# Investigations on the Theory of Riemann Zeta Function II:

## On the Riemann-Siegel Integral and Hardy's Z-Function

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*The heart of the prudent getteth knowledge; and the ear of the wise seeketh knowledge.*
*Proverb 18:15*

## ABSTRACT

We create new formulas for Riemann-Siegel Integral and Hardy's Z-function.

## 1. INTRODUCTION

We define Hardy's Z-function to be

(1.1)
$$Z(t) := e^{i\theta(t)} \zeta\left(\tfrac{1}{2} + it\right),$$

where $\theta(t)$ is given by

(1.2)
$$\theta(t) := \Im\left(\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{t}{2}\log\pi,$$

see [1, p. 47].

In this paper, we proved a new formula for Hardy's Z-function:

(1.3)
$$Z(t) = \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1}\left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right],$$

and a new formula for Riemann-Siegel integral:

(1.4)
$$\int_{C_N} \frac{(-x)^{-\frac{1}{2}+it}e^{-Nx}}{e^x - 1}dx = \frac{e^{2i\theta(t)}(2\pi)^{\frac{1}{2}+it}e^{\frac{t\pi}{2} - \frac{i\pi}{4}}\left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)}\left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right]$$

$$-2(2\pi)^{\frac{1}{2}+it}e^{i\theta(t)}e^{\frac{t\pi}{2} - \frac{i\pi}{4}}\left(1 - ie^{-\frac{i\pi}{4}}\right)\sum_{n=1}^{N}\left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right).$$

## 2. PRELIMINARES

THEOREM 1. *Let* $\operatorname{Re}(s) > 0$ *and* $s \neq 1$, *then*

(2.1)
$$\zeta(s) = \frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1},$$

*where* $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s, a)$ *is the Hurwitz zeta function.*

*Proof.* See 2. □

COROLLARY 1. *For* $\operatorname{Re}(s) > 0$, *then*

(2.2)
$$(2^s - 1)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-1/2}\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\left[2^s + \zeta\left(s, \frac{3}{2}\right)\right],$$

*and, for $0 < Re(s) < 1$, then*

$$(2.3) \qquad (2^{1-s} - 1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\,\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s} + \zeta\left(1 - s, \frac{3}{2}\right)\right],$$

*where $\Gamma(s)$ is the gamma function, $\zeta(s)$ is the Riemann zeta function and $\zeta(s, a)$ is the Hurwitz zeta function.*

*Proof.* See [2]. $\square$

THEOREM 2 (Approximate Functional Equation). *Let $x, y \in \mathbb{R}^+$ with $2\pi xy = |t|$; then for $s = \sigma + it$, with $0 \le \sigma \le 1$, we have*

$$(2.1) \qquad \zeta(s) = \sum_{n \le x}\frac{1}{n^s} + \chi(s)\sum_{n \le y}\frac{1}{n^{1-s}} + \mathcal{O}(x^{-\sigma}) + \mathcal{O}\left(|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}\right),$$

*where $\chi(s)$ is given by*

$$(2.2) \qquad \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s).$$

*Proof.* See [1, p. 47]. $\square$

For the Hardy's Z-function, we have the approximate

$$(2.3) \qquad Z(t) = 2\sum_{n=1}^{[x]}\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}} + \mathcal{O}\left(t^{-\frac{1}{4}}\right),$$

see [1, p. 48].

**The Riemann-Siegel Formula.** THEOREM 3. *For all $t \in \mathbb{R}$,*

$$(2.4)\ Z(t) = 2\sum_{n=1}^{N}\left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right) + \frac{e^{-i\theta(t)}e^{-\frac{t\pi}{2}}}{(2\pi)^{\frac{1}{2}+it}e^{-\frac{i\pi}{4}}\left(1 - ie^{-\frac{i\pi}{4}}\right)}\int_{C_N}\frac{(-x)^{-\frac{1}{2}+it}e^{-Nx}}{e^x - 1}\,dx,$$

*where $C_N$ is positively oriented closed contour containing all of the points $\pm 2\pi i N, \pm 2\pi i(N - 1), \dots, \pm 2\pi i$, and $0$. See [1, p. 48].*

## 3. LEMMA AND THEOREM

**Hardy's Z-Function.** LEMMA 1. *For all $t \in \mathbb{R}$, then*

$$(3.1) \qquad Z(t) = \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1}\left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right].$$

*Proof.* Let $s = \frac{1}{2} + it$ in (2.1)

$$(3.2) \qquad \zeta\left(\frac{1}{2} + it\right) = \frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it} - 1} + \frac{\zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)}{2^{\frac{1}{2}+it} - 1}.$$

Substituting (3.2) in (1.1), we find

$$Z(t) = \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1}\left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right]. \square$$

**The Riemann-Siegel Integral.** THEOREM 4. *For all $t \in \mathbb{R}$, then*

$$(3.3) \quad \int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx = \frac{e^{2i\theta(t)} (2\pi)^{\frac{1}{2}+it} e^{\frac{t\pi}{2}-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)\right]$$

$$- 2(2\pi)^{\frac{1}{2}+it} e^{i\theta(t)} e^{\frac{t\pi}{2}-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right) \sum_{n=1}^{N} \left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right),$$

*where $C_N$ is positively oriented closed contour containing all of the points $\pm 2\pi i N, \pm 2\pi i(N - 1), \ldots, \pm 2\pi i$, and $0$.*

*Proof.* We set (3.3) in the left-hand side of (2.4)

$$(3.4) \quad \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1}\left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)\right] = 2\sum_{n=1}^{N}\left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right)$$

$$+ \frac{e^{-i\theta(t)} e^{-\frac{t\pi}{2}}}{(2\pi)^{\frac{1}{2}+it} e^{-\frac{i\pi}{4}}\left(1 - ie^{-\frac{i\pi}{4}}\right)}\int_{C_N}\frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx,$$

thereby,

$$\int_{C_N}\frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx = \frac{e^{2i\theta(t)} (2\pi)^{\frac{1}{2}+it} e^{\frac{t\pi}{2}-\frac{i\pi}{4}}\left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)}\left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)\right]$$

$$- 2(2\pi)^{\frac{1}{2}+it} e^{i\theta(t)} e^{\frac{t\pi}{2}-\frac{i\pi}{4}}\left(1 - ie^{-\frac{i\pi}{4}}\right)\sum_{n=1}^{N}\left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right). \square$$

## REFERENCES

[1] Peter Borwein, Stephen Choi, Brendan Rooney, Andrea Weirathmueller, *The Riemann Hypothesis – A resource for the afficionado and virtuoso alike*, Springer, 2008.

[2] Edigles Guedes and Prof. Dr. Raja Rama Gandhi, *Investigations on the Theory of Riemann Zeta Function: New Functional Equation, Integral Representation and Laurent Expansion for Riemann's Zeta Function*, May 1, 2013.