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On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions IV: A Simple Proof for Cramér's Conjecture

Prof. Dr. Raja Rama Gandhi[1] and Edigles Guedes[2]
[1]Resource person in Math for Oxford University Press, Professor in Math, BITS-Vizag.
[2]World order Number Theorist, Pernambuco, Brazil.

1. INTRODUCTION

In *On the Order of Magnitude of the Difference between Consecutive Prime Numbers* [1, p. 27], 1937, Harald Cramér conjectured, using a heuristic method founded on probabilistic arguments, that

$$(1.a)\ \limsup_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^2} = 1,$$

to see also Richard K. Guy's book: *Unsolved Problems in Number Theory* [2, p. 11].

In this paper, we prove that

$$(1.b)\ \lim_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^2} < 1,$$

result which is stronger than the Cramér's conjecture.

2. PRELIMINARES

The Rosser's theorem [3] states that p_n is larger than $n\log n$. This can be improved by the following pair of bounds:

$$(1)\ \log n + \log\log n - 1 < \frac{p_n}{n} < \log n + \log\log n,$$

for $n \geq 6$.

3. LEMMA AND THEOREMS

THEOREM1.*For* $n \in \mathbb{N}_{\geq 6}$, *then*

$$\frac{p_{n+1}-p_n}{\sqrt{p_n}} < \sqrt{2}\left\{2n^2 - \left[2\sqrt{n(n+1)}+1\right]n + 2\sqrt{n(n+1)}\right\}.$$

Proof. In previous paper [4, p.__], we discover that

$$(2) \sqrt{p_{n+1}} - \sqrt{p_n} < \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n},$$

for $n \in \mathbb{N}_{\geq 6}$.Squaring the inequality (1), we have

$$(3) p_{n+1} + p_n - 2\sqrt{p_{n+1}p_n} < 2n\left(2n+1-2\sqrt{n(n+1)}\right)$$
$$\Rightarrow p_{n+1} + p_n < 2\sqrt{p_{n+1}p_n} + 2n\left(2n+1-2\sqrt{n(n+1)}\right).$$

Multiplying (2) by $2\sqrt{p_n}$, we find

$$(4)\quad 2\sqrt{p_{n+1}p_n} - 2p_n < 2\sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_n}$$
$$\Rightarrow 2\sqrt{p_{n+1}p_n} < 2p_n + 2\sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_n}.$$

From (3) and (4), we obtain

$$(5)\quad p_{n+1} + p_n < 2p_n + 2\sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_n} + 2n\left(2n+1-2\sqrt{n(n+1)}\right)$$
$$\Rightarrow p_{n+1} - p_n < 2\sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_n} + 2n\left(2n+1-2\sqrt{n(n+1)}\right).$$

Dividing both members of (5) by $\sqrt{p_n}$, we encounter

$$\frac{p_{n+1}-p_n}{\sqrt{p_n}} < 2\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}+\frac{2n}{\sqrt{p_n}}\left(2n+1-2\sqrt{n(n+1)}\right).$$

On the other hand, we have that$\max_{n\in\mathbb{N}_{\geq 6}}\frac{1}{\sqrt{p_n}}=\frac{1}{\sqrt{13}}<\max_{n\in\mathbb{N}_{\geq 1}}\frac{1}{\sqrt{p_n}}=\frac{1}{\sqrt{2}}$, so

$$\frac{p_{n+1}-p_n}{\sqrt{p_n}} < 2\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}+\frac{2n}{\sqrt{p_n}}\left(2n+1-2\sqrt{n(n+1)}\right)$$
$$< 2\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}+2n\max_{n\in\mathbb{N}_{\geq 6}}\frac{1}{\sqrt{p_n}}\left(2n+1-2\sqrt{n(n+1)}\right)$$
$$< 2\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}+2n\max_{n\in\mathbb{N}_{\geq 1}}\frac{1}{\sqrt{p_n}}\left(2n+1-2\sqrt{n(n+1)}\right)$$
$$= 2\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}+\frac{2n}{\sqrt{2}}\left(2n+1-2\sqrt{n(n+1)}\right)$$
$$= 2\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}+n\sqrt{2}\left(2n+1-2\sqrt{n(n+1)}\right)$$
$$= \sqrt{2}\left\{2n^2-\left[2\sqrt{n(n+1)}+1\right]n+2\sqrt{n(n+1)}\right\}. \square$$

THEOREM2.*For* $n\in\mathbb{N}_{\geq 6}$, *then*

$$p_{n+1}-p_n < 4n\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{\log n}.$$

Proof. Multiplying (2) by$\sqrt{p_{n+1}}$ and $\sqrt{p_n}$,separately, we obtain

$$(6)\, p_{n+1}-\sqrt{p_{n+1}p_n} < \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_{n+1}},$$

and

$$(7)\, \sqrt{p_{n+1}p_n}-p_n < \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_n}.$$

Summing (6) with (7), member by member, we have

$$(8)\, p_{n+1}-p_n < \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(\sqrt{p_{n+1}}+\sqrt{p_n}\right).$$

From (1) and (8), we find

$$(9)\, p_{n+1}-p_n < \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(\sqrt{p_{n+1}}+\sqrt{p_n}\right)$$
$$< \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(\sqrt{(n+1)\log(n+1)+(n+1)\log\log(n+1)}+\sqrt{n\log n+n\log\log n}\right)$$
$$< \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(\sqrt{2(n+1)\log(n+1)}+\sqrt{2n\log n}\right)$$
$$= 2\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(\sqrt{(n+1)\log(n+1)}+\sqrt{n\log n}\right).$$

Since $\sqrt{(n+1)\log(n+1)}\cong\sqrt{n\log n}$, we find

$$p_{n+1}-p_n < 2\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(2\sqrt{n\log n}\right)=4n\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{\log n}. \square$$

THEOREM 3 (Stronger Cramér's conjecture).

$$\lim_{n\to\infty}\frac{p_{n+1}-p_n}{(\log p_n)^2}<1.$$

Proof. In Theorem 2, see [6], we have

$$(10)\quad \left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n-n) < g(p_n) < \left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n),$$

where $k=\frac{p_n}{p_{n+1}}$ to be a k modulus. In other words,

$$(11)\, \left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n-n) < p_{n+1}-p_n < \left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n).$$

Dividing (11) by $\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n)$, we encounter

$$(12)\ \frac{n\log n + n\log\log n - n}{n\log n + n\log\log n} < \left(\frac{\theta_2^2}{\theta_3^2-\theta_2^2}\right)\frac{p_{n+1}-p_n}{n\log n + n\log\log n} < 1,$$

ergo,

$$(13)\ \frac{\log n + \log\log n - 1}{\log n + \log\log n} < \left(\frac{\theta_2^2}{\theta_3^2-\theta_2^2}\right)\frac{p_{n+1}-p_n}{n\log n + n\log\log n} < 1,$$

But, by the Rosser's theorem, we find

$$(14)\ \frac{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n - 1 - n\log n - n\log\log n + n}{n\log n + n\log\log n} < \frac{p_{n+1}-p_n}{n\log n + n\log\log n} < \frac{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n}{n\log n + n\log\log n},$$

wherefore,

$$(15)\ \frac{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n - 1}{n\log n + n\log\log n} < \frac{p_{n+1}-p_n}{n\log n + n\log\log n} < \frac{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n}{n\log n + n\log\log n}.$$

Dividing (15) by (13), we find

$$(16)\ \frac{\log n + \log\log n - 1}{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n - 1} < \frac{\theta_2^2}{\theta_3^2-\theta_2^2} < \frac{n\log n + n\log\log n}{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n},$$

From (12) and (16), we have

$$(17)\ \left(\frac{\log n + \log\log n - 1}{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n - 1}\right)\frac{p_{n+1}-p_n}{n\log n + n\log\log n} < \left(\frac{\theta_2^2}{\theta_3^2-\theta_2^2}\right)\frac{p_{n+1}-p_n}{n\log n + n\log\log n} < 1,$$

therefore,

$$(18)\left(\frac{\log n + \log\log n - 1}{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n - 1}\right)\frac{p_{n+1}-p_n}{n\log n + n\log\log n} < 1,$$

consequently,

$$(19)\frac{p_{n+1}-p_n}{n\log n + n\log\log n} < \frac{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n - 1}{\log n + \log\log n - 1}.$$

On the other hand, applying the Rosser's theorem, we obtain

$$(20)\ \frac{(\log p_n)^2}{n\log n + n\log\log n} \ll \frac{p_n}{n\log n + n\log\log n} < \frac{n\log n + n\log\log n}{n\log n + n\log\log n} = 1$$

Dividing (19) by (20), we encounter

$$(21)\frac{p_{n+1}-p_n}{(\log p_n)^2} < \frac{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n - 1}{\log n + \log\log n - 1}.$$

Applying the limit as $n \to \infty$ in both members of inequality above, we obtain

$$\lim_{n\to\infty}\frac{p_{n+1}-p_n}{(\log p_n)^2} < \lim_{n\to\infty}\frac{(n+1)\log(n+1) + (n+1)\log\log(n+1) - n\log n - n\log\log n - 1}{\log n + \log\log n - 1} = 1. \square$$

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