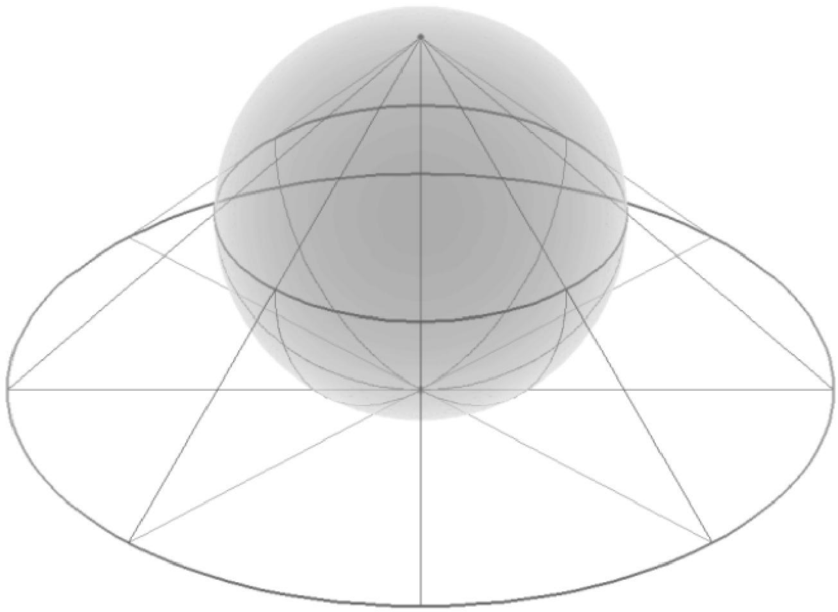

FIGURE 2.1. Stereographic projection, showing that $\hat{\mathbb{C}}$ is homeomorphic to S^2.

HOPF FIBRATION, HONGWAN LIU

Appendix VI

Boerdijk-Coxeter Helix

Periodic modication of the Boerdijk-Coxeter helix

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Abstract

The Boerdijk-Coxeter helix [1][2] is a helical structure of tetrahedra which possesses no non-trivial translational or rotational symmetries. In this document, we develop a procedure by which this structure is modified to obtain both translational and rotational (upon projection) symmetries along/about its central axis. We report the nding of several, distinct periodic structures, and focus on two particular forms related to the pentagonal and icosahedral aggregates of tetrahedra as well as Buckminster Fuller's \jitterbug transformation".

Abstract:

Helices and dense packing of spherical objects are two closely related problems. For instance, the Boerdijk-Coxeter helix, which is obtained as a linear packing of regular tetrahedra, is a very efficient solution to some close-packing problems. The shapes of biological helices result from various kinds of interaction forces, including steric repulsion. Thus, the search for a maximum density can lead to structures related to the Boerdijk-Coxeter helix. Examples are presented for the -helix structure in proteins and for the structure of the protein collagen, but there are other examples of helical packings at different scales in biology. Models based on packing efficiency related to the Boerdijk-Coxeter helix, explain, mainly from topological arguments, why the number of amino acids per turn is close to 3.6 in -helices and 2.7 in collagen.

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Sphere packing, helices and the polytope $\{3, 3, 5\}$

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Abstract. The packing of tetrahedra in face contact is well-known to be relevant to atomic clustering in many complex alloys. We briefly review some of the structures that can arise in this way, and introduce methods of dealing with the geometry of the polytope $\{3, 3, 5\}$, which is highly relevant to an understanding of these structures. Finally, we present a method of projection from S_3 to E_3 that enables coordinates for the key vertices of the collagen model of Sadoc and Rivier to be calculated.

6 The Boerdijk-Coxeter helices in $\{3, 3, 5\}$

Consider the effect of repeated action of

$$x \to p^2 x q \tag{6.1}$$

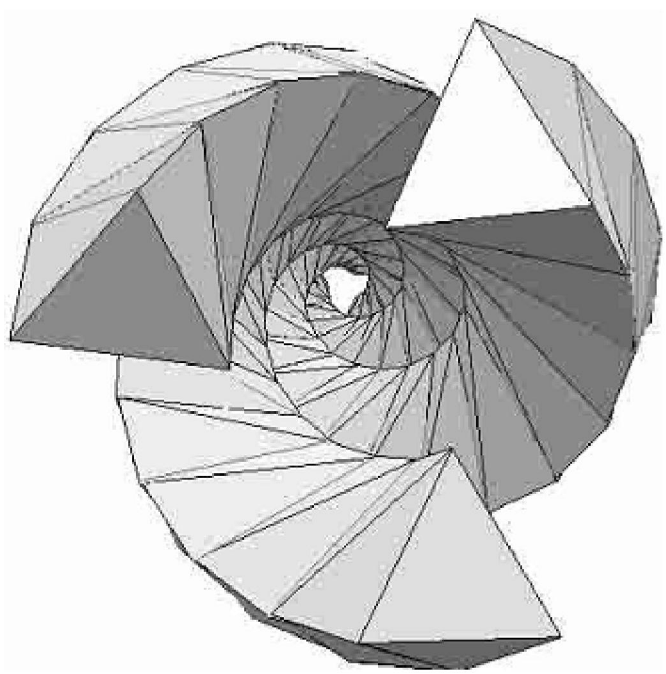
with $p = 1/2(\tau\ -\sigma\ 1\ 0)$, $q = 1/2(1\ 1\ 1\ 1)$; *i.e.*, $\theta = \pi/5$, $\varphi = \pi/3$, $\mathbf{n} \sim [1\ \tau\ 0]$, $\mathbf{m} \sim [1\ 1\ 1]$. Then

$$R = \frac{1}{2}\begin{pmatrix} -1 & -\tau & \sigma & 0 \\ 0 & -\sigma & -\tau & -1 \\ \tau & -1 & 0 & \sigma \\ -\sigma & 0 & -1 & \tau \end{pmatrix} . \tag{6.2}$$

Since $p^5 = q^3 = -1$, R has order 30. It will generate a sequence of 30 vertices (an orbit of the transformation R), starting from any vertex of $\{3,3,5\}$. For example, starting from **6** (0 0 0 1) we get the the vertices given by the columns of

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & -\sigma & 1 & 0 & 1 & -\sigma & 0 & 1 & 0 & 0 & -\sigma & \sigma & \\ 0 & -1 & \sigma & -1 & -\tau & -1 & -\tau & -\tau & -1 & -\tau & -1 & \sigma & -1 & 0 & 0 & \\ 0 & \sigma & 0 & -\sigma & 0 & 1 & 1 & -\sigma & \tau & 1 & 1 & \tau & -\sigma & 1 & 1 & \ldots \\ 2 & \tau & \tau & \tau & 1 & 1 & -\sigma & 0 & 0 & \sigma & -1 & -1 & -\tau & -\tau & -\tau & \end{pmatrix}$$

(... denotes a repetition of the fifteen given columns, with opposite sign.) in this sequence, every set of four consecutive vertices is the set of vertices of a regular tetrahedron with edge length $1/\tau$. That is, we have a *Boerdijk-Coxeter helix* consisting of 30 tetrahedral cells of $\{3,3,5\}$.



Bibliography

Wikipedia

Wolfram

Tony Smith

For online animations of the Hopf Fibration:

http://gbbservices.com/math/hopf.html

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'Other people, he said, see things and say why? But I dream things that never were and I ask, why not?"

Robert F. Kennedy, after George Bernard Shaw