Investigations on the Theory of Riemann Zeta Function I:

New Functional Equation, Integral Representation and Laurent Expansion for Riemann's Zeta Function

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Blessed {be} he that cometh in the name of the LORD: we have blessed you out of the house of the LORD.
Psalms 118:26

ABSTRACT.

We developed a new functional equation and a new integral representation for the Riemann zeta function.

1. INTRODUCTION

Our main goal is the development of these formulas:

$$(1.1)\qquad (2^{1-s}-1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s}+\zeta\left(1-s,\frac{3}{2}\right)\right],$$

$$(1.2)\quad \zeta(s)=\frac{2^s}{2^s-1}+\frac{3^{1-s}}{(2-2^{1-s})(s-1)}+\frac{2^{s-1}}{3^s(2^s-1)}+\frac{2}{2^s-1}\int_0^\infty\frac{\sin\left[s\tan^{-1}\left(\frac{2t}{3}\right)\right]}{\left[\left(\frac{3}{2}\right)^2+t^2\right]^{s/2}}\frac{dt}{e^{2\pi t}-1},$$

and

$$(1.3)\qquad \zeta(s)=\frac{2^s}{2^s-1}+\frac{1}{(s-1)(2^s-1)}+\frac{1}{2^s-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\left(\frac{3}{2}\right)(s-1)^n.$$

which converges more rapidly than the known Laurent expansion [see 5]:

$$(1.3)\qquad \zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(1)(s-1)^n.$$

2. LEMMAS AND THEOREMS

LEMMA 1. *For $n\in\mathbb{Z}_{\geq 1}$ and $\alpha\in\mathbb{R}-\{0,1\}$, then*

$$(2.1)\qquad \frac{1}{n}=\frac{\alpha}{(\alpha-1)n}-\frac{\alpha}{\alpha(\alpha-1)n}$$

Proof. We expand the right-hand side of (2.1)

$$\frac{\alpha}{(\alpha-1)n}-\frac{\alpha}{\alpha(\alpha-1)n}=\frac{\alpha}{\alpha-1}\left(\frac{1}{n}-\frac{1}{\alpha n}\right)$$

$$=\frac{\alpha}{(\alpha-1)}\left(\frac{\alpha-1}{\alpha n}\right)=\frac{1}{n}.\ \square$$

LEMMA 2. *For $n\in\mathbb{Z}_{\geq 1}$ and $m\in\mathbb{Z}_{>1}$, then*

$$(2.2)\qquad \frac{1}{n}=\frac{1}{n-1}-\frac{1}{(n-1)n}.$$

Proof. Let $\alpha = n$, in Lemma 1. □

THEOREM 1. *Let* $\operatorname{Re}(s) > 0$ *and* $s \neq 1$, *then*

$$\zeta(s) = \frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1}, \tag{2.3}$$

where $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s, a)$ *is the Hurwitz zeta function.*

Proof. For $\operatorname{Re}(s) > 0$, then [see 1]

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \Rightarrow \zeta(s) - \frac{1}{1 - 2^{1-s}} = \frac{1}{1 - 2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^{s-1}} \cdot \frac{1}{n}. \tag{2.4}$$

Substituting (2.2) in (2.4), we obtain

$$\zeta(s) - \frac{1}{1 - 2^{1-s}} = \frac{1}{1 - 2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^{s-1}} - \frac{1}{1 - 2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^s},$$

$$\zeta(s) + \frac{1}{1 - 2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^s} = \frac{1}{1 - 2^{1-s}} + \frac{1}{1 - 2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^{s-1}}. \tag{2.5}$$

On the other hand, in [2, p. 9], we encounter

$$\int_0^{\infty} e^{-nx} x^{s-1} dx = \frac{\Gamma(s)}{n^s}, \tag{2.6}$$

for $\operatorname{Re}(s) > 0$ and $n > 0$.

We set (2.6) in (2.5)

$$\zeta(s) + \frac{1}{(1 - 2^{1-s})\Gamma(s)} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n-1} \int_0^{\infty} e^{-nx} x^{s-1} dx$$
$$= \frac{1}{1 - 2^{1-s}} + \frac{1}{(1 - 2^{1-s})\Gamma(s)} \sum_{n=2}^{\infty} \frac{(-1)^{n-1} n}{n-1} \int_0^{\infty} e^{-nx} x^{s-1} dx,$$

$$\zeta(s) + \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^{\infty} \left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n-1} e^{-nx} \right) x^{s-1} dx$$
$$= \frac{1}{1 - 2^{1-s}} + \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^{\infty} \left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1} n}{n-1} e^{-nx} \right) x^{s-1} dx,$$

$$\zeta(s) - \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^{\infty} \log(e^{-x} + 1)\, e^{-x} x^{s-1} dx$$
$$= \frac{1}{1 - 2^{1-s}} - \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^{\infty} \left[\frac{(e^{-x} + 1) \log(e^{-x} + 1) + e^{-x}}{e^x + 1} \right] x^{s-1} dx,$$

$$\zeta(s) - \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \log(e^{-x}+1)\, e^{-x}\, x^{s-1}dx$$
$$= \frac{1}{1-2^{1-s}} - \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \left[\frac{(e^{-x}+1)\log(e^{-x}+1)}{e^x+1}\right] x^{s-1}dx$$
$$- \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{e^{-x}x^{s-1}}{e^x+1}dx,$$

$$\zeta(s) + \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \left[\frac{(e^{-x}+1)\log(e^{-x}+1)}{e^x+1}\right] x^{s-1}dx$$
$$- \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \log(e^{-x}+1)\, e^{-x}\, x^{s-1}dx$$
$$= \frac{1}{1-2^{1-s}} - \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{e^{-x}x^{s-1}}{e^x+1}dx,$$

$$\zeta(s) + \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \left[\frac{(e^{-x}+1)\log(e^{-x}+1)}{e^x+1} - \log(e^{-x}+1)\, e^{-x}\right] x^{s-1}dx$$
$$= \frac{1}{1-2^{1-s}} - \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{e^{-x}x^{s-1}}{e^x+1}dx,$$

$$\zeta(s) = \frac{1}{1-2^{1-s}} - \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{e^{-x}x^{s-1}}{e^x+1}dx,$$

$$\zeta(s) + \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{e^{-x}x^{s-1}}{e^x+1}dx = \frac{1}{1-2^{1-s}}. \tag{2.7}$$

We calculate

$$\int_0^\infty \frac{e^{-x}x^{s-1}}{e^x+1}dx = 2^{-s}\,\Gamma(s)\left(\zeta(s) - \zeta\left(s,\frac{3}{2}\right)\right), \tag{2.8}$$

for $\operatorname{Re}(s) > 0$.

Substituting (2.8) in (2.7), we have

$$\zeta(s) + \frac{1}{(1-2^{1-s})\Gamma(s)}\left\{2^{-s}\,\Gamma(s)\left[\zeta(s) - \zeta\left(s,\frac{3}{2}\right)\right]\right\} = \frac{1}{1-2^{1-s}},$$

$$\zeta(s) + \frac{\zeta(s) - \zeta\left(s,\frac{3}{2}\right)}{2^s-2} = \frac{1}{1-2^{1-s}},$$

$$\left(\frac{2^s-1}{2^s-2}\right)\zeta(s) = \frac{1}{1-2^{1-s}} + \frac{\zeta\left(s,\frac{3}{2}\right)}{2^s-2},$$

$$\left(\frac{2^s-1}{2^s-2}\right)\zeta(s) = \frac{2^s}{2^s-2} + \frac{\zeta\left(s,\frac{3}{2}\right)}{2^s-2},$$

$$(2^s-1)\,\zeta(s) = 2^s + \zeta\left(s,\frac{3}{2}\right),$$

$$\zeta(s) = \frac{2^s}{2^s-1} + \frac{\zeta\left(s,\frac{3}{2}\right)}{2^s-1}. \square$$

COROLLARY 1. *Let* $\zeta(s) = 0$ *and* $\mathrm{Re}(s) > 0$, *then*

$$2^s + \zeta\left(s, \frac{3}{2}\right) = 0 \tag{2.9}$$

holds.

Proof. If we assume that $\zeta(s) = 0$, in Theorem 1, then

$$\frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1} = 0 \Rightarrow 2^s + \zeta\left(s, \frac{3}{2}\right) = 0. \square$$

The Theorem 2 can be rewritten as follows:

COROLLARY 1.a. *For* $\mathrm{Re}(s) > 0$, *all the roots of* $\zeta(s)$ *satisfies the equation*

$$2^s + \zeta\left(s, \frac{3}{2}\right) = 0. \tag{2.10}$$

COROLLARY 2. *For* $\mathrm{Re}(s) > 0$, *then*

$$(2^s - 1)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-1/2}\,\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\left[2^s + \zeta\left(s, \frac{3}{2}\right)\right], \tag{2.11}$$

and, for $0 < Re(s) < 1$, *then*

$$(2^{1-s} - 1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\,\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s} + \zeta\left(1-s, \frac{3}{2}\right)\right], \tag{2.12}$$

where $\Gamma(s)$ *is the gamma function,* $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s, a)$ *is the Hurwitz zeta function.*

Proof. In [1] and [3, p. 136-144], we have

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\,\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\,\zeta(1-s) \Rightarrow \zeta(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\pi^{s-1/2}\,\zeta(1-s), \tag{2.12}$$

for all complex s, proved by Riemann (1859).

Substituting (2.12) in (2.3)

$$\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\pi^{s-1/2}\,\zeta(1-s) = \frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1} \tag{2.13}$$

$$\Rightarrow (2^s - 1)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-1/2}\,\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\left[2^s + \zeta\left(s, \frac{3}{2}\right)\right].$$

Let $1 - s \rightarrow s$ in (2.13)

$$(2^{1-s} - 1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\,\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s} + \zeta\left(1-s, \frac{3}{2}\right)\right]. \square$$

THEOREM 2. *Let* $\mathrm{Re}(s) > 1$, *then*

$$\zeta(s) = \frac{2^s}{2^s - 1} + \frac{3^{1-s}}{(2 - 2^{1-s})(s-1)} + \frac{2^{s-1}}{3^s(2^s - 1)} + \frac{2}{2^s - 1}\int_0^\infty \frac{\sin\left[s\tan^{-1}\left(\frac{2t}{3}\right)\right]}{\left[\left(\frac{3}{2}\right)^2 + t^2\right]^{s/2}}\frac{dt}{e^{2\pi t} - 1}, \tag{2.14}$$

where $\zeta(s)$ is the Riemann zeta function.

Proof. In [4], we encounter the Abel-Plana formula for Hurwitz zeta function, that is,

$$(2.15)\qquad \zeta(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}=\frac{\alpha^{1-s}}{s-1}+\frac{1}{2\alpha^s}+2\int_0^{\infty}\frac{\sin\left(s\tan^{-1}\frac{t}{\alpha}\right)}{(\alpha^2+t^2)^{s/2}}\frac{dt}{e^{2\pi t}-1}.$$

Let $\alpha=\frac{3}{2}$ in (2.15)

$$(2.16)\qquad \zeta\left(s,\frac{3}{2}\right)=\frac{3^{1-s}}{2^{1-s}(s-1)}+\frac{2^{s-1}}{3^s}+2\int_0^{\infty}\frac{\sin\left[s\tan^{-1}\left(\frac{2t}{3}\right)\right]}{\left[\left(\frac{3}{2}\right)^2+t^2\right]^{s/2}}\frac{dt}{e^{2\pi t}-1}.$$

Substituting (2.16) in (2.3), we complete the proof. □

THEOREM 3. *Let* $\mathrm{Re}(s)>1$, *then*

$$(2.17)\qquad \zeta(s)=\frac{2^s}{2^s-1}+\frac{1}{(s-1)(2^s-1)}+\frac{1}{2^s-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\left(\frac{3}{2}\right)(s-1)^n,$$

where $\zeta(s)$ is the Riemann zeta function and $\gamma_n(\alpha)$ are the Stieltjes constants.

Proof. In [5], we find the Laurent series expansion of the Hurwitz zeta function:

$$(2.18)\qquad \zeta(s,\alpha)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(\alpha)(s-1)^n.$$

Let $\alpha=\frac{3}{2}$ in (2.18)

$$(2.19)\qquad \zeta\left(s,\frac{3}{2}\right)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\left(\frac{3}{2}\right)(s-1)^n.$$

Substituting (2.19) in (2.3), we complete the proof. □

REFERENCES

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[4] https://en.wikipedia.org/wiki/Abel-Plana_formula, available in May 07, 2013.

[5] https://en.wikipedia.org/wiki/Stieltjes_constants, available in May 07, 2013.