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# AN ELEMENTARY PROOF OF OPPERMANN'S CONJECTURE

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**ABSTRACT.** We prove the Oppermann's conjecture: given an integer, $n > 1$, there is, at least, one prime between $n^2 - n$ and $n^2$, and, at least, another prime between $n^2$ and $n^2 + n$, using the prime-counting function and the Bertrand's Postulate.

## 1. INTRODUCTION

The Oppermann's conjecture, named after Ludvig Oppermann, in 1882, relates to distribution of the prime numbers. It states that, for any integer, $n > 1$, there is, at least, one prime between $n^2 - n$ and $n^2$, and, at least, another prime between $n^2$ and $n^2 + n$. We use the alternative statement:

Let $\pi(n)$ be the prime-counting function, that is, the number of prime numbers less than or equal to $n$. Then,

(1) $\pi(n^2 - n) < \pi(n^2) < \pi(n^2 + n)$,

for $n > 1$. This means that between the square of a number $n$ and the square of the same number plus (or minus) that number, there is a prime number. Or, equivalently,

$$\pi(n^2) - \pi(n^2 - n) > 0 \tag{2}$$

and

$$\pi(n^2 + n) - \pi(n^2) > 0. \tag{3}$$

## 2. LEMMAS AND THEOREMS

LEMMA 1. (Bertrand's Postulate, actually a Theorem) *For any integer $n > 3$, there always exists, at least, one prime number, $p$, with $n < p < 2n - 2$.*

A weaker, but more elegant formulation is:

LEMMA 2. (Weak Bertrand's Postulate) *For every $n > 1$ there is always, at least, one prime number, $p$, such that $n < p < 2n$.*

THEOREM 1. *For $n \geq 5$ and $n \in \mathbb{Z}_+$, then*

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$$(4)\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1},$$

$$(5)\pi(n) = 2 + \sum_{k=5}^{n} \frac{1 - \cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] + \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)},$$

$$(6)\pi(n) = 2 - \sum_{k=5}^{n} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\},$$

$$(7)\pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi i \Gamma(z)}{z}}\right)\left(1 - e^{-\frac{2\pi i}{z}}\right)}{2 - 2\cos\left(\frac{2\pi}{z}\right)}.$$

*Proof*. Part 1. In [Dickson, pp. 427], H. Laurent noted that

$$(8) \qquad f(z) = \frac{e^{\frac{2\pi i \Gamma(z)}{z}} - 1}{e^{-\frac{2\pi i}{z}} - 1} = \begin{cases} 0, \mathit{if}\ z\ \mathit{is}\ \mathit{composite} \\ 1, \mathit{if}\ z\ \mathit{is}\ \mathit{prime} \end{cases},$$

forz $\geq 5$ and $z \in \mathbb{Z}_+$.

Observe that

$$f(z) = e^{\frac{2\pi i}{z}} \frac{\cos\left[\frac{2\pi\Gamma(z)}{z}\right] + i \sin\left[\frac{2\pi\Gamma(z)}{z}\right] - 1}{1 - e^{\frac{2\pi i}{z}}}$$

$$= e^{\frac{2\pi i}{z}} \frac{\cos\left[\frac{2\pi\Gamma(z)}{z}\right] + i \sin\left[\frac{2\pi\Gamma(z)}{z}\right] - 1}{1 - \cos\left(\frac{2\pi}{z}\right) - i \sin\left(\frac{2\pi}{z}\right)}$$

$$= -\left[\cos\left(\frac{2\pi}{z}\right) + i \sin\left(\frac{2\pi}{z}\right)\right] \frac{1 - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] - i \sin\left[\frac{2\pi\Gamma(z)}{z}\right]}{1 - \cos\left(\frac{2\pi}{z}\right) - i \sin\left(\frac{2\pi}{z}\right)}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right) - i \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i \sin\left(\frac{2\pi}{z}\right)}$$
$$- \left\{\frac{i \sin\left(\frac{2\pi}{z}\right) - i \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i \sin\left(\frac{2\pi}{z}\right)}\right\}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$-\frac{i\left\{\sin\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right) - \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right)\right\}}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right] + i\left\{\sin\left(\frac{2\pi}{z}\right) - \sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]\right\}}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}.$$

Using the identity

$$\frac{a+bi}{c-di} = \frac{ac-bd}{c^2+d^2} + i\frac{bc+ad}{c^2+d^2},$$

we find the following real part

$$\Re[f(z)] = \frac{1 - \cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] + \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)} \tag{9}$$

$$= -\csc\left(\frac{\pi}{z}\right)\sin\left[\frac{\pi\Gamma(z)}{z}\right]\cos\left\{\frac{\pi[\Gamma(z)+1]}{z}\right\} \tag{10}$$

$$= \begin{cases} 0, if\ z\ is\ composite \\ 1, if\ z\ is\ prime \end{cases},$$

for$z \geq 5$ and $z \in \mathbb{Z}_+$.

The imaginary part is the following

$$\Im[f(z)] = -\frac{\sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right] - \sin\left[\frac{2\pi\Gamma(z)}{z}\right] - \sin\left(\frac{2\pi}{z}\right)}{2 - 2\cos\left(\frac{2\pi}{z}\right)} \tag{11}$$

$$= \frac{\sin\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right] - \sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)}.$$

It follows from (9) and (11) that

$$(12)\ f(z)=\frac{1-\cos\left(\frac{2\pi}{z}\right)-\cos\left[\frac{2\pi\Gamma(z)}{z}\right]+\cos\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}+i\frac{\sin\left(\frac{2\pi}{z}\right)+\sin\left[\frac{2\pi\Gamma(z)}{z}\right]-\sin\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{1-\cos\left(\frac{2\pi}{z}\right)+i\sin\left(\frac{2\pi}{z}\right)-\cos\left[\frac{2\pi\Gamma(z)}{z}\right]+i\sin\left[\frac{2\pi\Gamma(z)}{z}\right]+\cos\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]-i\sin\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{1-e^{-\frac{2\pi i}{z}}-e^{-\frac{2\pi i\Gamma(z)}{z}}+e^{-\left[\frac{2\pi i\Gamma(z)}{z}+\frac{2\pi i}{z}\right]}}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{-1\left(e^{-\frac{2\pi i}{z}}-1\right)+e^{-\frac{2\pi i\Gamma(z)}{z}}\left(e^{-\frac{2\pi i}{z}}-1\right)}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{\left(1-e^{-\frac{2\pi i\Gamma(z)}{z}}\right)\left(1-e^{-\frac{2\pi i}{z}}\right)}{2-2\cos\left(\frac{2\pi}{z}\right)}=\begin{cases}0, if\ z\ is\ composite\\ 1, if\ z\ is\ prime\end{cases},$$

for$z\geq 5$ and $z\in\mathbb{Z}_+$.

Part 2. The prime-counting function is the function counting the number of prime numbers less than or equal to some real number $x$. It is denoted by $\pi(x)$.From above definition, we have

$$\pi(x)=\sum_{p\leq x}1.$$

With the restriction for the positive integers and greater than or equal to five, it follows that

$$\pi(n)=\ 2+\sum_{k=5}^{n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

by (8),

$$\pi(n)=2+\sum_{k=5}^{n}\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left[\frac{2\pi\Gamma(k)}{k}\right]+\cos\left[\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right]}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

by (9),

$$\pi(n) = 2 - \sum_{k=5}^{n} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\}$$

by (10),

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}$$

by (12).□

COROLLARY 1. *For* $x \in \mathbb{R}_{\geq 5}$, *then*

$$(13)\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1},$$

$$(14)\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left[\frac{2\pi\Gamma(k)}{k}\right] + \cos\left[\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right]}{2 - 2\cos\left(\frac{2\pi}{k}\right)},$$

$$(15)\pi(x) = 2 - \sum_{k=5}^{\lfloor x \rfloor} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\},$$

$$(16)\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{\left(1 - e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}.$$

*Proof*. Is obvious by the definition of floor function: $\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \leq x\}$ and previous Theorem. □

THEOREM 2. (Oppermann's Theorem) *For any integer,* $n > 1$, *there is, at least, one prime number between* $n^2 - n$ *and* $n^2$, *and, at least, another prime number between* $n^2$ *and* $n^2 + n$.

*Proof*. Step 1. By use from (4), we find

$$(17)\pi(n^2 + n) = 2 + \sum_{k=5}^{n^2+n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1},$$

$$(18)\pi(n^2) = 2 + \sum_{k=5}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$

and

$$(19)\pi(n^2-n) = 2 + \sum_{k=5}^{n^2-n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}.$$

Step 2.Subtracting (18) from (17),we have

$$(20)\pi(n^2+n)-\pi(n^2) = 2 + \sum_{k=5}^{n^2+n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \left[2 + \sum_{k=5}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]$$

$$= 2 + \sum_{k=5}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} + \sum_{k=2n+1}^{n^2+n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \left[2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} + \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]$$

$$= \pi(2n) + \sum_{k=2n+1}^{n^2+n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \pi(n) - \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=2n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} + \sum_{k=n^2+1}^{n^2+n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \sum_{k=2n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=n^2+1}^{n^2+n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}.$$

By (8) we have the inequality

$$(21)\qquad 0 = \min_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right) \leq \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} \leq \max_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right) = 1.$$

From (20) and (21), it follows that

$$\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}\min_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)-\sum_{k=n+1}^{2n}\min_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)$$
$$\leq \pi(n^2+n)-\pi(n^2)$$
$$\leq \pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}\max_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)$$
$$-\sum_{k=n+1}^{2n}\max_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right),$$

$$\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}0-\sum_{k=n+1}^{2n}0\leq \pi(n^2+n)-\pi(n^2)$$
$$\leq \pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}1-\sum_{k=n+1}^{2n}1,$$

$$\pi(2n)-\pi(n)\leq \pi(n^2+n)-\pi(n^2)\leq \pi(2n)-\pi(n)+\ n-n,$$

$$\pi(2n)-\pi(n)\leq \pi(n^2+n)-\pi(n^2)\leq \pi(2n)-\pi(n),$$

for $n\in\mathbb{Z}_{\geq 5}$.

Step 3. For $n=2,\pi(2^2+2)-\pi(2^2)=\pi(6)-\pi(4)=3-2=1>0$; for $=3,\pi(3^2+3)-\pi(3^2)=\pi(12)-\pi(9)=5-4=1>0$ , for $n=4,\pi(4^2+4)-\pi(4^2)=\pi(20)-\pi(16)=8-6=2>0$.

Step 4.Subtracting (19) from (18),we have

$$(22)\pi(n^2)-\pi(n^2-n)=2+\sum_{k=5}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\left[2+\sum_{k=5}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]$$

$$=2+\sum_{k=5}^{2n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}+\sum_{k=2n+1}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$
$$-\left[2+\sum_{k=5}^{n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}+\sum_{k=n+1}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]$$

$$=\pi(2n)+\sum_{k=2n+1}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\pi(n)-\sum_{k=n+1}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=2n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \sum_{k=n+1}^{n^2-n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=2n+1}^{n^2-n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} + \sum_{k=n^2-n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$
$$- \sum_{k=2n+1}^{n^2-n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=n^2-n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}.$$

From (21) and (22), it follows that

$$\pi(2n) - \pi(n) + \sum_{k=n^2-n+1}^{n^2} \min_{k\in\mathbb{Z}_{\geq 5}} \left( \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right) - \sum_{k=n+1}^{2n} \min_{k\in\mathbb{Z}_{\geq 5}} \left( \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right)$$
$$\leq \pi(n^2) - \pi(n^2 - n)$$
$$\leq \pi(2n) - \pi(n) + \sum_{k=n^2-n+1}^{n^2} \max_{k\in\mathbb{Z}_{\geq 5}} \left( \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right)$$
$$- \sum_{k=n+1}^{2n} \max_{k\in\mathbb{Z}_{\geq 5}} \left( \frac{e^{\frac{2\pi i\Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right),$$

$$\pi(2n) - \pi(n) + \sum_{k=n^2-n+1}^{n^2} 0 - \sum_{k=n+1}^{2n} 0 \leq \pi(n^2) - \pi(n^2 - n)$$
$$\leq \pi(2n) - \pi(n) + \sum_{k=n^2-n+1}^{n^2} 1 - \sum_{k=n+1}^{2n} 1,$$

$$\pi(2n) - \pi(n) \leq \pi(n^2 + n) - \pi(n^2) \leq \pi(2n) - \pi(n) + n - n,$$

$$\pi(2n) - \pi(n) \leq \pi(n^2 + n) - \pi(n^2) \leq \pi(2n) - \pi(n),$$

for$n \in \mathbb{Z}_{\geq 5}$.

Step 5. For $n = 2, \pi(2^2) - \pi(2^2 - 2) = \pi(4) - \pi(2) = 2 - 1 = 1 > 0$; for $= 3, \pi(3^2) - \pi(3^2 - 3) = \pi(9) - \pi(6) = 4 - 3 = 1 > 0$ , for $n = 4, \pi(4^2) - \pi(4^2 - 4) = \pi(16) - \pi(12) = 6 - 5 = 1 > 0$. This is completes the proof. □

REFERENCES

[Dickson] Dickson, Leonard Eugene, *History of the Theory of Numbers, Volume I: Divisibility and Primality*, Dover, 2005.

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