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ON ANDRICA'S CONJECTURE, CRAMÉR'S CONJECTURE, GAPS BETWEEN PRIMES AND JACOBI THETA FUNCTIONS VI:

Lower bound for Differences between Consecutive Primes

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Abstract: In this paper we developed an interesting result by constructing lower bound for differences between consecutive primes.

Key words: Rieman Zeta-Function, Erdös

MSC: 11Y40, 62N10

1. INTRODUCTION

In book *The Riemann Zeta-Function: Theory and Applications* [1, p. 350], AleksandarIvic refers to a paper by P. Erdös [2], 1935, in which he proved, by ingenious method, which gives that:

$$(1.1) p_{n+1} - p_n > C\frac{\log p_n \log\log p_n}{\log\log\log p_n},$$

holds for infinitely many n and some absolute constant $C > 0$.In turn, R. A. Rankin [3], 1962/1963, refined the Erdös method, and he obtained:

$$(1.2) p_{n+1} - p_n > C\frac{\log p_n \log\log p_n \log\log\log\log p_n}{\log\log\log p_n}.$$

On the other hand, Harald Cramér [4, p. 41] established:

$$(1.3) p_{n+1} - p_n > \sqrt{p_n}\log p_n.$$

In this paper, we prove that

$$(1.4) p_{n+1} - p_n > C\left[\sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1\right],$$

Where with we conjecture that

$$(1.5) \qquad p_{n+1} - p_n \geq C\left\lfloor\sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1\right\rfloor,$$

Where $\lfloor x \rfloor$is the floor function. Finally, we proved that

$$(1.6) \qquad \frac{p_{n+1}}{p_n} \ll 1 + \frac{1}{p_n^{3258/20000}}.$$

2010 Mathematics Subject Classification. 11Y40, 62N10
Keywords: Rieman Zeta-Function, Erdös

2. THEOREM AND CONJECTURE

THEOREM 1. *Let* $n \in \mathbb{N}_{\geq 6}$ *and* C *any number, then*

$$p_{n+1} - p_n > C\left[\sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1\right]. \tag{2.1}$$

Proof. Step 1. In [5, p. 43], we have the inequality

$$2\sqrt{p_{n+1}p_n} < 2p_n + 2\sqrt{2}\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}\sqrt{p_n} \Rightarrow \tag{2.2}$$

$$\sqrt{p_{n+1}p_n} - \sqrt{2}\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}\sqrt{p_n} < p_n \Leftrightarrow$$

$$p_n > \sqrt{p_{n+1}p_n} - \sqrt{2}\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}\sqrt{p_n},$$

pursuant to

$$p_{n+1} > \sqrt{p_{n+2}p_{n+1}} - \sqrt{2}\left(\sqrt{n+2} - \sqrt{n+1}\right)\sqrt{n+1}\sqrt{p_{n+1}}. \tag{2.3}$$

Subtracting (2.2) of (2.3), we encounter

$$p_{n+1} - p_n > \sqrt{p_{n+2}p_{n+1}} - \sqrt{p_{n+1}p_n} - \sqrt{2}\left(\sqrt{n+2} - \sqrt{n+1}\right)\sqrt{n+1}\sqrt{p_{n+1}} + \sqrt{2}\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}\sqrt{p_n} = \tag{2.4}$$

$$= \sqrt{p_{n+1}}\left[\sqrt{p_{n+2}} - \sqrt{p_n} - \sqrt{2}\left(\sqrt{n+2} - \sqrt{n+1}\right)\sqrt{n+1} + \sqrt{2}\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}\sqrt{\frac{p_n}{p_{n+1}}}\right]$$

$$= \sqrt{p_{n+1}}\left[\sqrt{p_{n+2}} - \sqrt{p_n} + \sqrt{2}\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}\sqrt{\frac{p_n}{p_{n+1}}} - \sqrt{2}\left(\sqrt{n+2} - \sqrt{n+1}\right)\sqrt{n+1}\right]$$

$$= \sqrt{p_{n+1}}\left\{\sqrt{p_{n+2}} - \sqrt{p_n} + \sqrt{2}\left[\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}\sqrt{\frac{p_n}{p_{n+1}}} - \left(\sqrt{n+2} - \sqrt{n+1}\right)\sqrt{n+1}\right]\right\}.$$

We observe that, since $p_n < p_{n+1}$, then $\sqrt{\frac{p_n}{p_{n+1}}} \cong 1$, see Table 1. Therefore,

$$(2.5) p_{n+1} - p_n > c_1\sqrt{p_{n+1}}\{\sqrt{p_{n+2}} - \sqrt{p_n} + \sqrt{2}[(\sqrt{n+1} - \sqrt{n})\sqrt{n} - (\sqrt{n+2} - \sqrt{n+1})\sqrt{n+1}]\}$$

We note that $\sqrt{2}[(\sqrt{n+1} - \sqrt{n})\sqrt{n} - (\sqrt{n+2} - \sqrt{n+1})\sqrt{n+1}]$ tends rapidly to zero, see Table 2. Accordingly,

$$(2.6) p_{n+1} - p_n > c_2\sqrt{p_{n+1}}(\sqrt{p_{n+2}} - \sqrt{p_n}).$$

Step 2. Henceforth, we will use the *reduction ad absurdum* to prove that

$$(2.7) \sqrt{p_{n+1}}(\sqrt{p_{n+2}} - \sqrt{p_n}) > \sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1.$$

Firstly, we assume, by hypothesis, that

$$(2.8) \sqrt{p_{n+1}}(\sqrt{p_{n+2}} - \sqrt{p_n}) \leq \sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1.$$

Dividing by both members of inequality (2.8) by $\sqrt{p_{n+1}}$, we encounter

$$(2.9) \sqrt{p_{n+2}} - \sqrt{p_n} \leq \log p_{n+2} - \log p_n + \frac{1}{\sqrt{p_{n+1}}} \Rightarrow \sqrt{p_{n+2}} + \log p_n \leq \log p_{n+2} + \sqrt{p_n} + \frac{1}{\sqrt{p_{n+1}}}$$

The exponentiation of (2.9) gives us

$$(2.10) p_n e^{\sqrt{p_{n+2}}} \leq p_{n+2} e^{\sqrt{p_n} + \frac{1}{\sqrt{p_{n+1}}}} \Rightarrow \frac{p_n}{p_{n+2}} \leq e^{\sqrt{p_n} + \frac{1}{\sqrt{p_{n+1}}} - \sqrt{p_{n+2}}} = \frac{1}{e^{\sqrt{p_{n+2}} - \sqrt{p_n} - \frac{1}{\sqrt{p_{n+1}}}}} \Leftrightarrow e^{\sqrt{p_{n+2}} - \sqrt{p_n} - \frac{1}{\sqrt{p_{n+1}}}} \leq \frac{p_{n+2}}{p_n}.$$

For $n \in \mathbb{N}_{\geq 6}$, is easy to check that $e^{\sqrt{p_{n+2}} - \sqrt{p_n} - \frac{1}{\sqrt{p_{n+1}}}} \nleq \frac{p_{n+2}}{p_n}$, so our hypothesis is false. See Table 3. Hence, it follows that $\sqrt{p_{n+1}}(\sqrt{p_{n+2}} - \sqrt{p_n}) > \sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1$, as we wanted to demonstrate.

Step 3. From (2.6) and (2.7), we obtain

$$p_{n+1} - p_n > c_3\sqrt{p_{n+1}}(\sqrt{p_{n+2}} - \sqrt{p_n}) > c_4[\sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1]. \square$$

CONJECTURE1: *Let* $n \in \mathbb{N}_{\geq 6}$ *and* C *any number, then*

$$p_{n+1} - p_n \geq C\lfloor\sqrt{p_{n+1}}(\log p_{n+2} - \log p_n) + 1\rfloor,$$

where $\lfloor x \rfloor$ *is the floor function.*

THEOREM 2. *For* $n \in \mathbb{N}_{>9}$, *then*

(2.11) $$\frac{p_{n+1}}{p_n} \ll 1 + \frac{1}{p_n^{3258/20000}}.$$

Proof. Step 1. In [6], we have the inequality

$p_{n+1} - p_n \ll p_n^{8371/10000}$.

for $n > 9$; wherefore,

(2.12) $$(p_{n+1} - p_n)^2 \ll p_n^{16742/10000} = p_n \cdot p_n^{6742/10000} \Rightarrow \frac{(p_{n+1} - p_n)^2}{p_n} \ll p_n^{6742/10000}.$$

Multiplying (2.13) by $1/p_n$, we have

$$\frac{(p_{n+1} - p_n)^2}{p_n^2} \ll \frac{p_n^{6742/10000}}{p_n} = \frac{1}{p_n^{3258/10000}} \Rightarrow \frac{p_{n+1} - p_n}{p_n} \ll \frac{1}{p_n^{3258/20000}} \Leftrightarrow \frac{p_{n+1}}{p_n} \ll 1 + \frac{1}{p_n^{3258/20000}}. \square$$

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EDIGLES GUEDES, WORLD ORDER NUMBER THEORIST, PERNAMBUCO, BRAZIL

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Table 1

In this table, we have: first column: n; second column: p_n; third column: p_{n+1}; fourth column: $\sqrt{\frac{p_n}{p_{n+1}}}$.

1	2	3	0.81649658092772 6
2	3	5	0.7745966692414834
3	5	7	0.8451542547285166
4	7	11	0.7977240352174656
5	11	13	0.9198662110077999
6	13	17	0.8744746321952062
7	17	19	0.9459053029269173
8	19	23	0.9088932591463857
9	23	29	0.8905635565617213
10	29	31	0.9672041516493516
11	31	37	0.9153348228041135
12	37	41	0.9499679070317291
13	41	43	0.9764672918705589
14	43	47	0.9565007145952775
15	47	53	0.9416965821485117
16	53	59	0.947789578306157
17	59	61	0.9834699358669274
18	61	67	0.9541738631895289
19	67	71	0.971422653550444
20	71	73	0.9862062358989764
21	73	79	0.9612755239323388
22	79	83	0.9756060828611426
23	83	89	0.9657040279831711
24	89	97	0.957875656437659
25	97	101	0.9799979793876926
26	101	103	0.9902436691399974
27	103	107	0.9811303799342402
28	107	109	0.9907832134966705
29	109	113	0.9821414205253256
30	113	127	0.9432729082972536
31	127	131	0.9846144671164251
32	131	137	0.9778570343163892
33	137	139	0.992779688949853
34	139	149	0.9658601896963496
35	149	151	0.9933554081432372
36	151	157	0.9807055824713378
37	157	163	0.9814225308444268
38	163	167	0.9879513673210928
39	167	173	0.9825059384426867
40	173	179	0.9830973740822291
41	179	181	0.994459791164577
42	181	191	0.9734700709613991
43	191	193	0.9948051596666967
44	193	197	0.9897956513705651
45	197	199	0.994962184579755
46	199	211	0.9711477550225341
47	211	223	0.9727221292883055
48	223	227	0.9911502684384192
49	227	229	0.9956236113842678
50	229	233	0.9913791494810404

Table 2

In this table, we have: first column: n; second column: p_n; third column: p_{n+1}; fourth column: $\sqrt{2}\left[\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}-\left(\sqrt{n+2}-\sqrt{n+1}\right)\sqrt{n+1}\right]$.

1	2	3	1	−0.0498880527646590 2
2	3	5	2	−0.0206643080555071 33
3	5	7	2	−0.0113622723973075 39
4	7	11	4	−0.0071978097049787 28
5	11	13	2	−0.0049711351237533 685
6	13	17	4	−0.0036402919735864 622
7	17	19	2	−0.0027811933685415 27
8	19	23	4	−0.0021943026256446 892
9	23	29	6	−0.0017755468194908 478
10	29	31	2	−0.0014662727075000 364
11	31	37	6	−0.0012313610106788 257
12	37	41	4	−0.0010487333101246 67
13	41	43	2	−0.0009039412071891 466
14	43	47	4	−0.0007872059143510 131
15	47	53	6	−0.0006917168014632 202
16	53	59	6	−0.0006126119516689 652
17	59	61	2	−0.0005463451649797 271
18	61	67	6	−0.0004902805633063 993
19	67	71	4	−0.0004424258253372 8027
20	71	73	2	−0.0004012523893598 0053
21	73	79	6	−0.0003655716593042 0994
22	79	83	4	−0.0003344481254755 247
23	83	89	6	−0.0003071373321544 6514
24	89	97	8	−0.0002830408892466 4087
25	97	101	4	−0.0002616733774070 7524
26	101	103	2	−0.0002426376836745 23
27	103	107	4	−0.0002256063995075 3761
28	107	109	2	−0.0002103076355631 624
29	109	113	4	−0.0001965140941277 134
30	113	127	14	−0.0001840345714407 0722
31	127	131	4	−0.0001727072909492 946
32	131	137	6	−0.0001623946307265 2827
33	137	139	2	−0.0001529789207278 6562
34	139	149	10	−0.0001443590695666 9137
35	149	151	2	−0.0001364478381304 8636
36	151	157	6	−0.0001291696225546 9456
37	157	163	6	−0.0001224586402999 2982
38	163	167	4	−0.0001162574378696 884
39	167	173	6	−0.0001105156565456 7263
40	173	179	6	−0.0001051890063391 2058
41	179	181	2	−0.0001002384089361 3297
42	181	191	10	−0.0000956292788277 7912
43	191	193	2	−0.0000913309172054 9959
44	193	197	4	−0.0000873159994551 3705
45	197	199	2	−0.0000835601396421 7786
46	199	211	12	−0.0000800415191799 3991
47	211	223	12	−0.0000767405692366 0832
48	223	227	4	−0.0000736396979169 1185
49	227	229	2	−0.0000707230554040 0329
50	229	233	4	−0.0000679763308931 2138

Table 3

In this table, we have: first column: n; second column: p_n; third column: p_{n+1}; fourth column: p_{n+2}; fifth column: $e^{\sqrt{p_{n+2}}-\sqrt{p_n}-\frac{1}{\sqrt{p_{n+1}}}}$; sixth column: $\frac{p_{n+2}}{p_n}$

1	2	3	5	1.2769879565250648	2.5
2	3	5	7	1.5943831282749161	2.3333333333333335
3	5	7	11	2.018979811142102	2.2
4	7	11	13	1.93148400086736	1.8571428571428572
5	11	13	17	1.6974561315632921	1.5454545454545454
6	13	17	19	1.6666440322776312	1.4615384615384615
7	17	19	23	1.557855449819763	1.3529411764705883
8	19	23	29	2.2654002395712247	1.5263157894736843
9	23	29	31	1.7972136643906813	1.3478260869565217
10	29	31	37	1.6786542338744328	1.2758620689655173
11	31	37	41	1.9561160249205707	1.3225806451612903
12	37	41	43	1.3750666904876445	1.162162162162162
13	41	43	47	1.349901723827378	1.146341463414634
14	43	47	53	1.7803435894912953	1.2325581395348837
15	47	53	59	1.989991930133486	1.2553191489361701
16	53	59	61	1.4917514145156994	1.150943396226415
17	59	61	67	1.4566949606980872	1.1355932203389831
18	61	67	71	1.6384171984575573	1.1639344262295082
19	67	71	73	1.2712145825086658	1.0895522388059702
20	71	73	79	1.411994857324697	1.1126760563380282
21	73	79	83	1.5744736780049133	1.1369863013698631
22	79	83	89	1.5465435216096741	1.1265822784810127
23	83	89	97	1.8821682460643192	1.1686746987951808
24	89	97	101	1.6725675026772382	1.1348314606741574
25	97	101	103	1.222050327304741	1.0618556701030928
26	101	103	107	1.2161278086899345	1.0594059405940595
27	103	107	109	1.2149966200220241	1.058252427184466
28	107	109	113	1.209591563046201	1.0560747663551402
29	109	113	127	2.085584337616967	1.165137614678899
30	113	127	131	2.0681235086540846	1.1592920353982301
31	127	131	137	1.4160932327886468	1.078740157480315
32	131	137	139	1.2954617748935786	1.0610687022900764
33	137	139	149	1.5174583726544613	1.0875912408759123
34	139	149	151	1.5165778965597723	1.0863309352517985
35	149	151	157	1.2738319289305096	1.0536912751677852
36	151	157	163	1.490528744236453	1.0794701986754967
37	157	163	167	1.369653587708779	1.0636942675159236
38	163	167	173	1.3612709425074072	1.0613496932515338
39	167	173	179	1.4625940193792257	1.0718562874251496
40	173	179	181	1.253493057153984	1.046242774566474
41	179	181	191	1.4431920853320148	1.0670391061452513
42	181	191	193	1.4426222750115065	1.0662983425414365
43	191	193	197	1.1542055957637005	1.031413612565445
44	193	197	199	1.1537816517116122	1.0310880829015545
45	197	199	211	1.5208692983863397	1.0710659898477157
46	199	211	223	2.133162679087425	1.120603015075377
47	211	223	227	1.605949505652633	1.0758293838862558
48	223	227	229	1.1424660138807983	1.0269058295964126
49	227	229	233	1.1408076644118084	1.026431718061674
50	229	233	239	1.2987038194521876	1.0436681222707425