YANG-MILLS GAUGE THEORY AND THE FOUNDATIONS OF NUCLEAR PHYSICS

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Abstract: This is the first partial draft of a paper under development to further elaborate the author's thesis presented in several earlier-published papers, that baryons including protons and neutrons are Yang-Mills magnetic monopoles, and to respond to queries and comments received with respect to these earlier papers. This paper fully develops the non-linear aspects of Yang-Mills gauge theory and applies these to the inverses used to populate the Yang-Mills magnetic monopolies with quarks and turn them into baryons and give rise to QCD. We also show how the perturbations in these inverses, which arise from the non-linear theory, create a pseudo-mass term which is responsible for the short-range of the nuclear interaction, notwithstanding the zero-mass gluon gauge fields.

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I. Introduction

In a recent paper [1], the author presented the thesis that the non-vanishing magnetic monopoles of Yang-Mills theory are in fact synonymous with baryons. That is, magnetic monopoles, long-pursued since the time of James Clerk Maxwell have, in Yang-Mills incarnation, always been hiding in plain sight as baryons, and most importantly, as the protons and neutrons which rest at the center of the material universe.

Since the release of that paper, the author has been in communication with a number of people who have offered helpful comment and critique and asked for clarification of certain key points of development. At the same time, the original paper did at points allude to some "deeper analysis" which was consciously not detailed in [1], in order to achieve as much brevity as possible in a paper that was already 69 pages. Also, with the benefit of more than half a year of reflection on this original paper, as well as the confidence that the author has gained in the physical correctness of this thesis through the subsequent prediction to parts per 10^5 or 10^6 AMU accuracy of the empirical binding energies for fifteen (15) distinct nuclear isotopes namely, ^{2}H, ^{3}H, ^{3}He, ^{4}He, ^{6}Li, ^{7}Li, ^{7}Be, ^{8}Be, ^{10}B, ^{9}Be, ^{10}Be, ^{11}B, ^{11}C, ^{12}C and ^{14}N as well as the neutron minus proton mass difference as detailed in several subsequent papers [2], [3] [4], [5], [6], it became apparent to the author that some of the original material in [1] could be developed more simply, directly, and broadly.

Consequently, this paper revisits the main development in [1] of the thesis that baryons including protons and neutrons are magnetic monopoles, simplifies the development where that is possible, expands on matters that were alluded to but not fully presented at the time, and answers pertinent questions posed to the author by others who are attempting to understand the theoretical basis of this theory. In contrast to the author's other recent papers mentioned above in which the goal was to confirm this thesis with empirical predictions or retrodictions, this paper will not attempt to expand this already significant set of empirical points of contact between theory and experiment. Rather, this paper is a foundational paper, dealing with the very deepest theoretical physics issues which underlie the thesis that baryons are Yang-Mills magnetic monopoles, and in many ways, showing – as this paper's title suggests – how Yang-Mills gauge theory is indeed the theoretical foundation of nuclear physics. Included in this paper is a discussion of a number of issues pertaining to the so-called "Yang-Mills mass gap" [7] problem, most notably, the question of how and why quarks become confined, and how to generate a short-range nuclear force from a gauge field of gluons which are presumed to be massless. And, in many ways, while the first paper [1] neglected some of the non-linear features of Yang-Mills because the author had concluded that those features could be neglected in the zero-perturbation limit which the author was reviewing in [1], this paper neglects nothing. The development in this paper is intended to and does fully incorporate and develop *all* of the non-linear features of

Yang-Mills theory which distinguish such theories from Abelian theories such as classical electrodynamics and its quantized formulation QED.

Out of all the comments received by the author since the publication of [1], perhaps the most important and fundamental comment comes from a well-known author and teacher about gravitational and relativity theory:

> "One thing that appears doubtful to me is the way you handle the QCD gauge fields, replacing them by the fermion source currents from which they originate. It seems to me that your procedure involves a wholesale deletion of all the nonlinearities in these gauge fields, which would seem unacceptable, because these nonlinearities are essential for generating short-range forces from a zero-mass gauge field. How you expect to get short-range forces from your approach is a mystery to me."

The commenter is 100% on target to be concerned about exactly this issue. It is a first tier issue, and it concerned the author for over seven years. Indeed, this single issue was most responsible for it taking the author close to seven years from the time he first hypothesized in April 2005 that baryons are magnetic monopoles, until the time that he could present a fully-developed theory on the subject in December 2012 when [1] was completed.

The crux of the author's own self-critique during that period of time was the following: On the one hand, we were using the nonlinearities of Yang-Mills to show the existence of magnetic monopoles that do not vanish from combining the equation $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu$ relating the vector potential G^μ to the field density $F^{\mu\nu}$ with the equation $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ relating the field density $F^{\mu\nu}$ to magnetic source charges $P^{\sigma\mu\nu}$. Recall that magnetic sources *do* vanish identically, $P^{\sigma\mu\nu} = 0$, in Abelian gauge theory, most notably QED, by combining $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu$ with $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$. On the other hand, when it came to Maxwell's electric source charge equation $J^\mu = \partial_\sigma F^{\mu\sigma}$, the simplest way to populate the Yang-Mills magnetic monopoles with quarks was to entirely delete the nonlinearities of Yang Mills, as the above commenter correctly points out. To the author, doing so seemed to be mathematically inconsistent, and it took over seven years to understand how to overcome this inconsistency.

To restate in different language: the author was taking advantage of the nonlinearity of Yang-Mills theory to create non-zero chromo-magnetic monopoles, and then seeking to ignore this very same non-linearity when it came to the chromo-electric charges which the author wanted to use to populate these magnetic monopoles with quarks to turn them into baryons including protons and neutrons.. This inconsistency bothered the author greatly, and really kept his research stuck in the same place for almost seven years.

In May 2012, the author finally bit the bullet and did a complete calculation involving the electric charge equation with all the nonlinearities, and then inverted that equation to express the gauge fields in terms of the fermion source currents from which they originate. Happily, it was discovered that all the extra terms that were introduced into the electric charge equation

$J^{\mu}=\partial_{\sigma}F^{\mu\sigma}$ by these non-linearities have the form of a perturbation $-V=\left(\partial_{\sigma}G^{\sigma}+G^{\sigma}\partial_{\sigma}\right)+G_{\sigma}G^{\sigma}$ identical in form to that which is used in quantum electrodynamics, with the only difference being that the gauge fields were now Yang-Mill gauge fields. In other words, the nonlinearities that the author wanted to ignore, when they show up in the chromo-electric charge equation $J^{\mu}=\partial_{\sigma}F^{\mu\sigma}$, are simply a perturbation. So that would justify simply looking at everything in the zero perturbation $V\to 0$ limit. This in turn allowed the author to feel comfortable using the same inverted equation $G^{\mu}=-J^{\mu}/\left(k^{\alpha}k_{\alpha}-m^{2}\right)$ between sources and gauge fields that one would use in a simple linear theory like QED.

Stated differently, the author discovered that when used in the magnetic monopole equation $P^{\sigma\mu\nu}=\partial^{\sigma}F^{\mu\nu}+\partial^{\mu}F^{\nu\sigma}+\partial^{\nu}F^{\sigma\mu}$ the nonlinearities of Yang-Mills create non-vanishing monopoles; and when used in the electric charge equation $J^{\mu}=\partial_{\sigma}F^{\mu\sigma}$ the nonlinearities introduce a new term $-V=\left(\partial^{\sigma}G_{\sigma}+G_{\sigma}\partial^{\sigma}\right)+G^{\sigma}G_{\sigma}$ that is no more and no less than a perturbation. The former non-vanishing magnetic monopoles the author very much wanted, and the latter appearance of a perturbation gave the author the ability to neglect terms that he very much wanted to neglect but heretofore could not justify neglecting.

All of this "deeper analysis" was alluded to between [2.8] and [2.10] of [1]. But to have discussed this entire seven-year journey and all of the calculations that led to being fully comfortable using [2.9] in [2.5] of [1] would have added many additional pages, and would have diluted the overall development. So the author deferred that full exposition of this analysis at the time of preparing [1], but will now present this analysis, completely, in sections II through VI of the present paper.

Once the author realized that the non-linearities of Yang-Mills simply produce perturbations V in the electric charge equation $J^{\mu}=\partial_{\sigma}F^{\mu\sigma}$, and then decided to neglected the perturbations, that of course had other consequences, but these the author was willing to accept. The primary consequence was that if one was regarding these magnetic monopoles as protons and neutrons, then one would be throwing away a tremendous amount of the interaction and noise that occurs inside the protons and neutrons which is undoubtedly responsible in some fashion for giving these nucleons their observed masses and energies. So the author knew that when he finally calculated the energies of these Yang-Mills magnetic monopoles without the perturbations, he could not expect to get the entire mass of a proton or neutron, but would instead get other, much lower energies, and so would have to find a way to make sense of what these lower energies actually represented, physically. This is discussed to a fair degree following [11.12] of [1].

As the author now understands, energies that result with the perturbations (and also the "Higgs vacuum") turned off, correspond to *nuclear binding energies*, as opposed to full masses of the proton and neutron. But this all originated in finally understanding that the nonlinear terms that the author wanted to neglect in the electric charge equation $J^{\mu}=\partial_{\sigma}F^{\mu\sigma}$ were perturbation terms that *could* be neglected as long as he was willing to modify his understanding of the proton and neutron energies that would come out of this. In section II through VI to follow, we shall now present in detail, the "deeper analysis" referred to just after [2.9] of [1].

II. A Review of Classical and Quantum Electrodynamics

II.1 Magnetic Monopole Densities

Because this paper will be largely focused on all of the non-linearities that come about in Yang-Mills (non-Abelian) gauge theories, most notably Quantum Chromodynamics (QCD), it is helpful as a point of comparison to first review classical and quantum electrodynamics (QED), which are entirely linear (Abelian) gauge theories. We start classically, with Maxwell's equations for the electric charge density J^ν and the magnetic charge density $P^{\sigma\mu\nu}$, respectively:

$$J^\nu = \partial_\mu F^{\mu\nu}, \tag{2.1}$$

$$P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}. \tag{2.2}$$

With only these two equations, there is nothing which requires the magnetic charge density $P^{\sigma\mu\nu}$ to become zero. It is only after introducing an Abelian (commuting) vector potential G^μ, and relating this to the field strength tensor $F^{\mu\nu}$ according to the relationship:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu = \partial^{[\mu} G^{\nu]}, \tag{2.3}$$

that the magnetic charge density vanishes, $P^{\sigma\mu\nu} = 0$. Because this result is central to much of the discussion here, it is important to review exactly how this zeroing of the magnetic monopole charge density comes about.

The first step is to substitute (2.3) into (2.2), which yields:

$$\begin{aligned} P^{\sigma\mu\nu} &= \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} \\ &= \partial^\sigma\left(\partial^\mu G^\nu - \partial^\nu G^\mu\right) + \partial^\mu\left(\partial^\nu G^\sigma - \partial^\sigma G^\nu\right) + \partial^\nu\left(\partial^\sigma G^\mu - \partial^\mu G^\sigma\right). \\ &= \left[\partial^\sigma, \partial^\mu\right] G^\nu + \left[\partial^\mu, \partial^\nu\right] G^\sigma + \left[\partial^\nu, \partial^\sigma\right] G^\mu \end{aligned} \tag{2.4}$$

So, if the partial spacetime derivatives are commuting with one another, $\left[\partial^\mu, \partial^\nu\right] = 0$, then (2.4) is clearly equal to zero, by identity. But what happens when the derivatives do not commute, i.e., when $\left[\partial^\mu, \partial^\nu\right] \neq 0$? This is an important question, because in the curved spacetime which is responsible for gravitation, derivatives do not commute. Indeed, the Riemann curvature tensor $R^\sigma{}_{\alpha\mu\nu}$, which may be *defined* via the relationship

$$\left[\partial_{;\mu}, \partial_{;\nu}\right] G_\alpha \equiv R^\sigma{}_{\alpha\mu\nu} G_\sigma \tag{2.5}$$

for any non-vanishing vector G^μ, is in fact a direct measure of the degree to which these derivatives are non-commuting. This can be explicitly expanded to show the Christoffel symbols via the expression $\partial_{;\mu} G^\nu = \partial_\mu G^\nu + \Gamma^\nu_{\mu\sigma} G^\sigma$ for the covariant (;) derivative of a vector field. But one of the important geometric identities satisfied by the Riemann tensor, is the first Bianchi identity $R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma} = 0$, with a cycling of indexes identical to that which obtains in the magnetic monopole field equation (2.2). So if we ask "what happens to magnetic monopoles in curved spacetime?" the answer is obtained by substituting (2.5) together with the first Bianchi identity into (2.4) to yield:

$$P^{\sigma\mu\nu} = \left[\partial^{;\sigma}, \partial^{;\mu}\right] G^\nu + \left[\partial^{;\mu}, \partial^{;\nu}\right] G^\sigma + \left[\partial^{;\nu}, \partial^{;\sigma}\right] G^\mu = \left(R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma}\right) G^\tau = 0 . \tag{2.6}$$

This is a very important result, because it tells us that the vanishing of magnetic monopoles in Maxwell's theory (and to be discussed later, the confinement of quarks in QCD, see Section 1 of [1]) is brought about not only via the trivial relationship $\left[\partial^\mu, \partial^\nu\right] = 0$ for the commuting of derivatives in flat spacetime, but even in curved spacetime, by the very nature of the spacetime geometry itself. That is, the non-existence of magnetic monopoles in Maxwell's electrodynamics is a direct consequence of spacetime geometry, such that $P^{\sigma\mu\nu} = 0$ is a geometrically-rooted relationship. In the language of "differential forms," the combined relationships (2.4), (2.6) for $P^{\sigma\mu\nu} = 0$ are expressed compactly as $P = dF = ddG = 0$, and are discussed in geometric terms by saying that "the exterior derivative of an exterior derivative is zero," $dd = 0$, see, e.g., [8] §4.6.

Differential forms also provide a very helpful way to take volume and surface integrals while easily applying Gauss' / Stokes theorem, which theorem we write generally for any differential form *X*, as $\iint dX = \oint X$. Specifically, to express in integral form the absence of magnetic monopole densities specified in (2.4), (2.6), one writes $P = dF = ddG = 0$ as: (wedge products $\wedge$ in $\frac{1}{2!} F^{\mu\nu} dx_\mu \wedge dx_\nu = F^{\mu\nu} dx_\mu dx_\nu$ are considered to already have been summed)

$$\iiint P = \iiint dF = \iiint ddG = \oiint F = \oiint F^{\mu\nu} dx_\mu dx_\nu = \oiint dG = 0 . \tag{2.7}$$

One may extract Maxwell's magnetic charge equation in integral form, $\oiint \vec{B} \cdot d\vec{A} = 0$, from the space-space *ij* bivector components of $\oiint F^{\mu\nu} dx_\mu dx_\nu = 0$. While magnetic fields may flow across some surfaces, there is never a *net* flux of a magnetic field through any *closed* two dimensional surface. Faraday's inductive law $\oint \vec{E} \cdot d\vec{l} = -\iint (\partial \vec{B} / \partial t) \cdot d\vec{A}$ is extracted from the time-space 0*k* bivector components. While magnetic fields are often referred to as dipole fields, it is probably better to think of them as *aterminal* fields, i.e., as fields for which the field lines never end at any terminal locale.

II.2 Electric Charge Densities and their Inversion

We now turn to the electric charge equation (2.1). Substituting the Abelian (2.3) into (2.1) and engaging in some well-known gymnastics with the spacetime indexes yields:

$$\begin{aligned}J^{\nu} &= \partial_{\mu}F^{\mu\nu} = \partial_{\mu}\left(\partial^{\mu}G^{\nu} - \partial^{\nu}G^{\mu}\right) = \partial_{\mu}\partial^{\mu}G^{\nu} - \partial_{\mu}\partial^{\nu}G^{\mu} = \partial_{\sigma}\partial^{\sigma}G^{\nu} - \partial^{\mu}\partial^{\nu}G_{\mu} \\ &= g^{\mu\nu}\partial_{\sigma}\partial^{\sigma}G_{\mu} - \partial^{\mu}\partial^{\nu}G_{\mu} = \left(g^{\mu\nu}\partial_{\sigma}\partial^{\sigma} - \partial^{\mu}\partial^{\nu}\right)G_{\mu}\end{aligned}. \quad (2.8)$$

First, we introduce a "Proca mass" m for the gauge field, by hand, in the usual way, using $\partial_{\sigma}\partial^{\sigma} \to \partial_{\sigma}\partial^{\sigma} + m^2$ to rewrite (2.8) as:

$$J^{\nu} = \left(g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - \partial^{\mu}\partial^{\nu}\right)G_{\mu}. \quad (2.9)$$

The Proca mass serves three purposes. First, in circumstances where one is *not* concerned with gauge symmetry and renormalizability and simply wants to know the effect of mass m on the field equation (2.8), this tells us what that effect will be. Second, for circumstances where one *is* concerned with preserving gauge symmetry, and wants to be able to generate masses from a Lagrangian with gauge symmetry via spontaneous symmetry breaking, the Proca mass m operates as a "red flag" to tell us which masses we want to be able to introduce not by hand, but by symmetry breaking. In other words, terms with Proca masses eventually need to be replaced with mass terms hidden in the gauge symmetry, in more complete theories. Third, the configuration space operator in (2.8) has no inverse, which requires gauge fixing, see, e.g., [9], chapter III.4, while (2.9) with the Proca mass is easily invertible as we shall now see.

What is of interest here about the form of (2.9) is that it allows us to readily derive the *inverse* relationship $G_{\nu} \equiv I_{\mu\nu}J^{\nu}$ where $I_{\mu\nu}$ is a second rank inverse tensor. Because J^{ν} in (2.9) is equal to the entire configuration space operator $g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - \partial^{\mu}\partial^{\nu}$ operating on G_{μ}, we can separately specify (define) the inverse *such that*:

$$I_{\nu\tau}\left[g^{\mu\tau}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - \partial^{\mu}\partial^{\tau}\right] \equiv \delta^{\mu}{}_{\nu}. \quad (2.10)$$

We further surmise given (2.10), that $I_{\nu\tau} \equiv Ag_{\nu\tau} + B\partial_{\nu}\partial_{\tau}$ will specify the general form of the inverse, with A and B being unknowns. Substituting into (2.10) we obtain:

$$\left[Ag_{\nu\tau} + B\partial_{\nu}\partial_{\tau}\right]\left[g^{\mu\tau}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - \partial^{\mu}\partial^{\tau}\right] \equiv \delta^{\mu}{}_{\nu}. \quad (2.11)$$

Now we need to solve (2.11) for A and B. The solution is well known, but because we will need to solve similar, more difficult inverses later, let us do the full exercise.

Usually, this calculation is done in momentum space, but here, we will stay in configuration space at the outset. First, we expand and apply $g_{\nu\tau}g^{\mu\tau} = \delta^{\mu}{}_{\nu}$ to obtain:

$$A\delta^{\mu}{}_{\nu}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - Ag_{\nu\tau}\partial^{\mu}\partial^{\tau} + B\partial_{\nu}\partial_{\tau}g^{\mu\tau}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - B\partial_{\nu}\partial_{\tau}\partial^{\mu}\partial^{\tau} = \delta^{\mu}{}_{\nu}. \quad (2.12)$$

We see that to eliminate $\delta^{\mu}{}_{\nu}$, we must set $A = 1/\left(\partial_\sigma \partial^\sigma + m^2\right)$. We do so. With some rearranging and raising or lowering indexes to absorb the remaining $g_{\mu\nu}$, we next obtain:

$$B\partial_\nu \partial^\mu \left(\partial_\sigma \partial^\sigma + m^2\right) - B\partial_\nu \partial_\tau \partial^\mu \partial^\tau = \partial^\mu \partial_\nu / \left(\partial_\sigma \partial^\sigma + m^2\right). \tag{2.13}$$

Next, we work in flat spacetime, $R^{\sigma}{}_{\alpha\mu\nu} = 0$, so that via (2.5), the derivatives now commute, $\left[\partial^\mu, \partial_\nu\right] = 0$. Commuting the derivatives at will and further reducing including factoring out the $\partial_\nu \partial^\mu$ finally yields $B = \left(1/m^2\right)/\left(\partial_\sigma \partial^\sigma + m^2\right)$. Finally, substituting A and B into $I_{\nu\tau} \equiv A g_{\nu\tau} + B\partial_\nu \partial_\tau$ allows us to obtain the inverse:

$$I_{\nu\tau} = \frac{g_{\nu\tau} + \dfrac{\partial_\nu \partial_\tau}{m^2}}{\partial_\sigma \partial^\sigma + m^2} = \frac{-g_{\nu\tau} + \dfrac{k_\nu k_\tau}{m^2}}{k_\sigma k^\sigma - m^2}. \tag{2.14}$$

In the final expression, we have used the substitution $\partial^\mu \to ik^\mu$ to finally convert over into the more-commonly used momentum space inverse.

Given this inverse, we may now use $G_\nu = I_{\mu\nu} J^\nu$ to write:

$$G_\mu = I_{\mu\nu} J^\nu = \frac{g_{\mu\nu} + \dfrac{\partial_\mu \partial_\nu}{m^2}}{\partial_\sigma \partial^\sigma + m^2} J^\nu = \frac{-g_{\mu\nu} + \dfrac{k_\mu k_\nu}{m^2}}{p_\sigma p^\sigma - m^2} J^\nu \to \frac{-g_{\mu\nu} + \dfrac{k_\mu k_\nu}{m^2}}{p_\sigma p^\sigma - m^2 + i\varepsilon} J^\nu. \tag{2.15}$$

In the final expression, we apply the $i\varepsilon$ prescription for the on-shell poles at $k_\sigma k^\sigma - m^2 = 0$. Setting m=0 in the above shows clearly why the massless configuration space operator in (2.8) has no inverse: $k_\mu k_\nu / m^2 \to \infty$ as $m \to 0$. However, noting that the conservation of the electric current density is expressed as $\partial_\nu J^\nu = 0$, or $k_\nu J^\nu = 0$ in momentum space ([9] at I.5(4)), the above reduces with some final index gymnastics to the very simplified:

$$G^\mu = \frac{1}{\partial_\sigma \partial^\sigma + m^2} J^\mu = -\frac{1}{k_\sigma k^\sigma - m^2} J^\mu \to -\frac{1}{k_\sigma k^\sigma - m^2 + i\varepsilon} J^\mu. \tag{2.16}$$

If we set the Proca mass $m = 0$ and also denoting $k_\sigma k^\sigma = q^2$, this becomes $G^\mu = -J^\mu / \left(q^2 + i\varepsilon\right)$.

Finally, in Dirac theory, the electric source J^ν density in turn may be expressed in terms of fermion wavefunctions ψ. The Dirac equation tells us that $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$. For the adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$ the field equation is $i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = 0$. Adding yields $\partial_\mu \left(\overline{\psi}\gamma^\mu \psi\right) = 0$

as is well known. And because the conserved current is expressed by $\partial_\mu J^\mu = 0$ as already employed above, we identify the current density with $J^\mu = \overline{\psi}\gamma^\mu\psi$. So if we want to express the gauge field G^ν in (2.16) in terms of the fermion (e.g., electron) sources themselves, we may further rewrite (2.16) as:

$$G^\mu = \frac{1}{\partial_\sigma\partial^\sigma + m^2}\overline{\psi}\gamma^\mu\psi = -\frac{\overline{\psi}\gamma^\mu\psi}{k_\sigma k^\sigma - m^2} \to -\frac{\overline{\psi}\gamma^\mu\psi}{k_\sigma k^\sigma - m^2 + i\varepsilon}. \tag{2.17}$$

This relationship – and analogues to this relationship – plays a central role in developing the thesis that Yang-Mills magnetic monopoles are baryons, because we use gauge fields G^μ expressed in inverse form in the manner of (2.17) to "populate" the Yang-Mills magnetic monopoles with quarks and thereby turn them into baryons.

Now let's briefly look at the integral form of the field equation (2.8) using the language of differential forms. Whereas the magnetic monopole equation is $P = dF = ddG = 0$, the electric charge equation is $*J = d*F = d*dA$, where J is a current density three-form. The duality * operator is employed on the fields, $*F^{\mu\nu} = \frac{1}{2!}\varepsilon^{\mu\nu\sigma\tau}F_{\sigma\tau}$, hence $*F = *F^{\mu\nu}dx_\mu dx_\nu$, and also on the sources $*J^{\mu\nu\sigma} = \varepsilon^{\tau\mu\nu\sigma\tau}J_\tau$, hence $*J = J^{\mu\nu\sigma}dx_\mu dx_\nu dx_\sigma$, where $\varepsilon^{\mu\nu\sigma\tau}$ is the antisymmetric Levi-Civita tensor (summing wedge products as in (2.7)). Thus, in integral form, Maxwell's electric charge equation reads (contrast (2.7)):

$$\iiint *J = \iiint d*F = \iiint d*dA = \oiint *F = \oiint *F^{\mu\nu}dx_\mu dx_\nu = \oiint *dA. \tag{2.18}$$

In integral form, Maxwell's electric charge equation $\oiint \vec{E}\cdot d\vec{A} = Q$ is obtained from the space-space bivector components, while Ampere's law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \iint_S \frac{\partial\vec{E}}{\partial t}\cdot d\vec{A}$ with Maxwell's displacement current which established the light-speed propagation of electromagnetic signals, is extracted from the time-space bivector components of the above. The non-vanishing charge Q in the charge equation is what does bring into being, *electric* monopoles in which there *is* a *net* electric flux crossing *closed* two-dimensional surfaces. In contrast to aterminal magnetic fields, the electric fields do terminate at the source charge, e.g., electron.

II.3 Quantum Electrodynamics (QED)

So far we have focused on classical electrodynamics. We now turn to QED. Starting with the Lagrangian density $\mathfrak{L}$ for a field ϕ with a source density J, one produces the quantum field theory for ϕ, J by obtaining the transition amplitude $W(J)$ from the path integral:

$$Z = \int D\phi \exp\left((i/\hbar)\int d^4x\mathfrak{L}(\phi)\right) = \int D\phi \exp\left((i/\hbar)S(\phi)\right) \equiv \mathcal{C}\exp\left(iW(J)\right). \tag{2.19}$$

In the limit where the action $S(\phi)=\int d^4x\mathfrak{L}(\phi)\gg\hbar$, one may evaluate the path integral with the stationary phase (or steepest descent) approximation so as to determine the extremum of $S(\phi)$. (It is understood that $\int d^4x\mathfrak{L}(\phi)$ is an integral from $-\infty$ to $+\infty$ over all four spacetime coordinates x^μ.) This leads to the Euler-Lagrange equation: (see, e.g., [9] at I.3(8)-(9))

$$\partial^\sigma\left(\frac{\partial\mathfrak{L}}{\partial(\partial^\sigma\phi)}\right)-\frac{\partial\mathfrak{L}}{\partial\phi}=0. \tag{2.20}$$

So, if we know the classical equation for a field ϕ and wish to obtain its quantum counterpart, we first find the $\mathfrak{L}$ needed to make (2.20) to reproduce the classical field equation. For example, we find that Maxwell's classical field equation $J^\nu=\partial_\mu F^{\mu\nu}$ of (2.1) for the electric charge density J^μ may be reproduced via (2.20) by the Lagrangian density:

$$\mathfrak{L}_J=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-J^\mu G_\mu. \tag{2.21}$$

So the "only" thing we need to do is use (2.21) in (2.19) to obtain $W(J)$. But this is not always an "easy" thing to do, and for some theories, such as Yang-Mills, it has to date proven impossible to obtain an exact, strictly analytical evaluation of $W(J)$ in (2.19). That is the reason for such things as perturbation and lattice gauge theory, see, e.g., section VII.1 of [9]. But for QED an analytical calculation is possible, and as a template for later more difficult calculations involving Yang-Mills theory, we shall review this (known) calculation in detail.

In general, one solves (2.19) to deduce $W(J)$ from a given Lagrangian density $\mathfrak{L}(\varphi)$, using what Zee [9] in Appendix A refers to as the "central identity of quantum field theory" (we have reversed the sign for J because we are using the electrodynamic convention in which the units of charge (electrons) are negative whereas Zee uses a positive charge sign convention):

$$\int D\phi\exp\left(-\tfrac{1}{2}\phi\cdot K\cdot\phi-V(\phi)-J\cdot\phi\right)=\mathcal{C}\exp\left(V(\delta/\delta J)\right)\exp\left(\tfrac{1}{2}J\cdot K^{-1}\cdot J\right), \tag{2.22}$$

with the quadratic terms in ϕ in (2.19), (2.22) converted over to $W(J)$ via the Gaussian integral:

$$\int dx\exp\left(-\tfrac{1}{2}Ax^2-Jx\right)=\left(-2\pi/A\right)^{.5}\exp\left(J^2/2A\right). \tag{2.23}$$

Basically, one starts with (2.22), takes the Lagrangian density $\mathfrak{L}(\varphi)$ of the theory under consideration, applies whatever tricks or resourcefulness one can muster to put at least part of the Lagrangian in the general quadratic form $-\tfrac{1}{2}\phi\cdot K\cdot\phi+J\cdot\phi$ from (2.22) which then maps over to $-\tfrac{1}{2}Ax^2+Jx$ in (2.23), and takes all the remaining terms and puts them into $V(\varphi)$. For QED, as we shall see, $V(\phi)=0$ because this is a linear theory, which makes things comparatively simple.

First, we use the compact form $F^{\mu\nu}=\partial^{[\mu}G^{\nu]}$ of the Abelian field strength (2.3) to write (2.21) as:

$$\mathfrak{L}_J=-\tfrac{1}{4}\partial^{[\mu}G^{\nu]}\partial_{[\mu}G_{\nu]}-J^{\mu}G_{\mu}\,. \tag{2.24}$$

This means that the action for $\phi=A_\mu$ is specified by:

$$S\left(G_\mu\right)=\int d^4x\mathfrak{L}_J\left(G_\mu\right)=\int d^4x\left(-\tfrac{1}{4}\partial^{[\mu}G^{\nu]}\partial_{[\mu}G_{\nu]}-J^{\mu}G_{\mu}\right). \tag{2.25}$$

This does already contain the term $J\cdot\phi\rightarrow J^{\mu}A_{\mu}$ of (2.22), but does not yet contain the term $-\tfrac{1}{2}\phi\cdot K\cdot\phi$. The "resourcefulness" that we apply to obtain $-\tfrac{1}{2}\phi\cdot K\cdot\phi$, is to integrate by parts.

First, we use the product rule written as $\partial^{\mu}(ab)=\partial^{\mu}ab+a\left(\partial^{\mu}b\right)$ with $a=G^{\nu}$, $b=\partial_{[\mu}G_{\nu]}$ to obtain $\partial^{\mu}\left(G^{\nu}\partial_{[\mu}G_{\nu]}\right)=\partial^{\mu}G^{\nu}\partial_{[\mu}G_{\nu]}+G^{\nu}\partial^{\mu}\partial_{[\mu}G_{\nu]}$. It is simple to then construct the antisymmetric expression from the upper (contravariant) indexes:

$$\partial^{[\mu}\left(G^{\nu]}\partial_{[\mu}G_{\nu]}\right)=\partial^{[\mu}G^{\nu]}\partial_{[\mu}G_{\nu]}+G^{[\nu}\partial^{\mu]}\partial_{[\mu}G_{\nu]}\,. \tag{2.26}$$

We isolate $\partial^{[\mu}G^{\nu]}\partial_{[\mu}G_{\nu]}$ above and use it in (2.25) to write:

$$\begin{aligned}S\left(G_\mu\right)&=\int d^4x\mathfrak{L}\left(G_\mu\right)=\int d^4x\left(-\tfrac{1}{4}\partial^{[\mu}G^{\nu]}\partial_{[\mu}G_{\nu]}-J^{\mu}G_{\mu}\right)\\&=\int d^4x\left(-\tfrac{1}{2}\partial^{\mu}\left(G^{\nu}\partial_{[\mu}G_{\nu]}\right)+\tfrac{1}{4}G^{[\nu}\partial^{\mu]}\partial_{[\mu}G_{\nu]}-J^{\mu}G_{\mu}\right)\\&=\int d^4x\left(-\tfrac{1}{2}\partial^{\mu}\left(G^{\nu}\partial_{[\mu}G_{\nu]}\right)+\tfrac{1}{2}G^{\nu}\partial^{\mu}\partial_{\mu}G_{\nu}-\tfrac{1}{2}G^{\mu}\partial^{\nu}\partial_{\mu}G_{\nu}-J^{\mu}G_{\mu}\right)\\&=\int d^4x\left(-\tfrac{1}{2}\partial^{\mu}\left(G^{\nu}\partial_{[\mu}G_{\nu]}\right)+\tfrac{1}{2}G_{\mu}\left(g^{\mu\nu}\partial_{\sigma}\partial^{\sigma}-\partial^{\mu}\partial^{\nu}\right)G_{\nu}-J^{\mu}G_{\mu}\right)\end{aligned}. \tag{2.27}$$

Above, the final line is arrived at from the third line by simple index gymnastics, and we do one flat spacetime derivative commutation $\left[\partial^{\mu},\partial^{\nu}\right]=0$ by assuming $R^{\sigma}{}_{\alpha\mu\nu}=0$, again see (2.5). As in (2.9), we introduce a Proca mass "by hand" by setting $\partial_{\sigma}\partial^{\sigma}\rightarrow\partial_{\sigma}\partial^{\sigma}+m^2$, so this becomes:

$$S\left(G_\mu\right)=\int d^4x\left(-\tfrac{1}{2}\partial^{\mu}\left(G^{\nu}\partial_{[\mu}G_{\nu]}\right)+\tfrac{1}{2}G_{\mu}\left(g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma}+m^2\right)-\partial^{\mu}\partial^{\nu}\right)G_{\nu}-J^{\mu}A_{\mu}\right). \tag{2.28}$$

We see that $\tfrac{1}{2}G_{\mu}\left(g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma}+m^2\right)-\partial^{\mu}\partial^{\nu}\right)G_{\nu}$ is now a term of the form $-\tfrac{1}{2}\phi\cdot K\cdot\phi$ that fits (2.22), with $-K=g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma}+m^2\right)-\partial^{\mu}\partial^{\nu}$ and $\phi=G$. The term $-\tfrac{1}{4}\int d^4x\partial^{[\mu}\left(G^{\nu]}\partial_{[\mu}G_{\nu]}\right)$ which remains can be removed by imposing $G^{\nu}\left(x^{\mu}=\infty\right)=G^{\nu}\left(x^{\mu}=-\infty\right)=0$ as boundary conditions upon the gauge potential, so that $G^{\nu}\Big|_{A^{\nu}\left(x^{\mu}=-\infty\right)}^{A^{\nu}\left(x^{\mu}=+\infty\right)}=0$ for each of the coordinates $x^{\mu}=\left(t,x,y,z\right)$. Thus, with $d^4x=dx^0dx^1dx^2dx^3$, we may calculate that:

$$\begin{aligned}
&\int d^4x\partial^{\mu}(G^{\nu}\partial_{[\mu}G_{\nu]})=\int d^4xg^{\mu\sigma}\frac{\partial}{\partial x^{\sigma}}(G^{\nu}\partial_{[\mu}G_{\nu]}) \\
&\quad=\int dx^0dx^1dx^2dx^3g^{\mu0}\frac{\partial}{\partial x^0}(G^{\nu}\partial_{[\mu}G_{\nu]})+\int dx^0dx^1dx^2dx^3g^{\mu1}\frac{\partial}{\partial x^1}(G^{\nu}\partial_{[\mu}G_{\nu]}) \\
&\quad+\int dx^0dx^1dx^2dx^3g^{\mu2}\frac{\partial}{\partial x^2}(G^{\nu}\partial_{[\mu}G_{\nu]})+\int dx^0dx^1dx^2dx^3g^{\mu3}\frac{\partial}{\partial x^3}(G^{\nu}\partial_{[\mu}G_{\nu]}) \\
&\quad=\int dx^1dx^2dx^3g^{\mu0}\partial_{[\mu}G_{\nu]}\left(G^{\nu}\Big|_{G^{\nu}(t=-\infty)}^{G^{\nu}(t=+\infty)}\right)+\int dx^0dx^2dx^3g^{\mu1}\partial_{[\mu}G_{\nu]}\left(G^{\nu}\Big|_{G^{\nu}(x=-\infty)}^{G^{\nu}(x=+\infty)}\right) \\
&\quad+\int dx^0dx^1dx^3g^{\mu2}\partial_{[\mu}G_{\nu]}\left(G^{\nu}\Big|_{G^{\nu}(y=-\infty)}^{G^{\nu}(y=+\infty)}\right)+\int dx^0dx^1dx^2g^{\mu3}\partial_{[\mu}G_{\nu]}\left(G^{\nu}\Big|_{G^{\nu}(z=-\infty)}^{G^{\nu}(z=+\infty)}\right) \\
&\quad=0
\end{aligned}, \tag{2.29}$$

Now, with (2.29), (2.28) reduces to:

$$S\left(G_{\mu}\right)=\int d^4x\left(\tfrac{1}{2}G_{\mu}\left(g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma}+m^2\right)-\partial^{\mu}\partial^{\nu}\right)G_{\nu}-J^{\mu}G_{\mu}\right). \tag{2.30}$$

This is finally quadratic in G_{ν}. So we place this into path integral (2.19) with $\hbar=1$ as:

$$Z=\int DG_{\mu}\exp i\int d^4x\left(\tfrac{1}{2}G_{\mu}\left(g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma}+m^2\right)-\partial^{\mu}\partial^{\nu}\right)G_{\nu}-J^{\mu}G_{\mu}\right)\equiv\mathcal{C}\exp\left(iW\left(J\right)\right). \tag{2.31}$$

Comparing (2.22) with $V=0$ while

In (2.22), we set $V=0$ and scale $K\to -iK$ hence $K^{-1}\to iK^{-1}$, and $J\to iJ$, to write:

$$\int D\phi\exp i\left(\tfrac{1}{2}\phi\cdot K\cdot\phi-J\cdot\phi\right)=\mathcal{C}\exp-i\left(\tfrac{1}{2}J\cdot K^{-1}\cdot J\right) \tag{2.32}$$

Comparing (2.31) we see correspondences $J\to J^{\mu}$, $\phi\to G^{\mu}$, $K\to g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma}+m^2\right)-\partial^{\mu}\partial^{\nu}$. Additionally, $K\to g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma}+m^2\right)-\partial^{\mu}\partial^{\nu}$ is *identical* in form to the configuration space operator that appeared in (2.9) and that we inverted in (2.10) through (2.14). Thus, from (2.14), we already know that:

$$K^{-1}\to I_{\nu\tau}=\frac{g_{\nu\tau}+\frac{\partial_{\nu}\partial_{\tau}}{m^2}}{\partial_{\sigma}\partial^{\sigma}+m^2}=\frac{-g_{\nu\tau}+\frac{k_{\nu}k_{\tau}}{m^2}}{k_{\sigma}k^{\sigma}-m^2}, \tag{2.33}$$

Therefore, using all of the foregoing correspondences in (2.32), and making certain all the spacetime indexes are properly represented and also including the conjugate source density $J^{\mu *}(k)$, we obtain:

$$\int DG \exp i\left(\tfrac{1}{2}G_\mu\left(g^{\mu\nu}\left(\partial_\sigma\partial^\sigma+m^2\right)-\partial^\mu\partial^\nu\right)G_\nu - J^\mu G_\mu\right)=\mathcal{C}\exp\left(-\frac{1}{2}iJ^{\mu *}(k)\frac{-g_{\mu\nu}+\frac{k_\mu k_\nu}{m^2}}{k_\sigma k^\sigma-m^2}J^\nu(k)\right),\quad(2.34)$$

Comparing (2.34) to (2.31) and migrating $\int d^4x \to \int d^4k/(2\pi)^4$ from (2.31) as a result of the Fourier transform into momentum space, allows us to pick off the amplitude by setting:

$$\begin{aligned}Z&=\int D\phi \exp i\int d^4x\left(\tfrac{1}{2}G_\mu\left(g^{\mu\nu}\left(\partial_\sigma\partial^\sigma+m^2\right)-\partial^\mu\partial^\nu\right)G_\nu - J^\mu G_\mu\right)\\ &\equiv \mathcal{C}\exp\left(iW(J)\right)=\mathcal{C}\exp\left(-\frac{1}{2}i\int\frac{d^4k}{(2\pi)^4}J^\mu\frac{-g_{\mu\nu}+\frac{k_\mu k_\nu}{m^2}}{k_\sigma k^\sigma-m^2}J^\nu\right)\end{aligned}\quad .\quad(2.35)$$

which means that:

$$W(J)=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}J^{\mu *}(k)\frac{g_{\mu\nu}-\frac{k_\mu k_\nu}{m^2}}{k_\sigma k^\sigma-m^2+i\varepsilon}J^\nu(k)=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}J^{\mu *}(k)\frac{1}{k_\sigma k^\sigma-m^2+i\varepsilon}J_\mu(k).\quad(2.36)$$

In the final expression, we again employ $k_\nu J^\nu=0$ as in (2.16) to impose the conservation of the electric source charge density, and have also added the $+i\varepsilon$ term. This should be compared to I.5(5) in [9], to which it is identical. This expression of course, tells us that like electric charges repel.

In (2.36), the term

$$\pi_{\mu\nu}=iI_{\nu\tau}=i\frac{-g_{\mu\nu}+\frac{k_\mu k_\nu}{m^2}}{k_\sigma k^\sigma-m^2+i\varepsilon}\quad(2.37)$$

is the propagator for vector bosons exchanged between the two source currents. In a linear theory such as QED in which the path integral can be solved exactly using an action such as (2.30) which is quadratic in the fields with no higher-order terms, e.g., ϕ^3, ϕ^4, etc., we find comparing (2.15) that the inverse is the same as the propagator, $\pi_{\mu\nu}=iI_{\nu\tau}$, up to a constant factor. This is not in general true, however, for a theory such as Yang-Mills theory which contains higher-order field terms, as we shall see in sections IV and V.

II.4 Does the Concurrence between Configuration Space Operators for the Field Equation and the Path Integral Continue to Apply in Yang-Mills Theory?

Momentarily, we will begin to examine Yang-Mills theory, which does contain higher order gauge field terms, specifically, G^3, G^4. The development in this section will serve as a foundation for the more complicated considerations which arise once these non-linearities are introduced. But the single most important point to be kept in mind from all of the development in this section is the following:

We do not expect that the Yang-Mills inverses will be the same as the Yang-Mills propagators up to a constant factor, that is, we do not expect that $\pi_{\mu\nu} \neq iI_{\nu\tau}$ in Yang-Mills theory. But, there is another very important concurrence that we have seen which may apply, and as we shall establish, which does apply to Yang-Mills theory just as it does to electrodynamics. That concurrence is established by comparing $J^{\nu} = \left(g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - \partial^{\mu}\partial^{\nu}\right)G_{\mu}$, which is the field equation (2.9), to $S\left(A_{\mu}\right) = \int d^4x\left(\tfrac{1}{2}G_{\mu}\left(g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - \partial^{\mu}\partial^{\nu}\right)G_{\nu} - J^{\mu}G_{\mu}\right)$, which is the electrodynamic action (2.30) arrived at following integration-by-parts that is fed into the path integral. It is clear that in both (2.9) and (2.30), one comes across $g^{\mu\nu}\left(\partial_{\sigma}\partial^{\sigma} + m^2\right) - \partial^{\mu}\partial^{\nu}$, which is the *exact same configuration space operator*.

Given this observation, we now raise the following question: In Yang-Mills theory, are we to also expect that the configuration space operator in the classical field equation for chromo-electric charges will be equal to the configuration space operator in the action that is fed into the path integral to quantize Yang-Mills theory? This is not a trivial question, and in fact, in the process of answering this question, we are led into a number of potentially-fruitful directions. In the discussion to follow, we shall establish that the answer to the question is: *yes, these configuration space operators <u>are</u> the same, even in Yang-Mills*. This, in turn, will reveal that for the chromo-electric charge equation which is the analog to (2.1), the impact of Yang-Mills theory is to add a perturbative term of the form $-V = \left(\partial^{\sigma}G_{\sigma} + G_{\sigma}\partial^{\sigma}\right) + G^{\sigma}G_{\sigma}$ which when set to zero allows us in turn to use a simple inverse of the form (2.17) to populate the Yang-Mills magnetic monopoles with quark wavefunctions, and thereby turn these magnetic monopoles into baryons. Subsequent to spontaneous symmetry breaking to ensure topological stability, see section 6 through 8 of [1], these baryons become identified with protons and neutrons which in this zero-perturbation approximation yield – not the entire proton and neutron masses themselves – but rather, extremely accurate retrodictions for the nuclear *binding energies* developer by the author in [2], [4], [5], [6]. In this light, the nuclear binding energies themselves are sending us very clear signals about what is going on inside of the protons and neutrons, when all of the perturbative and vacuum effects are neglected. Put in yet another way: while perturbations and the vacuum do affect the overall neutron and proton masses and the dynamics *inside* the proton and neutron, they have absolutely no effect (at least to parts in 10^5 AMU or higher) on the binding energies, and on the *difference* between the neutron and proton masses, all of which are strictly *external* manifestations of the physics of protons and neutrons.

III. Classical Yang-Mills Theory, and Why its Magnetic Monopoles Look Very Much Like Credible Baryon Candidates

III.1 The Profound Importance of Non-Commuting Objects in Physics

In many ways, many of the key developments which occurred in 20th century physics can be summarized by the simple fact that many physical objects which in the 19th century were assumed to be commuting objects came to be recognized as *non-commuting*. Consider some examples: In the 19th century, one assumed that position commuted with momentum, i.e., that $[x, p_x] = 0$. Then, in the 20th century, Heisenberg taught that in fact they do not commute, i.e., that $[x, p_x] = i\hbar$, and that the very same angular momentum constant $\hbar$ discovered by Planck in 1901 was the measure of the degree to which $[x, p_x] \neq 0$. The consequences of this non-commutativity are profound, and extend to the uncertainty principle which represents a Fourier transform between position and momentum in which the minimal uncertainty spread applies to Gaussian distributions. As another example, although Riemann in the 1860s had already considered parallel transport for pure geometry, Einstein applied this to the physics of gravitation, which in many ways can be summarized by (2.5) already used several times here, namely, $\left[\partial_{;\mu}, \partial_{;\nu}\right] G_\alpha \equiv R^\sigma{}_{\alpha\mu\nu} G_\sigma$: In curved spacetime, derivatives do not commute, and the Riemann tensor is the measure of the degree to which they do not commute. As another example, we learned in the 20th century to use commutators to understand which quantities are conserved and observable, and which are not. Thus, for example, $[L, H] \neq 0$ and $[S, H] \neq 0$ state that orbital angular momentum *L* and intrinsic spin *S* do not separately commute with the Hamiltonian *H*, and so are not conserved or observable. But, we learned, their sum $[(L+S), H] = 0$ *does* commute with the Hamiltonian, which also informs us that the total of spin plus orbital angular momentum is a conserved observable. As a final example, Dirac, in the process of trying to develop a non-trivial linear expression for the spacetime metric interval $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ found cause to develop the gamma-matrix operators γ^μ, which also do not commute, $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$. This bilinear operator is central to understanding the polarization and magnetization of fermions with spin ½.

While it took Planck's using a quantized energy relationship $E = nhf$ in 1901 to fit together the separate curves of Wien and Rayleigh-Jeans followed by Einstein's 1905 explanation of the photoelectric effect in terms of light quanta leading eventually to Heisenberg's 1925-1927 formulation of the commutator relation $[x, p_x] = i\hbar$ and uncertainty to kick the quantum revolution into high gear, the mathematical foundations of much of 20th century physics had already been presciently laid by William Rowan Hamilton in 1843. It was then, almost 70 years prior to Planck's discovery, that Hamilton used his penknife to carve the quaternion relationship $i^2 = j^2 = k^2 = ijk = -1$ into the Brougham Bridge in Dublin Ireland. Quaternions were designed to extend into the three space dimensions of the observed physical universe, the imaginary number $i^2 = -1$ which had gained acceptance through the work of Euler and Gauss. These quaternions are *non-commuting* numbers, and they were specifically designed to compactly summarize the effects of rotations in three space dimensions, and the fact that rotations do not commute. The modern representation of these quaternions is embodied in the 2x2 Pauli spin matrices $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = I$, which are Hermitian, which have the commutation relationship $\left[\sigma_i, \sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$, and which were also developed circa 1925. So, these $\left[\sigma_i, \sigma_j\right] \neq 0$ are clearly non-commuting. These σ_i matrices also specify the Yang-Mills gauge group SU(2), which is the simplest non-Abelian group. By the way, if one wishes to take some of the mystery or consternation out of axial and left-right chiral relationships involving γ^5, it is useful to think of Dirac's $i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = 1$ as simply a generalization of Hamilton's $ijk = -1$ quaternion relationship into spacetime physics.

Yang-Mills gauge theories, developed in 1954, rest mathematically upon a generalization of what Hamilton first conceived in Dublin in 1843, and what Pauli developed in 2x2 matrix form in 1925, into SU(N) matrices of any NxN dimensionality. Normalized such that $Tr\left(\lambda^i\lambda^j\right) = \frac{1}{2}\delta^{ij}$, the N^2-1 generators $\lambda^i; i = 1,2,3...N^2-1$ of any Yang-Mills gauge group SU(N) maintain the commutator relationship $\left[\lambda_i, \lambda_j\right] = if_{ijk}\lambda_k$, where f_{ijk} are the group structure relationships. This generalizes the Pauli relationship which becomes $\left[\sigma_i, \sigma_j\right] = i\varepsilon_{ijk}\sigma_k$ once we normalize to $Tr\left(\sigma^i\sigma^j\right) = \frac{1}{2}\delta^{ij}$. Each generator is an NxN matrix and so can be written $\lambda^i{}_{AB}; A, B = 1,2,3...N$, but in general it is simpler to suppress these A, B indexes and simply keep in mind at all times that these indexes are implicitly there.

Physically, an SU(N) gauge theory extending Maxwell's electrodynamics into non-Abelian domains is developed from these generators rooted in the Hamiltonian quaternions in the following way: first, one posits a set of N^2-1 vector potentials (gauge fields) $G^{i\mu}$; $i = 1,2,3...N^2-1$. Next, one sums these with the generators to form $G^\mu{}_{AB} \equiv \lambda^i{}_{AB}G^{i\mu}$ which with A, B indexes implicit is normally written as $G^\mu \equiv \lambda^i G^{i\mu}$. This is an NxN matrix of spacetime 4-vectors. Similarly, one forms a set of N^2-1 field strength tensors $F^{i\mu\nu}$, each of which is a bivector with a "chromo-electric" field $\mathbf{E_i}$ and a chromo-magnetic field $\mathbf{B_i}$. We then use these to form $F^{\mu\nu}_{AB} \equiv \lambda^i_{AB}F^{i\mu\nu}$ which is an NxN Yang-Mills matrix of 4x4 antisymmetric second rank tensors (bivectors). Finally, *in very important contrast to (2.3),* we specify the NxN field strength matrix $F^{\mu\nu}$ in terms of the NxN gauge field matrix G^μ as:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i\left[G^\mu, G^\nu\right] = \partial^{[\mu}G^{\nu]} - i\left[G^\mu, G^\nu\right] \tag{3.1}$$

Once again, we see a commutator, this time $\left[G^\mu, G^\nu\right] \neq 0$, which we take to be non-vanishing. *Everything that differentiates Yang-Mills gauge theory from an Abelian gauge theory such as QED, originates solely and exclusively from the fact that these gauge field / vector potential matrices do not commute, i.e., from the fact that* $\left[G^\mu, G^\nu\right] \neq 0$. Yang-Mills theories, simply put, are gauge theories in which the vector potentials $G^\mu \equiv \lambda^i G^{i\mu}$ do not self-commute $\left[G^\mu, G^\nu\right] \neq 0$.

During the 20th century, when non-commuting objects became profoundly important to the development of physics from canonical commutation to gravitation to Dirac theory to conservation laws and observables, the non-commuting $\left[G^{\mu}, G^{\nu}\right] \neq 0$ of Yang-Mills theory deserve a prominent place of importance which is still evolving and will continue to evolve until such time as Yang-Mills theory is fully understood in all aspects. One of those aspects, is that baryons themselves are the magnetic monopoles of Yang-Mills gauge theory, which means that Yang-Mills theory underlies any viable theory of nuclear matter and hence, of the materiality of the material universe.

III.2 Non-Linear, Non-Commuting Gauge Fields, and Gauge Theory on Steroids

There are several different, fully equivalent ways in which one can think about Yang-Mills gauge theories, and depending on circumstance, the way that one chooses can make a big difference in whether a calculation or conceptualization is reasonably clean and simple, or messy and obtuse. *The first way to think about Yang-Mills is that of (3.1), as a theory in which the gauge fields do not commute.* The word "Abelian" is a synonym for "commuting," and so as a non-Abelian gauge theory, Yang-Mills theories are simply theories of non-commuting gauge field. As we shall review momentarily, this leads very directly to non-vanishing magnetic monopole source charges.

But first it is worth being reminded how to expand out (3.1) using $F^{\mu\nu} = \lambda^i F^{i\mu\nu}$, $G^{\mu} = \lambda^i G^{i\mu}$ and $\left[\lambda_i, \lambda_j\right] = i f_{ijk} \lambda_k$. We find while renaming summed indexes as needed that:

$$\begin{aligned} \lambda^i F^{i\mu\nu} &= \partial^{\mu} \lambda^i G^{i\nu} - \partial^{\nu} \lambda^i G^{i\mu} - i\left[\lambda^i G^{i\mu}, \lambda^j G^{j\nu}\right] \\ &= \lambda^i \partial^{\mu} G^{i\nu} - \lambda^i \partial^{\nu} G^{i\mu} - i\left[\lambda^i, \lambda^j\right] G^{i\mu} G^{j\nu} \\ &= \lambda^i \partial^{\mu} G^{i\nu} - \lambda^i \partial^{\nu} G^{i\mu} + f^{kji} \lambda^i G^{k\mu} G^{j\nu} \end{aligned} \tag{3.2}$$

The λ^i is then factored out from all terms, leaving, after more renaming, the perhaps more-familiar expression:

$$F^{i\mu\nu} = \partial^{\mu} G^{i\nu} - \partial^{\nu} G^{i\mu} + f^{ijk} G^{j\mu} G^{k\nu} = \partial^{[\mu} G^{i\nu]} + f^{ijk} G^{j\mu} G^{k\nu} \tag{3.3}$$

If we now use (3.3) to form a Lagrangian density akin to the pure field terms in (2.21), we obtain the also familiar:

$$\begin{aligned} \mathfrak{L} = -\tfrac{1}{4} F^{i\mu\nu} F_{i\mu\nu} &= -\tfrac{1}{4}\left(\partial^{[\mu} G^{i\nu]} + f^{ijk} G^{j\mu} G^{k\nu}\right)\left(\partial_{[\mu} G_{i\nu]} + f_{ilm} G_{l\mu} G_{m\nu}\right) \\ &= -\tfrac{1}{4} \partial^{[\mu} G^{i\nu]} \partial_{[\mu} G_{i\nu]} - \tfrac{1}{2} f_{ijk} \partial^{[\mu} G^{i\nu]} G_{j\mu} G_{k\nu} - \tfrac{1}{4} f^{ijk} f_{ilm} G^{j\mu} G^{k\nu} G_{l\mu} G_{m\nu} \end{aligned} \tag{3.4}$$

The first term, $-\tfrac{1}{4} \partial^{[\mu} G^{i\nu]} \partial_{[\mu} G_{i\nu]}$, a "harmonic oscillator" term, is quadratic in the gauge fields, and is fully analogous and indeed identical in form to the QED term $-\tfrac{1}{4} \partial^{[\mu} G^{\nu]} \partial_{[\mu} G_{\nu]}$ in (2.24) prior to the integration by parts. But the remaining terms $-\frac{1}{2} f_{ijk}\partial^{[\mu}G^{i\nu]}G_{j\mu}G_{k\nu}$ and $-\frac{1}{4} f^{ijk} f_{ilm}G^{j\mu}G^{k\nu}G_{l\mu}G_{m\nu}$, the "perturbation" terms, represent vertices with three and four interacting gauge fields. This is unprecedented in QED, and makes Yang-Mills a *non-linear* theory. *So the second way to think about Yang-Mills theory is that of (3.4), in which the gauge fields do* <u>*not*</u> *act like photons and forego interactions one another like ships passing in the night. Rather, the Yang-Mills gauge fields fully interact with one another as well as with their fermion (current) sources.* Unfortunately, doing exact calculations with (3.4) is difficult, and in general we will find it unhelpful to split (3.4) into harmonic and perturbative parts as is done in perturbative gauge theory, or to spoil the Lorentz invariance as in lattice gauge theory. Another approach is needed.

A third way to think about Yang-Mills gauge theory is to expand the commutator in (3.1) and then reconsolidate using gauge covariant derivatives $D^\mu \equiv \partial^\mu - iG^\mu$, as such:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - iG^\mu G^\nu + iG^\nu G^\mu = \left(\partial^\mu - iG^\mu\right)G^\nu - \left(\partial^\nu - iG^\nu\right)G^\mu = D^\mu G^\nu - D^\nu G^\mu = D^{[\mu}G^{\nu]} \quad (3.5)$$

We compare $F^{\mu\nu} = D^{[\mu}G^{\nu]}$ above to the Abelian field strength $F^{\mu\nu} = \partial^{[\mu}G^{\nu]}$ and see that the *only* difference is that the ordinary derivative is replaced by $\partial^\mu \to D^\mu = \partial^\mu - iG^\mu$. This is actually a very pedagogically-useful observation: Consider that gauge theory first originates when one has a field equation or a Lagrangian for a scalar ϕ or fermion ψ field which includes a term $\partial_\mu\phi$ or $\partial_\mu\psi$. One then subjects the field to the *local* gauge (phase) transformation $\phi \to e^{i\,a(x)}\phi$ or $\psi \to e^{i\,a(x)}\psi$ and insists that the field equation or Lagrangian remain invariant. What does one do to ensure that invariance? Make the replacement $\partial^\mu \to D^\mu = \partial^\mu - iG^\mu$. So now, one changes $\partial_\mu\phi \to D_\mu\phi$ and $\partial_\mu\psi \to D_\mu\psi$ with the consequence that ϕ or ψ now acquires an interaction with the gauge field G^μ.

So if we start with an Abelian gauge theory such as QED for which $F^{\mu\nu} = \partial^{[\mu}G^{\nu]}$, we can easily turn it into a non-Abelian gauge theory by replacing $\partial^\mu \to D^\mu = \partial^\mu - iG^\mu$ so that $F^{\mu\nu} = D^{[\mu}G^{\nu]}$. As a consequence, the gauge field G^ν acquires an interaction with the gauge field G^μ, i.e., the gauge field now starts to interact non-linearly with itself! This says the same thing as (3.4), with the exception that in the form of (3.5), the pure gauge term in the Lagrangian is the much cleaner (the ½ rather than ¼ owes to the $Tr\left(\lambda^i\lambda^j\right) = \frac{1}{2}\delta^{ij}$ normalization):

$$\mathfrak{L} = -\frac{1}{2}\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\mathrm{Tr}D^{[\mu}G^{\nu]}D_{[\mu}G_{\nu]}\,. \quad (3.6)$$

Given that (3.4) and (3.6) state *exactly the same physics*, it should be clear that (3.6) is a much easier expression to work with than (3.4). *This is a third way to think about Yang-Mills theories: A non-Abelian gauge theory is simply an Abelian gauge theory for which gauge theory has been applied to gauge theory. Or, perhaps with a bit more color (pun intended), Yang-Mills gauge theory is* <u>*gauge theory on steroids*</u>. As we shall soon see, the question posed at the end of section II, whether in Yang-Mills theory we to should expect the configuration space operator in the field equation to be equal to the configuration space operator in the action that is fed into the path integral, boils down to a question of just how steroidal Yang-Mills theory really is.

We shall find that when it comes to the $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ of (2.2) for magnetic monopole sources, it is most helpful to view Yang-Mills theory in the form of (3.1), as a theory on which the gauge field does not self-commute, that is, to think about the "non-Abelian" view of Yang-Mills theory. But, when it comes to the $J^\nu = \partial_\mu F^{\mu\nu}$ of (2.1) for electric charge sources, the more convenient view is that of (3.6), in which we view Yang-Mills as gauge theory on steroids.

III.3 Magnetic Monopole Sources in Yang-Mills Gauge Theory

With the foregoing, let's get right down to business and use the "non-commuting" field strength of (3.1) in (2.2). With the help once again of $\left[\partial_{;\mu}, \partial_{;\nu}\right] G_\alpha \equiv R^\sigma{}_{\alpha\mu\nu} G_\sigma$ from (2.5), see also (2.6), together with the first Bianchi identity $R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma} = 0$, we obtain:

$$\begin{aligned}
P^{\sigma\mu\nu} &= \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} \\
&= \partial^\sigma \left(\partial^{[\mu} G^{\nu]} - i\left[G^\mu, G^\nu \right] \right) + \partial^\mu \left(\partial^{[\nu} G^{\sigma]} - i\left[G^\nu, G^\sigma \right] \right) + \partial^\nu \left(\partial^{[\sigma} G^{\mu]} - i\left[G^\sigma, G^\mu \right] \right) \\
&= \left[\partial^{;\sigma}, \partial^{;\mu} \right] G^\nu + \left[\partial^{;\mu}, \partial^{;\nu} \right] G^\sigma + \left[\partial^{;\nu}, \partial^{;\sigma} \right] G^\mu - i\partial^\sigma \left[G^\mu, G^\nu \right] - i\partial^\mu \left[G^\nu, G^\sigma \right] - i\partial^\nu \left[G^\sigma, G^\mu \right] \\
&= \left(R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma} \right) G^\tau - i\partial^\sigma \left[G^\mu, G^\nu \right] - i\partial^\mu \left[G^\nu, G^\sigma \right] - i\partial^\nu \left[G^\sigma, G^\mu \right] \\
&= 0 - i\left(\partial^\sigma \left[G^\mu, G^\nu \right] + \partial^\mu \left[G^\nu, G^\sigma \right] + \partial^\nu \left[G^\sigma, G^\mu \right] \right)
\end{aligned} \quad .(3.7)$$

Here, the term $\left(R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma} \right) G^\tau$ once again vanishes as in QED with the able assistance of the spacetime geometry itself. As developed in section II.1, this is why there are no magnetic monopoles in QED. But, solely and directly as a result of the fact that $\left[G^\mu, G^\nu \right] \neq 0$, due to the remaining terms $-i\left(\partial^\sigma \left[G^\mu, G^\nu \right] + \partial^\mu \left[G^\nu, G^\sigma \right] + \partial^\nu \left[G^\sigma, G^\mu \right] \right) \neq 0$, *these magnetic monopoles are non-vanishing.* So if one believes in Yang-Mills gauge theory, one must also believe that the magnetic monopoles (3.7) exist somewhere, in some form, in the physical universe. What form they exist in is an open question. Whether they are topologically unstable objects that can only be observed for a small fraction of a second in a high energy accelerator; whether they can be made stable via spontaneous symmetry breaking and are hiding in plain sight as baryons and most notably as protons and neutrons (which the author contends is the case); or whether they are something else, is an open question at this point. But the non-commuting nature of the Yang-Mills gauge fields compels us to take these monopoles (3.7) seriously and ask: what are they, and where and how can we find them?

The above gets even more interesting when considered in differential forms language. The relationship (3.1) now takes on the compacted form $F = dG - iG^2$. As a result, (3.7) is written compactly as $P = dF = d\left(dG - iG^2\right) = -idG^2$, where $\left(R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma} \right) G^\tau$ is responsible for $dd=0$, "the exterior derivative of an exterior derivative is zero." So now, in integral form, the Yang-Mills magnetic monopole equation, in contrast to (2.7), is

$$\iiint P = \iiint dF = \iiint d\left(dG - iG^2\right) = -i\iiint dG^2 = \oiint F = \oiint dG - i\oiint G^2 = -i\oiint G^2 \quad . \tag{3.8}$$

Let us especially focus on the first and next-to-last expressions which we expand to write as (the final reduction to $-3i\oiint\left[G^\mu, G^\nu\right]dx_\mu dx_\nu$ involves a renaming of indexes together with recognizing that $dx_\sigma dx_\mu dx_\nu$ emerges from the wedge product $dx_\sigma \wedge dx_\mu \wedge dx_\nu$ which is antisymmetric under the interchange of any two adjacent dx_μ):

$$\begin{aligned}\iiint P &= \iiint P^{\sigma\mu\nu} dx_\sigma dx_\mu dx_\nu \\ &= \iiint\left(R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma}\right)G^\tau dx_\sigma dx_\mu dx_\nu - i\iiint\left(\partial^\sigma\left[G^\mu, G^\nu\right] - \partial^\mu\left[G^\nu, G^\sigma\right] - \partial^\nu\left[G^\sigma, G^\mu\right]\right)dx_\sigma dx_\mu dx_\nu \\ &= \oiint dG - i\oiint G^2 = 0 - 3i\oiint\left[G^\mu, G^\nu\right]dx_\mu dx_\nu\end{aligned} \quad . \tag{3.9}$$

So we see that *inside* the monopole volume, $\iiint\left(R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma}\right)G^\tau dx_\sigma dx_\mu dx_\nu$ describes the coupling of individual the N^2-1 gauge fields $G^{i\tau}$ of $G^\tau = \lambda^i G^{i\tau}$ to the spacetime geometry, and that this coupling via $R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma} = 0$ conspires to result in $\oiint dG = 0$, which is also deduced by comparing the final two expressions in (3.8). So the geometry couples to the gauge fields in a manner that prevents the gauge fields from flowing in and out across closed surfaces enclosing the monopole for exactly the same reasons that there are no magnetic monopoles at all in Abelian gauge theory.

And finally, making (3.7) even more interesting, as detailed in section 1 of [1], if we perform a local transformation $F \to F' = F - dG$ on the field strength *F*, which in expanded form is written as $F^{\mu\nu} \to F^{\mu\nu\prime} = F^{\mu\nu} - \partial^{[\nu}G^{\mu]}$, then we find from (3.8) as a direct result of $\oiint dG = 0$, that:

$$\iiint P = \oiint F \to \oiint F' = \oiint\left(F - dG\right) = \oiint F \quad . \tag{3.10}$$

This means that the flow of the field strength $\oiint F = -i\oiint G^2$ across a two dimensional surface is invariant under the local gauge-like transformation $F^{\mu\nu} \to F^{\mu\nu\prime} = F^{\mu\nu} - \partial^{[\nu}G^{\mu]}$.

Now, as much as the MIT Bag Model reviewed in, e.g., [10] section 18 has certain inelegant features such as the *ad hoc* introduction of backpressures to force confinement, this model very correctly makes one very important point that deserves utmost attention beyond the specifics of any particular model of confinement: *focus carefully on what flows and importantly does not flow across any closed two-dimensional surface.* This is why the integral form of Maxwell's equations is so vital to any sensible discussion of confinement. The confinement of gauge fields (which in SU(3) QCD are represented by the eight gluons of $G^\tau = \lambda^i G^{i\tau}$ with

$i=1,2,3...8$) is symbolically specified by $\oiint \text{Gluons}=0$. Similarly, the confinement of individual quarks (which are represented by the SU(3) Dirac wavefunction $\psi_A;\, A=1,2,3$ with three color eigenstates *R*, *G*, *B*) is specified symbolically by $\oiint \text{Quarks}=0$. Different theories may have different ways to achieve these two symbolic confinements, but in the end, one should pay close attention to the two-dimensional closed surface integrals and carefully examine what does and does not flow across these closed surfaces. Equations (3.8) through (3.10) contain a lot of information about what does and does not flow across the closed $\oiint$ surface of a Yang-Mills magnetic monopole, so as taught by the MIT Bag Model, we should study these equations carefully to see if these magnetic monopoles exhibit any attributes of confined gluons and quarks, or interactions via mesons.

A first point is made by $\iiint\left(R_\tau{}^{\nu\sigma\mu}+R_\tau{}^{\sigma\mu\nu}+R_\tau{}^{\mu\nu\sigma}\right)G^\tau dx_\sigma dx_\mu dx_\nu$ which leads to $\oiint dG=0$ in (3.9) and which are the exact same expressions which yield the absence of magnetic monopoles entirely, from Abelian electrodynamics, review (2.6) and (2.7). The term $\iiint\left(R_\tau{}^{\nu\sigma\mu}+R_\tau{}^{\sigma\mu\nu}+R_\tau{}^{\mu\nu\sigma}\right)G^\tau dx_\sigma dx_\mu dx_\nu$, which is the term that contains an *individual* gauge field $G^\tau=\lambda^i G^{i\tau}$, zeros out as a direct result of its coupling through the Riemannian geometry in the configuration of the first Bianchi identity, and upon Gauss' / Stokes' integration yields $\oiint dG=0$. So the question, in the context of the MIT bag model, is whether this term is to be interpreted as telling us that gauge fields (gluons in SU(3) QCD) are confined, which means that there is never a *net* flow of gauge fields across any *closed* surface surrounding a Yang-Mills magnetic monopole. As is the case with electrodynamics, Yang-Mills magnetic fields (and gluons in QCD) can and do flow, in net, through *open* surfaces, but because magnetic fields are aterminal fields, an outward flux over one portion of a closed surface is always cancelled by an inward flux across another portion of the closed surface. This is strengthened by the fact displayed in (3.10) that $\oiint F \to \oiint F' = \oiint F$ is invariant under the transformation $F\to F'=F-dG$, i.e., $F^{\mu\nu}\to F^{\mu\nu}{}'=F^{\mu\nu}-\partial^{[\nu}G^{\mu]}$ which renders the gauge fields (gluons in QCD) *not observable* with respect to net flux through the closed surface. *This would mean as argued in section 1 of [1] that gauge fields are confined in Yang-Mills theory <u>for the exact same geometric reasons</u> that magnetic monopoles <u>do not exist at all</u> in Abelian gauge theory.*

A second point is made by the fact that $\iiint P^{\sigma\mu\nu}dx_\sigma dx_\mu dx_\nu=-3i\oiint\left[G^\mu,G^\nu\right]dx_\mu dx_\nu$ in (3.9) is really the telling us the crux of what *does* net flow across closed surfaces of a Yang-Mills magnetic monopole. The only thing that flows are these $-3i\left[G^\mu,G^\nu\right]$ entities, whatever they turn out to represent. If these $-3i\left[G^\mu,G^\nu\right]$ do *not* turn out to represent individual quarks, then what (3.9) would be telling us, in the sense of the MIT bag model, is that neither individual gluons nor individual quarks net flow across the closed surface of a Yang-Mills magnetic monopole, $\oiint \text{Gluons}=0$ and $\oiint \text{Quarks}=0$. But what we also know is that baryons interact via meson exchange, and that mesons have a color wavefunction of the form $\overline{R}R+\overline{G}G+\overline{B}B$. So mesons *should* be permitted to flow in and out of baryons, that is, we should also have $\oiint \text{Mesons}\neq 0$.

So if we can show that $-3i\oiint\left[G^{\mu},G^{\nu}\right]dx_{\mu}dx_{\nu}$ represents meson flow, then these magnetic monopoles would forbid net quark and gluon flows but permit net meson flow, and we would have some very strong formal reasons for identifying Yang-Mills magnetic monopoles with baryons. Additionally, the factor of "3" which also emerges here, although it comes for the three additive terms in the middle line of (3.9), also signifies the number of colors of quark in QCD, the number of quarks in a baryon, and the number of terms in the meson color wavefunction $\overline{R}R+\overline{G}G+\overline{B}B$. So this "3" is a very strong hint – on top of the fact that $P^{\sigma\mu\nu}$ itself has three spacetime indexes and contains three additive terms – that there is some very definitive "three-ness" associated with these Yang-Mills magnetic monopoles. This could save us having to simply *postulate* three quarks per baryon as is presently done in QCD and instead *require* us to have three quarks per baryon upon which we then impose QCD as an Exclusion Principle, thereby answering the unanswered question as to why baryons contain three quarks and not some other number. These symmetry relationships are what led the author in April 2005 to begin taking seriously, the thesis that these non-vanishing magnetic monopoles originating from the non-commuting gauge fields of Yang-Mills gauge theory might be baryons.

But so far, beyond this number "3," there is no hint in this present development of any quarks in the magnetic monopole (3.9). So we need to now see if there is some way to "populate" these magnetic monopoles with quarks. This draws our attention back to (2.17), which would allow us to replace the gauge field in (3.7) with the source currents from which they originate, and then perhaps start to develop those source currents into quark currents. But can we do this? That is the question raised in the comment reported in the introduction: "One thing that appears doubtful to me is the way you handle the QCD gauge fields, replacing them by the fermion source currents from which they originate. It seems to me that your procedure involves a wholesale deletion of all the nonlinearities in these gauge fields. . ." It is this question that blocked the author from further development of the thesis that baryons are Yang-Mills magnetic monopoles from a full seven years from April 2005 until May 2012 before finally becoming convinced that the magnetic monopoles could in fact be formally populated with quarks in this way (which as a byproduct combines Maxwell's two equations into one equation with a field strength $z_1=12$ which is the same strength as the equation $R_{\mu\nu}=0$ for pure geometry, see Einstein's final paper [11], page 159). Let us now explore the specific problem which was the source of this "block," and well as the rationale and findings that ultimately allowed the author to overcome this block.

III.4 Electric Charge Sources in Yang-Mills Gauge Theory, and a First Pass to Populate Yang-Mills Magnetic Monopoles with Quarks and Demonstrate Why these Monopoles Appear to be Baryons

Let us now use the "steroidal" form of the field strength $F^{\mu\nu}=D^{[\mu}G^{\nu]}$ of (3.5) in Maxwell's charge equation (2.1) to obtain:

$$J^{\nu}=\partial_{\mu}F^{\mu\nu}=\partial_{\mu}D^{[\mu}G^{\nu]}=\partial_{\mu}\left(D^{\mu}G^{\nu}-D^{\nu}G^{\mu}\right)=\left(g^{\mu\nu}\partial_{\sigma}D^{\sigma}-\partial^{\mu}D^{\nu}\right)G_{\mu}. \tag{3.11}$$

Here, we have engaged in exactly the same index gymnastics used in (2.8), and see that (3.11) and (2.8) have exactly the same form, other than that (3.11) contains two appearances of the gauge-covariant derivative $D^\mu \equiv \partial^\mu - iG^\mu$ together with two appearances of the ordinary derivative ∂^μ. So (3.11) is on "partial steroids." We raise the question – to be answered in the affirmative when we consider doing an integration-by-parts precedent to developing quantum Yang-Mills theory – whether the remaining ordinary derivatives should *also* be gauge covariant, such that $J^\nu = \left(g^{\mu\nu} D_\sigma D^\sigma - D^\mu D^\nu\right) G_\mu$, so as to place (3.11) onto "full steroids." But for now, let us stick with (3.11) which the author used at face value for the development in section 2 of [1] while mentioning also in section 2 of [1] a perturbation tensor $-V^{\mu\nu} = i\left(\partial^\mu G^\nu + G^\nu \partial^\mu\right) + G^\mu G^\nu$ and a "deeper analysis." That "deeper analysis" is what we are in the midst of presenting here.

Given the above, we now use $D^\mu \equiv \partial^\mu - iG^\mu$ and also introduce a Proca mass via $\partial_\sigma \partial^\sigma \to \partial_\sigma \partial^\sigma + m^2$ as we did in (2.9), to write (3.11) as

$$J^\nu = \left(g^{\mu\nu} \partial_\sigma \left(\partial^\sigma - iG^\sigma\right) - \partial^\mu \left(\partial^\nu - iG^\nu\right)\right) G_\mu = \left(g^{\mu\nu} \left(\partial_\sigma \partial^\sigma + m^2 - i\partial_\sigma G^\sigma\right) - \partial^\mu \partial^\nu + i\partial^\mu G^\nu\right) G_\mu . \quad (3.12)$$

Contrasting to (2.9), we see that the "partial steroidal" (3.11) introduces the additional terms $i\left(-g^{\mu\nu} \partial_\sigma G^\sigma + \partial^\mu G^\nu\right) G_\mu$ that were not in (2.9), and that the configuration space operator to be inverted is now the more complicated $g^{\mu\nu} \left(\partial_\sigma \partial^\sigma + m^2 - i\partial_\sigma G^\sigma\right) - \partial^\mu \partial^\nu + i\partial^\mu G^\nu$. So we expect that the inverse corresponding to (2.14) will contain some additional terms and that the Yang-Mills counterparts to the reduced (2.16) and (2.17) will also have additional terms which capture the non-linear / non-commuting / gauge-steroidal nature of Yang-Mills theory.

Nonetheless, notwithstanding the new Yang-Mills terms in (3.12), let us at this juncture throw caution to the winds, and simply to explore the basic symmetry features of the magnetic monopole (3.7) which do not change based on (3.12) versus (2.9). Specifically, let us see what we obtain if despite (3.12), we substitute (2.17) in the form of $G^\mu = -\overline{\psi}\gamma^\mu \psi \left(k_\sigma k^\sigma - m^2\right)$, into (3.7). The result is as follows, and the development in the rest of this subsection may be considered a revised and simplified version of the derivation in sections 2,3 and 5 of [1]:

$$\begin{aligned} P^{\sigma\mu\nu} &= -i\left(\partial^\sigma \left[G^\mu, G^\nu\right] + \partial^\mu \left[G^\nu, G^\sigma\right] + \partial^\nu \left[G^\sigma, G^\mu\right]\right) \\ &= -i\left(\partial^\sigma \left[\frac{\overline{\psi}\gamma^\mu \psi}{k_\tau k^\tau - m^2}, \frac{\overline{\psi}\gamma^\nu \psi}{k_\tau k^\tau - m^2}\right] + \partial^\mu \left[\frac{\overline{\psi}\gamma^\nu \psi}{k_\tau k^\tau - m^2}, \frac{\overline{\psi}\gamma^\sigma \psi}{k_\tau k^\tau - m^2}\right] + \partial^\nu \left[\frac{\overline{\psi}\gamma^\sigma \psi}{k_\tau k^\tau - m^2}, \frac{\overline{\psi}\gamma^\mu \psi}{k_\tau k^\tau - m^2}\right]\right) . \quad (3.13) \\ &= -i\frac{1}{k_\tau k^\tau - m^2}\left(\partial^\sigma \frac{\overline{\psi}\gamma^{[\mu} \psi \overline{\psi}\gamma^{\nu]} \psi}{k_\tau k^\tau - m^2} + \partial^\mu \frac{\overline{\psi}\gamma^{[\nu} \psi \overline{\psi}\gamma^{\sigma]} \psi}{k_\tau k^\tau - m^2} + \partial^\nu \frac{\overline{\psi}\gamma^{[\sigma} \psi \overline{\psi}\gamma^{\mu]} \psi}{k_\tau k^\tau - m^2}\right) \end{aligned}$$

Then, we note the spin sum relationship which is often normalized (but not here) such that $N^2 = \mathrm{E} + m$. This spin sum *prior to normalization* is:

$$\sum_{\text{spins}} u\bar{u} = \frac{N^2}{E+m}\left(\not{p}+m\right). \tag{3.14}$$

Also seeing the emergent $\psi\bar{\psi} = u\bar{u}$ in each of the three terms in (3.13), we take the spin sums of all three of these terms in (3.13), and use (3.14) in (3.13) to write:

$$P^{\sigma\mu\nu} = -i\frac{1}{k_\tau k^\tau - m^2}\frac{N^2}{E+m}\left(\partial^\sigma \frac{\bar{\psi}\gamma^{[\mu}\left(\not{p}+m\right)\gamma^{\nu]}\psi}{k_\tau k^\tau - m^2} + \partial^\mu \frac{\bar{\psi}\gamma^{[\nu}\left(\not{p}+m\right)\gamma^{\sigma]}\psi}{k_\tau k^\tau - m^2} + \partial^\nu \frac{\bar{\psi}\gamma^{[\sigma}\left(\not{p}+m\right)\gamma^{\mu]}\psi}{k_\tau k^\tau - m^2}\right). \tag{3.15}$$

Next, we keep in mind that the fermion propagator

$$\frac{\not{p}+m}{\rho^\tau\rho_\tau - m^2} = \frac{\not{p}+m}{\left(\not{p}+m\right)\left(\not{p}-m\right)} = \left(\not{p}-m\right)^{-1}, \tag{3.16}$$

while also noting the appearance of $\left(\not{p}+m\right)/\left(k_\tau k^\tau - m^2\right)$ throughout (3.15) which is very similar in form to (3.16). So, if we can find some rationale (see section 3 of [1]) to associate the k^τ with ρ^τ, then we will have introduced propagating fermion wavefunctions into the monopole $P^{\sigma\mu\nu}$. Observing that the $1/\left(k_\tau k^\tau - m^2\right)$ represents propagation for a Proca-massive vector boson with *three* degrees of freedom and that fermions have *four* degrees of freedom, we shift one degree of freedom from the leading $1/\left(k_\tau k^\tau - m^2\right)$ over to the fermions by setting m=0 to turn that leading term into a massless bosons propagator. That is, for each term in (3.15), we shift:

$$\frac{1}{k_\tau k^\tau - m^2}\partial^\sigma \frac{\bar{\psi}\gamma^{[\mu}\left(\not{p}+m\right)\gamma^{\nu]}\psi}{k_\tau k^\tau - m^2} \Rightarrow \frac{1}{k_\tau k^\tau}\partial^\sigma \frac{\bar{\psi}\gamma^{[\mu}\left(\not{p}+m\right)\gamma^{\nu]}\psi}{p_\tau p^\tau - m^2}. \tag{3.17}$$

and now take ρ^τ to represent the fermion four-momentum. It should be clear that both parts of (3.17) contain a total of six degrees of freedom; they have just been shifted from a 3+3 to a 2+4 configuration not dissimilarly to how a degree of freedom is shifted from a Higgs scalar to a massless gauge boson to create massive vector bosons using the Goldstone mechanism. Thus, following this shifting of degrees of freedom, (3.15) becomes:

$$P^{\sigma\mu\nu} = -i\frac{1}{k_\tau k^\tau}\frac{N^2}{E+m}\left(\partial^\sigma \frac{\bar{\psi}\gamma^{[\mu}\left(\not{p}+m\right)\gamma^{\nu]}\psi}{p_\tau p^\tau - m^2} + \partial^\mu \frac{\bar{\psi}\gamma^{[\nu}\left(\not{p}+m\right)\gamma^{\sigma]}\psi}{p_\tau p^\tau - m^2} + \partial^\nu \frac{\bar{\psi}\gamma^{[\sigma}\left(\not{p}+m\right)\gamma^{\mu]}\psi}{p_\tau p^\tau - m^2}\right). \tag{3.18}$$

If we now normalize such that $N^2 = \left(E+m\right)k_\tau k^\tau$, then via (3.16) we can reduce (3.18) to:

$$P^{\sigma\mu\nu} = -i\left(\partial^\sigma\left(\bar{\psi}\gamma^{[\mu}\left(\not{p}-m\right)^{-1}\gamma^{\nu]}\psi\right) + \partial^\mu\left(\bar{\psi}\gamma^{[\nu}\left(\not{p}-m\right)^{-1}\gamma^{\sigma]}\psi\right) + \partial^\nu\left(\bar{\psi}\gamma^{[\sigma}\left(\not{p}-m\right)^{-1}\gamma^{\mu]}\psi\right)\right), \tag{3.19}$$

which contains three additive terms each containing a propagating fermion wavefunction.

Finally, keeping in mind that this is Yang-Mills theory, so that these fermion wavefunctions $\psi = \psi_A;\, A = 1,2,3...N$ actually contain N eigenstates for SU(N), and because (3.19) contains three propagating appearances of $\psi = \psi_A;\, A = 1,2,3...N$, we select the specific Yang-Mills gauge group SU(3) with generators $\lambda^i;\, i = 1,2,3...8$, the eight gauge bosons in $G^\mu = \lambda^i G^{i\mu}$, and three fermion eigenstates. Finally, we name the three eigenstates R=red, G=green, B=blue for the first (1), second (2) and third (3) terms in (3.19) respectively, and enforce Fermi-Dirac Exclusion as among the three appearances of the fermion wavefunction in (3.19) by setting:.

$$\psi_{(1)} \equiv \left|\lambda^8 = \tfrac{1}{\sqrt{3}}; \lambda^3 = 0\right\rangle = \begin{pmatrix}\psi_R\\0\\0\end{pmatrix}; \psi_{(2)} \equiv \left|\lambda^8 = -\tfrac{1}{2\sqrt{3}}; \lambda^3 = \tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\\psi_G\\0\end{pmatrix}; \psi_{(3)} \equiv \left|\lambda^8 = -\tfrac{1}{2\sqrt{3}}; \lambda^3 = -\tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\0\\\psi_B\end{pmatrix} . \quad (3.20)$$

This means that:

$$\psi_{(1)}\overline{\psi}_{(1)} = \begin{pmatrix}\psi_R\overline{\psi}_R & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}; \quad \psi_{(2)}\overline{\psi}_{(2)} = \begin{pmatrix}0 & 0 & 0\\ 0 & \psi_G\overline{\psi}_G & 0\\ 0 & 0 & 0\end{pmatrix}; \quad \psi_{(3)}\overline{\psi}_{(3)} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \psi_B\overline{\psi}_B\end{pmatrix}. \quad (3.21)$$

Then we use (3.21) to show the explicit 3x3 matrix character of $P^{\sigma\mu\nu} = P^{\sigma\mu\nu}_{AB}$:

$$P^{\sigma\mu\nu}_{AB} = -i\frac{1}{k_\tau k^\tau - m^2}\begin{pmatrix}\partial^\sigma \dfrac{\overline{\psi}_R\gamma^{[\mu}\psi_R\overline{\psi}_R\gamma^{\nu]}\psi_R}{k_\tau k^\tau - m^2} & 0 & 0\\ 0 & \partial^\mu \dfrac{\overline{\psi}_G\gamma^{[\nu}\psi_G\overline{\psi}_G\gamma^{\sigma]}\psi_G}{k_\tau k^\tau - m^2} & 0\\ 0 & 0 & \partial^\nu \dfrac{\overline{\psi}_B\gamma^{[\sigma}\psi_B\overline{\psi}_B\gamma^{\mu]}\psi_B}{k_\tau k^\tau - m^2}\end{pmatrix} . \quad (3.22)$$

Then, repeating the same steps that brought us from (3.13) to (3.19), we may turn this into:

$$P^{\sigma\mu\nu}_{AB} = -i\begin{pmatrix}\partial^\sigma\left(\overline{\psi}_R\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\psi_R\right) & 0 & 0\\ 0 & \partial^\mu\left(\overline{\psi}_G\gamma^{[\nu}\left(\not{p}_G - m_G\right)^{-1}\gamma^{\sigma]}\psi_G\right) & 0\\ 0 & 0 & \partial^\nu\left(\overline{\psi}_B\gamma^{[\sigma}\left(\not{p}_B - m_B\right)^{-1}\gamma^{\mu]}\psi_B\right)\end{pmatrix} . \quad (3.23)$$

where $\not{p}_C, m_C; C = R, G, B$ now represent the daggered momentum $\not{p} = \gamma^\tau p_\tau$ and mass m of each of each of the three fermion eigenstates. The trace equation $\mathrm{Tr}P^{\sigma\mu\nu} = P^{\sigma\mu\nu}_{AA}$ is then easily deduced to be:

$$\mathrm{Tr}P^{\sigma\mu\nu} = -i\left(\partial^{\sigma}\left(\overline{\psi}_R\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\psi_R\right) + \partial^{\mu}\left(\overline{\psi}_G\gamma^{[\nu}\left(\not{p}_G - m_G\right)^{-1}\gamma^{\sigma]}\psi_G\right) + \partial^{\nu}\left(\overline{\psi}_B\gamma^{[\sigma}\left(\not{p}_B - m_B\right)^{-1}\gamma^{\mu]}\psi_B\right)\right). \quad (3.24)$$

This is now the fully-developed Yang-Mills magnetic monopole, populated with three colored quarks, and it is formally equivalent to [5.5] of [1].

If we now associate each color wavefunction with the spacetime index in the related ∂^{σ} operator in (3.24), i.e., $\sigma \sim R$, $\mu \sim G$ and $\nu \sim B$, and keeping in mind that $\mathrm{Tr}P^{\sigma\mu\nu}$ is antisymmetric in all spacetime indexes, we express this antisymmetry with wedge products as $\sigma \wedge \mu \wedge \nu \sim R \wedge G \wedge B = R[G,B] + G[B,R] + B[R,G]$. This is the exact colorless wavefunction that is expected of a baryon. Indeed, the antisymmetric character of the spacetime indexes in a magnetic monopole should have been a good tipoff that magnetic monopoles would naturally make good baryons.

Furthermore, if we apply Gauss' / Stokes' theorem to (3.24) and also include from (3.9) the finding that $\oiint \mathrm{Tr}G^2 = 3\oiint \mathrm{Tr}\left[G^{\mu}, G^{\nu}\right] dx_{\mu} dx_{\nu}$, we find that:

$$\begin{aligned}\iiint \mathrm{Tr}P &= \oiint \mathrm{Tr}F = -i\oiint \mathrm{Tr}G^2 = -3i\oiint \mathrm{Tr}\left[G^{\mu}, G^{\nu}\right] dx_{\mu} dx_{\nu} \\ &= -i\oiint\left(\overline{\psi}_R\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\psi_R + \overline{\psi}_G\gamma^{[\mu}\left(\not{p}_G - m_G\right)^{-1}\gamma^{\nu]}\psi_G + \overline{\psi}_B\gamma^{[\mu}\left(\not{p}_B - m_B\right)^{-1}\gamma^{\nu]}\psi_B\right) dx_{\mu} dx_{\nu}\end{aligned} \quad .(3.25)$$

What is the color wavefunction for the $-3i\left[G^{\mu}, G^{\nu}\right]$ entities? By inspection, $\overline{R}R + \overline{G}G + \overline{B}B$. So *quarks do* <u>*not*</u> *net flow in and out of closed two-dimensional surfaces surrounding Yang-Mills magnetic monopoles, except in the colorless combination of a meson!* So (3.25) validates the suspicion expressed at the end of section III.3 that the appearance of a "3" in front of $\left[G^{\mu}, G^{\nu}\right]$ has something to do with there being three colors of quark inside the magnetic monopole.

So returning to the MIT bag model, we now see that for the magnetic monopole (3.24) with surface flux (3.25), 1) the color wavefunction is that of a baryon, namely $R[G,B] + G[B,R] + B[R,G]$; 2) from (3.9) and (3.10), $\oiint \mathrm{Gluons} = 0$; 3) from (3.25), $\oiint \mathrm{Mesons} \neq 0$ and 4) $\oiint \mathrm{Quarks}=0$ except in the colorless combination $\overline{R}R + \overline{G}G + \overline{B}B$ of a meson. Thus, on a formal basis, with the MIT Bag Model telling us to look at what flows across the surface of any theoretical entity proposed to be a baryon, and we see that the Yang-Mills magnetic monopole has *precisely the required formal symmetries and boundary flows required for a baryon.*

Of course, we still need to make these baryons topologically stable and see how to use them to represent protons and neutrons which are the most important baryons, see section 6 through 8 of [1], and we need to calculate their energies to see if they make sense in relation to empirical data, see sections 11 and 12 of [1]. Insofar as topological stability, we simply note that the trace equation (3.24) is non-vanishing, but that $\mathrm{Tr}P^{\sigma\mu\nu} = \mathrm{Tr}\left(\lambda^{i}_{AB} P^{i\sigma\mu\nu}\right) = 0$ *if we regard the gauge group as SU(3)*, because all of λ^{i} are traceless. In other words, the left and right sides of

(3.24) do not match up because one side is traceless and the other is not, if we assume the simple group SU(3). It is on this basis that we introduce the product group $SU(3)_C \times U(1)_{B-L}$, and then obtain the monopole (3.24) from spontaneous symmetry breaking from larger SU(4) gauge groups with a $B-L$ (baryon minus lepton number) generator which yields the quantum numbers required to turn these baryons into proton and neutrons and ensure that these magnetic monopoles are topologically stable. Again, these details are in sections through 8 of [1], and need little if any elaboration or modification here.

For the moment, the question now becomes this: In light of (3.12), can we, and if so under what circumstances can we, and with what consequences can we, substitute (2.17) in the form $G^{\mu} = -\overline{\psi}\gamma^{\mu}\psi\left(k_{\sigma}k^{\sigma} - m^2\right)$ into (3.7) to arrive at (3.24) and (3.25) which have all the essential required symmetries of a baryon? The development from (3.13) to (3.25), somewhat-perfected retrospectively, expresses the author's essential thinking about this subject in 2005. But it took seven more years for the author to become comfortably-convinced that replacing the gauge fields with the fermion source currents from which they originate, using the *Abelian* (2.17) which deletes certain non-linear aspects of Yang-Mills theory, is indeed a proper replacement. And, it was not until late-2012 that the author understand that the consequence of this replacement is that once these nascent baryons were turned into protons and neutrons, we would discover that by this replacement, we had deleted *all but the binding energies* of these nascent baryons, which theoretical binding energies would turn out to match up with near parts-per-million precision in AMU to experimentally-observed nuclear binding energies. The discussion following will explain how the author, over time, became comfortable that this was indeed a justifiable replacement which effectively combines both of Maxwell's equations into a single equation with field strength $z_1 = 12$, just as that of $R_{\mu\nu} = 0$ for pure spacetime geometry.

IV. The Yang-Mills Lagrangian Density for Chromo-Electric Source Charges, and its Configuration Space Operators

Using the compacted "gauge theory on steroids" view of (3.5), (3.6), the Lagrangian density for Yang-Mills gauge theories with non-vanishing chromo-electric source charges J^{μ} is:

$$\begin{aligned}\mathfrak{L}_J &= \mathrm{Tr}\left(-\tfrac{1}{2}F^{\mu\nu}F_{\mu\nu} - 2J^{\mu}G_{\mu}\right) = \mathrm{Tr}\left(-\tfrac{1}{2}D^{[\mu}G^{\nu]}D_{[\mu}G_{\nu]} - 2J^{\mu}G_{\mu}\right) = \mathrm{Tr}\left(-D^{\mu}G^{\nu}D_{[\mu}G_{\nu]} - 2J^{\mu}G_{\mu}\right) \\ &= -\tfrac{1}{2}F_{AB}^{\mu\nu}F_{BA\mu\nu} - 2J_{AB}^{\mu}G_{AB\mu}\end{aligned} \quad .(4.1)$$

This is just like the QED density (2.21), other than a doubling of the numeric coefficient because of the normalization $Tr\left(\lambda^i\lambda^j\right) = \tfrac{1}{2}\delta^{ij}$ and a trace because each of $G_{AB}^{\mu} = \lambda_{AB}^{i}G^{i\mu}$, $J_{AB}^{\mu} = \lambda_{AB}^{i}J^{i\mu}$ and $F_{AB}^{\mu\nu} = \lambda_{AB}^{i}F^{i\mu\nu}$ are now all NXN matrices for any gauge group SU(N) (or variants such as SU(N)xU(1) which as noted at the end of section III are required to impart topological stability to the magnetic monopoles). Contrast the highly-compacted matrix form (3.6) with the expanded form Lagrangian density (3.4). As noted at (2.22), sometimes the negative sign in front of $-2J^{\mu}G_{\mu}$, which represents the convention of a "negative" electric charge in electrodynamics, is reversed to establish a positive sign convention for Yang-Mills chromo-electric sources. Here, however, we shall maintain the electrodynamic convention so that all the Yang-Mills equations can be directly compared in all respects to the Abelian electrodynamic equations. When we use the term "chromo" when speaking about these electric or magnetic charges, this simply denotes that these charge densities are analogous to those from electrodynamics, except insofar as there are now N^2-1 of them for any group SU(N) via the relationships $J^{\mu}_{AB}=\lambda^i_{AB}J^{i\mu}$ and $P^{\sigma\mu\nu}_{AB}=\lambda^i_{AB}P^{i\sigma\mu\nu}$. For $SU(3)_C$ (or $SU(3)_C\times U(1)_{B-L}$), these "chromo" charges coincide with the charges of QCD color.

The goal in this next phase of development is to identify the configuration space operator or operators which are associated with (4.1), both for purposes of obtaining an inverse equation $G_\nu \equiv I_{\mu\nu}J^\nu$, and for purposes of obtaining an action to use in the path integral to quantize Yang-Mills field theory. Recall that for electrodynamics, we used the configuration space operator $g^{\mu\nu}\left(\partial_\sigma\partial^\sigma+m^2\right)-\partial^\mu\partial^\nu$ from the field equation (2.9) to specify the inverse $I_{\mu\nu}$ of $G_\nu \equiv I_{\mu\nu}J^\nu$ in (2.10), which inverse we then explicitly derived in (2.14). And further recall that we used the Lagrangian density (2.21) to specify an action (2.25) which we then integrated by parts to obtain the $g^{\mu\nu}\left(\partial_\sigma\partial^\sigma+m^2\right)-\partial^\mu\partial^\nu$ in the action (2.30). This operator then became part of the quadratic expression $\frac{1}{2}G_\mu\left(g^{\mu\nu}\left(\partial_\sigma\partial^\sigma+m^2\right)-\partial^\mu\partial^\nu\right)G_\nu-J^\mu G_\mu$ used in the path integral in (2.31) to derive the QED amplitude $W(J)$ in (2.36). *Recall also, importantly, that these two configuration space operators were <u>identical</u> to one another.* This will *not* be the case for the similarly-derived Yang-Mills configuration space operators.

Specifically, starting from Lagrangian density (4.1), we shall now do *exactly the same calculations* for Yang-Mills gauge theory which we earlier did for electrodynamics. But as we shall see, this will lead to *two different configuration space operators*. We will then be tasked with comparing these two different operators to determine which one is more suitable to use for obtaining the inverse $G_\nu \equiv I_{\mu\nu}J^\nu$ to replace the gauge fields G^μ in the magnetic monopole field equation with the sources from which they originate, as we did in (3.13) to develop the "first draft" baryon of (3.24). In the process, we shall come to understand how it is that by populating the Yang-Mills magnetic monopole with fermion sources using the linear inverse (2.17) of QED, we were in fact simply describing in (3.24), a baryon with all perturbations removed, which as noted at the very end of section III, is a baryon with all but the binding energies removed. We will thereafter proceed further to develop a complete baryon which now *includes* the perturbations and non-linearity of Yang-Mills gauge theory, for which (3.24) expresses the special case in which the perturbation is set to zero.

IV.1 The Configuration Space Operator Derived via the Euler Lagrange Equation

We first apply the Euler-Lagrange (2.20) to the Lagrangian density (4.1) to obtain the classical field equation. We expect to find $J^\nu=\partial_\mu F^{\mu\nu}$ of (2.1) which in Yang-Mills theory really is $J^\nu_{AB}=\partial_\mu F^{\mu\nu}_{AB}$, but the exercise is worthwhile.

We first expand out the Lagrangian density (4.1) with renamed indexes, using $F^{\mu\nu} = D^{[\mu}G^{\nu]}$ from (3.5) as well as $D^{\mu} = \partial^{\mu} - iG^{\mu}$, as such:

$$\begin{aligned}\mathfrak{L}_J &= \mathrm{Tr}\left(-\tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta} - 2J^{\alpha}G_{\alpha}\right) = \mathrm{Tr}\left(-\tfrac{1}{2}D^{[\alpha}G^{\beta]}D_{[\alpha}G_{\beta]} - 2J^{\alpha}G_{\alpha}\right) = \mathrm{Tr}\left(-D^{\alpha}G^{\beta}D_{[\alpha}G_{\beta]} - 2J^{\alpha}G_{\alpha}\right)\\ &= \mathrm{Tr}\left(-D^{\alpha}G^{\beta}D_{\alpha}G_{\beta} + D^{\alpha}G^{\beta}D_{\beta}G_{\alpha} - 2J^{\alpha}G_{\alpha}\right)\\ &= \mathrm{Tr}\left(-\left(\partial^{\alpha}G^{\beta} - iG^{\alpha}G^{\beta}\right)\left(\partial_{\alpha}G_{\beta} - iG_{\alpha}G_{\beta}\right) + \left(\partial^{\alpha}G^{\beta} - iG^{\alpha}G^{\beta}\right)\left(\partial_{\beta}G_{\alpha} - iG_{\beta}G_{\alpha}\right) - 2J^{\alpha}G_{\alpha}\right)\\ &= \mathrm{Tr}\begin{pmatrix} -2J^{\alpha}G_{\alpha} + G^{\alpha}G^{\beta}G_{\alpha}G_{\beta} - G^{\alpha}G^{\beta}G_{\beta}G_{\alpha} \\ -\partial^{\alpha}G^{\beta}\partial_{\alpha}G_{\beta} + \partial^{\alpha}G^{\beta}\partial_{\beta}G_{\alpha} + i\partial^{\alpha}G^{\beta}G_{\alpha}G_{\beta} - i\partial^{\alpha}G^{\beta}G_{\beta}G_{\alpha} + iG^{\alpha}G^{\beta}\partial_{\alpha}G_{\beta} - iG^{\alpha}G^{\beta}\partial_{\beta}G_{\alpha}\end{pmatrix}\end{aligned} \quad .(4.2)$$

Now, let's use this in the Euler-Lagrange equation (2.20) written as:

$$\begin{aligned}&\frac{\partial \mathfrak{L}_J}{\partial G_{\nu}} = \partial_{\mu}\left(\frac{\partial \mathfrak{L}_J}{\partial\left(\partial_{\mu}G_{\nu}\right)}\right)\\ &= \frac{\partial}{\partial G_{\nu}}\mathrm{Tr}\left(-2J^{\alpha}G_{\alpha} + G^{\alpha}G^{\beta}G_{\alpha}G_{\beta} - G^{\alpha}G^{\beta}G_{\beta}G_{\alpha}\right)\\ &= \partial_{\mu}\left(\frac{\partial}{\partial\left(\partial_{\mu}G_{\nu}\right)}\mathrm{Tr}\begin{pmatrix}-\partial^{\alpha}G^{\beta}\partial_{\alpha}G_{\beta} + \partial^{\alpha}G^{\beta}\partial_{\beta}G_{\alpha}\\ +i\partial^{\alpha}G^{\beta}G_{\alpha}G_{\beta} + iG^{\alpha}G^{\beta}\partial_{\alpha}G_{\beta} - i\partial^{\alpha}G^{\beta}G_{\beta}G_{\alpha} - iG^{\alpha}G^{\beta}\partial_{\beta}G_{\alpha}\end{pmatrix}\right)\end{aligned} \quad . \quad (4.3)$$

Using the product rule for derivatives and then applying the derivatives with proper index gymnastics, this becomes, on a term-by-term correspondence to (4.3):

$$\begin{aligned}&\mathrm{Tr}\left(-2J^{\nu} + \begin{pmatrix}G^{\beta}G^{\nu}G_{\beta} + G^{\alpha}G_{\alpha}G^{\nu} + G^{\nu}G^{\beta}G_{\beta} + G^{\alpha}G^{\nu}G_{\alpha}\\ -G^{\beta}G_{\beta}G^{\nu} - G^{\alpha}G^{\nu}G_{\alpha} - G^{\alpha}G^{\nu}G_{\alpha} - G^{\nu}G^{\beta}G_{\beta}\end{pmatrix}\right)\\ &= \partial_{\mu}\left(\mathrm{Tr}\left(-\partial^{\mu}G^{\nu} - \partial^{\mu}G^{\nu} + \partial^{\nu}G^{\mu} + \partial^{\nu}G^{\mu} + iG^{\mu}G^{\nu} + iG^{\mu}G^{\nu} - iG^{\nu}G^{\mu} - iG^{\nu}G^{\mu}\right)\right)\end{aligned} \quad . \quad (4.4)$$

Reducing, all of the G^3 terms (e.g., $G^{\beta}G^{\nu}G_{\beta}$) cancel, and we are left with:

$$\mathrm{Tr}J^{\nu} = \partial_{\mu}\left(\mathrm{Tr}\left(\partial^{[\mu}G^{\nu]} - iG^{[\mu}G^{\nu]}\right)\right) = \mathrm{Tr}\partial_{\mu}\left(\partial^{[\mu} - iG^{[\mu}\right)G^{\nu]} = \mathrm{Tr}\partial_{\mu}D^{[\mu}G^{\nu]} = \mathrm{Tr}\partial_{\mu}F^{\mu\nu}. \quad (4.5)$$

We thus see the electrodynamic field equation $J^{\nu} = \partial_{\mu}F^{\mu\nu}$ is recovered from the Yang-Mills Lagrangian, but in trace form $\mathrm{Tr}J^{\nu} = \mathrm{Tr}\partial_{\mu}F^{\mu\nu}$. Going in the reverse direction, this means that we can indeed start off with a classical field equation (3.11) for a Yang-Mills chromo-electric source charge density with field strength $F^{\mu\nu} = D^{[\mu}G^{\nu]}$, take the trace of each side to deduce that $\mathrm{Tr}J^{\nu} = \mathrm{Tr}\partial_{\mu}F^{\mu\nu}$, and know that this trace equation will be reproduced by applying the Euler-LaGrange equation in the form of (4.3) to the Yang-Mills Lagrangian density (4.1).

As a result, we confirm via (3.11) that $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$, or with a hand-added Proca mass, $g^{\mu\nu}\left(\partial_\sigma D^\sigma + m^2\right) - \partial^\mu D^\nu$, is the configuration space operator of $J^\nu = \partial_\mu D^{[\mu} A^{\nu]}$ which is the *field equation* (4.5) of the Yang-Mills Lagrangian density (4.1) deduced consistently from the Euler-Lagrange equation (2.20). This is in contrast to the configuration space operator $g^{\mu\nu}\left(\partial_\sigma \partial^\sigma + m^2\right) - \partial^\mu \partial^\nu$ of QED obtained in (2.9), and so we see that the rightmost derivatives have gone from $\partial^\mu \to D^\mu$ while the leftmost derivatives have not. Taking the lead from (2.10), this means to if we wish to develop an inverse equation $G_\nu \equiv I'_{\mu\nu} J^\nu$, we will have to find a new inverse $I'_{\nu\tau}$ *defined* such that:

$$\begin{aligned}&\left[g^{\tau\mu}\left(\partial_\sigma D^\sigma + m^2\right) - \partial^\tau D^\mu\right] I'_{\nu\tau} = \left[g^{\tau\mu}\left(\partial_\sigma\left(\partial^\sigma - iG^\sigma\right) + m^2\right) - \partial^\tau\left(\partial^\mu - iG^\mu\right)\right] I'_{\nu\tau} \\ &= \left[g^{\tau\mu}\left(\partial_\sigma \partial^\sigma - i\partial_\sigma G^\sigma + m^2\right) - \partial^\tau \partial^\mu + i\partial^\tau G^\mu\right] I'_{\nu\tau} \equiv \delta^\mu{}_\nu\end{aligned}. \tag{4.6}$$

Note that we have thus defined $G_\nu \equiv I'_{\mu\nu} J^\mu$ so as to reverse the order of operation of the covariant spacetime indexes in relation to $G_\nu \equiv I_{\mu\nu} J^\nu$ used for electrodynamics between (2.9) and (2.10). In electrodynamics, $I_{\mu\nu} = I_{\nu\mu}$ so the ordering does not matter. In Yang-Mills theory, the order does matter and this choice of convention will be illustrative later on.

Before we proceed to calculate this inverse which we shall do in section 5, we can see from the field equation $J^\nu = \partial_\mu D^{[\mu} G^{\nu]}$, or from $g^{\mu\nu}\left(\partial_\sigma D^\sigma + m^2\right) - \partial^\mu D^\nu$, that we have come upon a form of "*partial* minimal coupling principle" in which the dynamical field equations retain their form, with the exception that *some but not all* of the ordinary derivatives of electrodynamics are replaced by gauge covariant derivatives in the field strength. That is, going from Abelian to non-Abelian gauge theory:

$$J^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \partial^{[\mu} G^{\nu]} = \left(g^{\mu\nu}\partial_\sigma \partial^\sigma - \partial^\mu \partial^\nu\right) G_\mu \Rightarrow J^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu D^{[\mu} G^{\nu]} = \left(g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu\right) G_\mu . \tag{4.7}$$

Equation (4.7) to the right, with its *partial* minimal coupling, now begs the question: why is there not a *full* minimal coupling principle? Why do we replace $\partial_\mu \phi \to D_\mu \phi$ and $\partial_\mu \psi \to D_\mu \psi$ for theories of scalar and Dirac fermions to arrive at Abelian gauge theories in the first place, and then replace $\partial^{[\mu} G^{\nu]} \to D^{[\mu} G^{\nu]}$ to make the Abelian gauge theory non-Abelian, but then stop short of making the final replacement $\partial_\mu D^{[\mu} G^{\nu]} \to D_\mu D^{[\mu} G^{\nu]}$ in the field equation (4.7)? We do note that the ∂_μ rather than D_μ in (4.7) results from the outer ∂_μ in the Euler-Lagrange equation, so that *if* the field equation were to in fact be the *fully* minimally-coupled $J^\nu = D_\mu D^{[\mu} G^{\nu]}$, then we would have to use a modified version of the Euler-Lagrange equation that reads:

$$D_\mu\left(\frac{\partial\mathfrak{L}_J}{\partial\left(\partial_\mu G_\nu\right)}\right)-\frac{\partial\mathfrak{L}_J}{\partial G_\nu}=0\,. \tag{4.8}$$

So the question we shall largely be exploring is whether this form of *full* minimal coupling principle is justified for Yang-Mills (non-Abelian) gauge theories in relation to Abelian gauge theories such as electrodynamics.

IV.2 The Configuration Space Operator derived via Integration-by-Parts of the Yang-Mills Action

Back in (2.26) we used the product rule $\partial^\mu(ab)=(\partial^\mu a)b+a(\partial^\mu b)$ with $a=G^\nu$, $b=\partial_{[\mu}G_{\nu]}$ to obtain $\partial^\mu\left(G^\nu\partial_{[\mu}G_{\nu]}\right)=\partial^\mu G^\nu\partial_{[\mu}G_{\nu]}+G^\nu\partial^\mu\partial_{[\mu}G_{\nu]}$ for integration by parts in the action $S\left(G_\mu\right)$. For Yang-Mills theory the product rules are a bit trickier because of the gauge-covariant derivatives. Specifically, we now need to keep in mind for any *a*, *b* that:

$$D^\mu(ab)=\left(\partial^\mu-iG^\mu\right)(ab)=\partial^\mu ab+a\partial^\mu b-iG^\mu ab\ . \tag{4.9}$$

The extra term $-iG^\mu ab$ is wholly a creature of the gauge-covariant derivative, and does not exist for an ordinary derivative. So with the same assignments $a=G^\nu$, $b=D_{[\mu}G_{\nu]}$, (4.9) becomes:

$$D^\mu(G^\nu D_{[\mu}G_{\nu]})=\partial^\mu G^\nu D_{[\mu}G_{\nu]}+G^\nu\partial^\mu D_{[\mu}G_{\nu]}-iG^\mu G^\nu D_{[\mu}G_{\nu]}=D^\mu G^\nu D_{[\mu}G_{\nu]}+G^\nu\partial^\mu D_{[\mu}G_{\nu]}\,. \tag{4.10}$$

This should be contrasted with (2.26). Noting that the Yang-Mills Lagrangian density (4.1) contains a term $-\tfrac{1}{2}D^{[\mu}G^{\nu]}D_{[\mu}G_{\nu]}=-D^\mu G^\nu D_{[\mu}G_{\nu]}$, we now restructure (4.10) in terms of $D^\mu G^\nu D_{[\mu}G_{\nu]}$. The full calculation is instructive, with index gymnastics starting on the fifth line:

$$\begin{aligned}
D^\mu G^\nu D_{[\mu}G_{\nu]}&=D^\mu(G^\nu D_{[\mu}G_{\nu]})-G^\nu\partial^\mu D_{[\mu}G_{\nu]}\\
&=\left(\partial^\mu-iG^\mu\right)(G^\nu D_{[\mu}G_{\nu]})-G^\nu\partial^\mu D_{[\mu}G_{\nu]}\\
&=\partial^\mu\left(G^\nu D_{[\mu}G_{\nu]}\right)-iG^\mu G^\nu D_{[\mu}G_{\nu]}-G^\nu\partial^\mu D_{[\mu}G_{\nu]}\\
&=\partial^\mu\left(G^\nu D_{[\mu}G_{\nu]}\right)-iG^\mu G^\nu D_\mu G_\nu+iG^\mu G^\nu D_\nu G_\mu-G^\nu\partial^\mu D_\mu G_\nu+G^\nu\partial^\mu D_\nu G_\mu\ .\\
&=\partial^\mu\left(G^\nu D_{[\mu}G_{\nu]}\right)+\left(-iG^\sigma G^\nu D_\sigma+iG^\nu G^\sigma D_\sigma-G^\nu\partial^\sigma D_\sigma+G^\sigma\partial^\nu D_\sigma\right)G_\nu\\
&=\partial^\mu\left(G^\nu D_{[\mu}G_{\nu]}\right)+\left(G^\sigma D^\nu D_\sigma-G^\nu D^\sigma D_\sigma\right)G_\nu\\
&=\partial^\mu\left(G^\nu D_{[\mu}G_{\nu]}\right)+G_\mu\left(-g^{\mu\nu}D_\sigma D^\sigma+D^\nu D^\mu\right)G_\nu
\end{aligned} \tag{4.11}$$

Then, we hand-add a Proca mass as has been done previously, $\partial_\sigma\partial^\sigma\to\partial_\sigma\partial^\sigma+m^2$, so that

$$D^{\mu}G^{\nu}D_{[\mu}G_{\nu]} = \partial^{\mu}\left(G^{\nu}D_{[\mu}G_{\nu]}\right) + G_{\mu}\left(-g^{\mu\nu}\left(D_{\sigma}D^{\sigma}+m^2\right)+D^{\nu}D^{\mu}\right)G_{\nu}\ . \tag{4.12}$$

Contrasting with (2.27) in which we uncovered the Abelian configuration space operator $g^{\mu\nu}\partial_{\sigma}\partial^{\sigma} - \partial^{\nu}\partial^{\mu}$ (and in which we commuted $\left[\partial^{\mu},\partial^{\nu}\right]=0$ by assuming $R^{\sigma}{}_{\alpha\mu\nu}=0$), we find in the above the analogous operator $g^{\mu\nu}\left(D_{\sigma}D^{\sigma}+m^2\right)-D^{\nu}D^{\mu}$ (and are not at liberty to commute $D^{\nu}D^{\mu}$ because these are not commuting). This is a *fully* minimally-coupled configuration space operator, in which *every* ordinary spacetime derivative has been replaced by a gauge-covariant derivative, that is, $\partial^{\mu}\rightarrow D^{\mu}$ everywhere in the configuration space operator. This is now a gauge theory on complete steroids. And of equal interest, the only ordinary derivative remaining in the final line of (4.12) is in the term $\partial^{\mu}\left(G^{\nu}D_{[\mu}G_{\nu]}\right)$, which is perfectly-situated to allow this term to be zeroed out by boundary conditions imposed during integration by parts. So, let's continue.

We next expand Lagrangian density (4.1) and combine with (4.12), thus:

$$\begin{aligned}\mathfrak{L}_J &= \mathrm{Tr}\left(-D^{\mu}G^{\nu}D_{[\mu}G_{\nu]}-2J^{\mu}G_{\mu}\right)\\ &= \mathrm{Tr}\left(-\partial^{\mu}\left(G^{\nu}D_{[\mu}G_{\nu]}\right)+G_{\mu}\left(g^{\mu\nu}\left(D_{\sigma}D^{\sigma}+m^2\right)-D^{\nu}D^{\mu}\right)G_{\nu}-2J^{\mu}G_{\mu}\right)\end{aligned}. \tag{4.13}$$

From this we form the Yang-Mills action:

$$S\left(G_{\mu}\right)=\int d^4x\mathrm{Tr}\left(-\partial^{\mu}\left(G^{\nu}D_{[\mu}G_{\nu]}\right)+G_{\mu}\left(g^{\mu\nu}\left(D_{\sigma}D^{\sigma}+m^2\right)-D^{\nu}D^{\mu}\right)G_{\nu}-2J^{\mu}G_{\mu}\right) \tag{4.14}$$

which should be contrasted directly with the Abelian action (2.28). Aside from the trace and factor of 2 that emerges from the normalization $Tr\left(\sigma^i\sigma^j\right)=\frac{1}{2}\delta^{ij}$, this has *exactly* the same form as (2.28) and is now has a *full minimal coupling* $\partial^{\mu}\rightarrow D^{\mu}$ (gauge theory on steroids) everywhere except in the term $\partial^{\mu}\left(G^{\nu}D_{[\mu}G_{\nu]}\right)$. But, as noted, this is perfect, because if we again impose $G^{\nu}\left(x^{\mu}=\infty\right)=G^{\nu}\left(x^{\mu}=-\infty\right)=0$ as boundary conditions upon the gauge potential, a calculation identical in form to (2.29) which needn't even be repeated here, clearly informs us that $\int d^4x\mathrm{Tr}\partial^{\mu}\left(G^{\nu}D_{[\mu}G_{\nu]}\right)=0$. This means that (4.14) simplifies down to:

$$S\left(G_{\mu}\right)=\mathrm{Tr}\int d^4x\left(G_{\mu}\left(g^{\mu\nu}\left(D_{\sigma}D^{\sigma}+m^2\right)-D^{\nu}D^{\mu}\right)G_{\nu}-2J^{\mu}G_{\mu}\right), \tag{4.15}$$

to be contrasted to the Abelian action in (2.30). *This is the action that one then uses to quantize Yang-Mills theory*, which will be explored in detail in section ??? (to be added).

IV.3 The Yang-Mills Perturbation Tensor

Let us now take the configuration space operator in (4.15) and expand this out fully, thus:

$$\begin{aligned} g^{\mu\nu}\left(D_\sigma D^\sigma + m^2\right) - D^\nu D^\mu &= g^{\mu\nu}\left(\left(\partial_\sigma - iG_\sigma\right)\left(\partial^\sigma - iG^\sigma\right) + m^2\right) - \left(\partial^\nu - iG^\nu\right)\left(\partial^\mu - iG^\mu\right) \\ &= g^{\mu\nu}\left(\partial_\sigma\partial^\sigma + m^2 - i\left(\partial_\sigma G^\sigma + G^\sigma\partial_\sigma\right) - G_\sigma G^\sigma\right) - \partial^\nu\partial^\mu + i\left(\partial^\nu G^\mu + G^\nu\partial^\mu\right) + G^\nu G^\mu \\ &= g^{\mu\nu}\left(\partial_\sigma\partial^\sigma + m^2 + V\right) - \partial^\nu\partial^\mu - V^{\nu\mu} \end{aligned} \quad . \tag{4.16}$$

Very importantly, we see that the configuration space operator contains a non-symmetric tensor (especially because $\left[G^\mu, G^\nu\right] \neq 0$) is a hallmark of Yang-Mills):

$$V^{\mu\nu} \equiv -i\left(\partial^\mu G^\nu + G^\mu \partial^\mu\right) - G^\mu G^\nu \tag{4.17}$$

which we shall refer to as the "perturbation tensor," as well as its trace

$$V = V^\sigma{}_\sigma = -i\left(\partial_\sigma G^\sigma + G^\sigma \partial_\sigma\right) - G_\sigma G^\sigma \tag{4.18}$$

which we shall refer to as the "perturbation scalar" (scalar in the spacetime sense, this is still an NxN Yang-Mills matrix) Why? Because (4.18) has *exactly* the same form as the electromagnetic perturbation! (See, e.g., [12] eq. [4.4])

The emergence of these perturbations in (4.15) from the integration by parts is very important. This means that if we take the zero-perturbation limit in which $V^{\nu\mu} \to 0$, then (4.16) reduces to $g^{\mu\nu}\left(\partial_\sigma\partial^\sigma + m^2\right) - \partial^\nu\partial^\mu$, which is identical to the operator that we inverted from (2.10) to (2.16), and then used to introduce fermion sources in (2.17). Then, it was (2.17) that we used to inject fermions into the Yang-Mills magnetic monopole when we threw caution to the winds in (3.13), which led us to a magnetic monopole $\mathrm{Tr}P^{\sigma\mu\nu}$ in (3.24) and its integral form $\iiint \mathrm{Tr}P = \oiint \mathrm{Tr}F$ in (3.25) which contains all the symmetries of a baryon. So, if we can justify the use of the configuration space operator (4.17) to take inverses, we will have established that (3.24) represents a Yang-Mills magnetic monopole *in the zero-perturbation limit*. And, once we find the inverse for (4.16), we will have the means to generalize the magnetic monopole baryon (3.24) to include circumstances where $V^{\nu\mu} \neq 0$. In those circumstances, we should be able to find a more general equation for (3.24) which includes (3.24) as well as additional terms including $V^{\nu\mu}$ and V, and which reduces precisely to (3.24) once we set $V^{\nu\mu} = 0$.

If (4.16) is the correct operator to use in the classical chromo-electric field equation, this would mean that the correct classical field equation for a Yang-Mills chromo-electric charge is not (4.7), but rather is:

$$J^\nu = D_\mu F^{\mu\nu} = D_\mu D^{[\mu} G^{\nu]} = \left(g^{\mu\nu}\left(D_\sigma D^\sigma + m^2\right) - D^\mu D^\nu\right) G_\mu \, , \tag{4.19}$$

and it would also mean that the Euler-Lagrange equation needs a dose of steroids and for non-Abelian gauge theories should indeed be promoted to include the minimal coupling in (4.8).

Now let's turn to the inverse of this operator. Given that $V^{\tau\mu}$ is not symmetric, left-right order matters and it is important to set this up correctly. In particular, we carefully establish ordering by writing (4.19) to *define* the inverse $I''_{\tau\nu}$ as:

$$J^{\mu}=\left(g^{\mu\nu}\left(D_{\sigma}D^{\sigma}+m^{2}\right)-D^{\nu}D^{\mu}\right)G_{\nu}\equiv\left(g^{\mu\nu}\left(D_{\sigma}D^{\sigma}+m^{2}\right)-D^{\nu}D^{\mu}\right)I''_{\tau\nu}J^{\tau}=\delta^{\mu}{}_{\tau}J^{\tau}, \tag{4.20}$$

Therefore, compare to (4.6) and (2.10), the inverse operator $I''_{\tau\nu}$ for $G_{\nu}=I''_{\tau\nu}J^{\tau}$ is:

$$\left[g^{\mu\tau}\left(D_{\sigma}D^{\sigma}+m^{2}\right)-D^{\tau}D^{\mu}\right]I''_{\nu\tau}=\left[g^{\mu\tau}\left(\partial_{\sigma}\partial^{\sigma}+V+m^{2}\right)-\partial^{\tau}\partial^{\mu}-V^{\tau\mu}\right]I''_{\nu\tau}=\delta^{\mu}{}_{\nu}. \tag{4.21}$$

And, *if* it is in fact correct to apply such a minimal coupling principle to Yang-Mills theory, there is one other consequence as well: the magnetic monopole field equation (2.2), see also (3.7), needs to also be given its own dose of steroids, and should be promoted to:

$$\begin{aligned}P^{\sigma\mu\nu}&=D^{\sigma}F^{\mu\nu}+D^{\mu}F^{\nu\sigma}+D^{\nu}F^{\sigma\mu}=D^{\sigma}D^{[\mu}G^{\nu]}+D^{\mu}D^{[\nu}G^{\sigma]}+D^{\nu}D^{[\sigma}G^{\mu]}\\&=\left[D^{\sigma},D^{\mu}\right]G^{\nu}+\left[D^{\mu},D^{\nu}\right]G^{\sigma}+\left[D^{\nu},D^{\sigma}\right]G^{\mu}\\&=-i\left(D^{\sigma}\left[G^{\mu},G^{\nu}\right]+D^{\mu}\left[G^{\nu},G^{\sigma}\right]+D^{\nu}\left[G^{\sigma},G^{\mu}\right]\right)\end{aligned}. \tag{4.22}$$

So, which is it? Is the partially minimally-coupled classical field equation for Yang-Mills theory $J^{\nu}=\partial_{\mu}F^{\mu\nu}$ obtained in (4.5) via the ordinary Euler Lagrange equation, which has the inverse specified (4.6)? Or, is the fully minimally-coupled classical field equation (4.19), $J^{\nu}=D_{\mu}F^{\mu\nu}$, which was deduced once we integrated the Yang-Mills action by parts and found a configuration space operator (4.16) which in all respects is identical to that of Abelian gauge theory, which a minimal coupling principle in which we simply replace *all* ordinary derivatives in the configuration space operators and the field equations and even the Euler-Lagrange equation as in (4.8), with gauge-covariant derivatives via $\partial^{\mu}\to D^{\mu}$?

V. A Tale of Two Inverses

V.1. Symmetries of the Yang-Mills Perturbation Tensor

Let us first look as some of the symmetries of the perturbation tensor operator (4.17). We start by looking at the operation of taking two successive Yang-Mills gauge-covariant derivatives as is done in the $D^{\nu}D^{\mu}$ term of (4.16). Using (4.17):

$$D^{\mu}D^{\nu}=\left(\partial^{\mu}-iG^{\mu}\right)\left(\partial^{\nu}-iG^{\nu}\right)=\partial^{\mu}\partial^{\nu}-i\partial^{\mu}G^{\nu}-iG^{\mu}\partial^{\nu}-G^{\mu}G^{\nu}=\partial^{\mu}\partial^{\nu}+V^{\mu\nu}. \tag{5.1}$$

A question which is always of interest is to find the commutator of these two derivatives:

$$\left[D^{\mu}, D^{\nu}\right] = \left[\partial^{\mu}, \partial^{\nu}\right] + V^{[\mu\nu]}. \tag{5.2}$$

In flat spacetime where $\left[\partial^{\mu}, \partial^{\nu}\right] = 0$, see (2.5), this simply boils down to

$$V^{[\mu\nu]} = V^{\mu\nu} - V^{\nu\mu} = \left[D^{\mu}, D^{\nu}\right]. \tag{5.3}$$

So $V^{[\mu\nu]}$ is synonymous with the commutator of the Yang-Mills covariant derivatives. In curved spacetime, using (5.2) to operate on a vector field G^{σ} and combining with (2.5) we obtain:

$$\left[D^{;\mu}, D^{;\nu}\right] G^{\sigma} = \left[\partial^{;\mu}, \partial^{;\nu}\right] G^{\sigma} + V^{[\mu\nu]} G^{\sigma} = \left(R_{\tau}{}^{\sigma\mu\nu} + \delta_{\tau}{}^{\sigma} V^{[\mu\nu]}\right) G^{\tau}. \tag{5.4}$$

So the anti-symmetrized $\delta_{\tau}{}^{\sigma} V^{[\mu\nu]}$ plays a role in Yang-Mills theory that is not dissimilar to that played by the Riemann tensor $R_{\tau}{}^{\sigma\mu\nu}$ in gravitational theory: each is a "curvature" measure of the degree to which the spacetime derivatives do or do not commute! Applying (5.4) to the magnetic monopole on steroids, (4.22), the curvature terms vanish as in (2.6) via $R_{\tau}{}^{\nu\sigma\mu} + R_{\tau}{}^{\sigma\mu\nu} + R_{\tau}{}^{\mu\nu\sigma} = 0$, and so we obtain simply:

$$P^{\sigma\mu\nu} = V^{[\sigma\mu]} G^{\nu} + V^{[\mu\nu]} G^{\sigma} + V^{[\nu\sigma]} G^{\mu}. \tag{5.5}$$

In (5.5), we clearly see the role of $V^{[\mu\nu]}$ as an operator: The non-vanishing magnetic monopole arises via the index-cyclical application of the antisymmetric perturbation operator $V^{[\mu\nu]}$ to the Yang-Mills gauge fields G^{σ}.

In contrast, the term corresponding to $V^{\mu\nu}$ in (3.11) and (4.6) which is derived via the ordinary, non-steroidal Euler-Lagrange equation (2.20), is $\partial^{\mu} D^{\nu} = \partial^{\mu}\partial^{\nu} - i\partial^{\mu} G^{\nu}$. It will be seen that $-i\partial^{\mu} G^{\nu}$ corresponds to the very first term in (4.17). There is no particular apparent significance to the spacetime commutator $\partial^{[\mu} D^{\nu]} = -i\partial^{[\mu} G^{\nu]}$.

Next, let us examine the behavior of $V^{\mu\nu}$ under a gauge transformation. In Yang-Mills theory, in matrix form, a gauge field G^{μ} transforms according to:

$$G^{\mu} \to G^{\mu} + \partial^{\mu}\theta + i\left[\theta, G^{\mu}\right], \tag{5.6}$$

where $\theta_{AB}\left(x^{\mu}\right) = \lambda^{i}_{AB}\theta^{i}$ is an NxN matrix for SU(N) and contains the $i = 1, 2, 3 ... N^2 - 1$ local gauge parameters $\theta^{i}\left(x^{\mu}\right)$. So $V^{\mu\nu}$ in (4.17) will transform as:

$$
\begin{aligned}
V^{\mu\nu} \to V'^{\mu\nu} &= -i\left(\partial^\mu\left(G^\nu + \partial^\nu\theta + i\left[\theta, G^\nu\right]\right) + \left(G^\mu + \partial^\mu\theta + i\left[\theta, G^\mu\right]\right)\partial^\mu\right) \\
&\quad -\left(G^\mu + \partial^\mu\theta + i\left[\theta, G^\mu\right]\right)\left(G^\nu + \partial^\nu\theta + i\left[\theta, G^\nu\right]\right) \\
&= -i\left(\partial^\mu G^\nu + G^\mu\partial^\nu\right) - G^\mu G^\nu \\
&\quad - iD^\mu\left(\partial^\nu\theta + i\left[\theta, G^\nu\right]\right) - i\left(\theta + i\left[\theta, G^\mu\right]\right)D^\nu - \left(\partial^\mu\theta + i\left[\theta, G^\mu\right]\right)\left(\partial^\nu\theta + i\left[\theta, G^\nu\right]\right)
\end{aligned}
\tag{5.7}
$$

To simplify the appearance of (5.7), we define a new four-component object:

$$
i\Lambda^\mu \equiv \partial^\mu\theta + i\left[\theta, G^\mu\right] \tag{5.8}
$$

and use this together with (4.17) to condense (5.7) down to:

$$
V^{\mu\nu} \to V'^{\mu\nu} = V^{\mu\nu} + D^\mu\Lambda^\nu + \Lambda^\nu D^\mu + \Lambda^\mu\Lambda^\nu \tag{5.9}
$$

In contrast, for the non-steroidal (3.11) and (4.6), the transformation is:

$$
-i\partial^\mu G^\nu \to -i\partial^\mu G'^\nu = -i\partial^\mu\left(G^\nu + \partial^\nu\theta + i\left[\theta, G^\nu\right]\right) = -i\partial^\mu G^\nu + \partial^\mu\Lambda^\nu . \tag{5.10}
$$

This contains only the first term of (5.9), with $D^\mu\Lambda^\nu \to \partial^\mu\Lambda^\nu$ dropped back to an ordinary derivative.

From (5.9) it is helpful to examine the gauge transformation law for the *anti*-commutator $V^{\{\mu\nu\}}$, which is:

$$
V^{\{\mu\nu\}} \to V'^{\{\mu\nu\}} = V^{\{\mu\nu\}} + D^{\{\mu}\Lambda^{\nu\}} + \Lambda^{\{\nu}D^{\mu\}} + \Lambda^{\{\mu}\Lambda^{\nu\}} . \tag{5.11}
$$

Similarly, we form the anticommutator for (5.10), which gauge transforms as:

$$
-i\partial^{\{\mu}G^{\nu\}} \to -i\partial^{\{\mu}G'^{\nu\}} = -i\partial^{\{\mu}G^{\nu\}} + \partial^{\{\mu}\Lambda^{\nu\}} . \tag{5.12}
$$

It is worth noting that these two *gauge transformations* (5.11), (5.12) have certain similarities in form to the behavior of the symmetric gravitational field $h^{\mu\nu}$ in the linear approximation of gravitational theory under a *general coordinate transformation* $x^\mu \to x^\mu + \Lambda^\mu$, which behavior is (e.g. [13], eq. [3.49]):

$$
h^{\mu\nu} \to h^{\mu\nu}{}' = h^{\mu\nu} + \partial^{\{\mu}\Lambda^{\nu\}} . \tag{5.13}
$$

Here, we see more similarity between the non-steroidal transformation (5.12) and the gravitational field transformation (5.13) than between the steroidal (5.11) and the gravitational (5.13). But this actually argues in favor of the steroids: we know that gravitational theory has nothing to do with Yang-Mills, but we also suspect that a more complete theory of gravitation should achieve some connection with non-Abelian gauge fields. If one were to employ in gravitational theory the same "full minimal coupling" that we are examining here in Yang-Mills theory and thereby change $\partial^\mu \to D^\mu = \partial^\mu - iG^\mu$ (really, $\partial^\mu \to D^\mu = \partial^\mu - iG^\mu$), then to the degree that $\partial^{\{\mu}D^{\nu\}} = \partial^{\{\mu}\partial^{\nu\}} - i\partial^{\{\mu}G^{\nu\}}$ leads to the gauge transformation law (5.12) which tracks (5.13) in form, an fully-steroidal anticommutator:

$$\left\{D^\mu, D^\nu\right\} = \left\{\partial^\mu, \partial^\nu\right\} - i\partial^{\{\mu}G^{\nu\}} - iG^{\{\mu}\partial^{\nu\}} - G^{\{\mu}G^{\nu\}} = \left\{\partial^\mu, \partial^\nu\right\} + V^{\{\mu\nu\}}. \tag{5.14}$$

leads to (5.11). This would perhaps imply that the gravitational field, in Yang-Mills theory 1) would become an operator, and 2) would transform according to

$$h^{\mu\nu} \to h^{\mu\nu}\,' = h^{\mu\nu} + D^{\{\mu}\Lambda^{\nu\}} + \Lambda^{\{\mu}D^{\nu\}} + \Lambda^{\{\mu}\Lambda^{\nu\}}, \tag{5.15}$$

i.e., that the linear gravitational field $h^{\mu\nu}$ would transform exactly the same way under a general coordinate transformation as the symmetrized perturbation $V^{\{\mu\nu\}}$ of (5.11) transforms under a gauge transformation.

It is surprising, and perhaps pregnant, that a gauge transformation acting on the spin-1 gauge fields of a symmetrized perturbation $V^{\{\mu\nu\}}$ produce the same effect as a gravitational gauge transformation acting on the linear gravitational field $h^{\mu\nu}$, and at least raises the question whether there is some deep physical connection between the symmetric perturbation $V^{\{\mu\nu\}}$ and the linear gravitational field $h^{\mu\nu}$, and between Yang-Mills theory and linear gravitational theory, each of which in their own separate domains, are non-linear theories in which spacetime derivatives are non-commuting, see also the related (5.4).

V.2 Calculation of the Fully-Minimally-Coupled Yang-Mills Inverses

With the foregoing background, it is time to calculate the inverses $I''_{\nu\tau}$ and $I'_{\nu\tau}$ of the two configuration space operators (4.21) and (4.6) which we are in the midst of comparing here. To save time and space, however, it is only really necessary to calculate the inverse specified in (4.21), because (4.6) is simply the special case of (4.21) in which $V^{\mu\nu} \to -i\partial^\mu G^\nu$, i.e., in which $V^{\mu\nu}$ is replaced by its first term only, $-i\partial^\mu G^\nu$.

So, similarly to what we did with starting at (2.11), we surmise that from studying (4.21) that the inverse will now be of the general form $I''_{\nu\tau} \equiv g_{\nu\tau}A + \partial_\nu\partial_\tau B + V_{\nu\tau}C$. We therefore place this into (4.21):

$$\left[g^{\mu\tau}\left(\partial_\sigma\partial^\sigma + V + m^2\right) - \partial^\tau\partial^\mu - V^{\tau\mu}\right]\left[g_{\tau\nu}A + \partial_\nu\partial_\tau B + V_{\nu\tau}C\right] = \delta^\mu{}_\nu. \tag{5.16}$$

As in (4.20) and (4.21), we are being very careful with left-right placement. Recognizing that *A*, *B* and a newly-required *C* will themselves be matrices, we define these to right-multiply each of

$g_{\tau\nu}, \partial_\tau\partial_\nu, V_{\tau\nu}$, respectively, thus taking them out of the middle between the two bracketed sets of terms in the above. Now we solve for *A*, *B* and *C*. As for (2.12) we first expand and then apply $\delta^\mu{}_\nu = g^{\tau\mu} g_{\tau\nu}$ and absorb the remaining metric tensors $g_{\mu\nu}$ to write:

$$\delta^\mu{}_\nu = \begin{bmatrix} \delta^\mu{}_\nu \left(\partial_\sigma\partial^\sigma + V + m^2\right) A + \left(\partial_\sigma\partial^\sigma + V + m^2\right)\partial_\nu\partial^\mu B + \left(\partial_\sigma\partial^\sigma + V + m^2\right) V_\nu{}^\mu C \\ -\left(\partial_\nu\partial^\mu + V_\nu{}^\mu\right) A - \left(\partial^\tau\partial^\mu\partial_\nu\partial_\tau + V^{\tau\mu}\partial_\nu\partial_\tau\right) B - \left(\partial^\tau\partial^\mu V_{\nu\tau} + V^{\tau\mu} V_{\nu\tau}\right) C \end{bmatrix}. \tag{5.17}$$

We match up $\delta^\mu{}_\nu$ with $\delta^\mu{}_\nu \left(\partial_\sigma\partial^\sigma + V + m^2\right)$, and after cancelling the Kronecker delta, write this matchup as $\left(\partial_\sigma\partial^\sigma + V + m^2\right) A = 1$. Then because $\partial_\sigma\partial^\sigma + V + m^2$ is an NxN Yang-Mills matrix due to *V*, we multiply from the left by $\left(\partial_\sigma\partial^\sigma + V + m^2\right)^{-1}$ to write:

$$A = \left(\partial_\sigma\partial^\sigma + V + m^2\right)^{-1} \cdot 1 = \left(D_\sigma D^\sigma + m^2\right)^{-1} \cdot 1 \equiv 1/"\left(\partial_\sigma\partial^\sigma + V + m^2\right)" = 1/"\left(D_\sigma D^\sigma + m^2\right)". \tag{5.18}$$

There is a very important point now to be made. Because $V = V^\sigma{}_\sigma = -i\left(\partial_\sigma G^\sigma + G^\sigma \partial_\sigma\right) - G_\sigma G^\sigma$ in (4.18) contains Yang-Mills matrices $G^\sigma = \lambda^i_{AB} G^{i\sigma}$ and so is an NxN matrix, we cannot blithely put a term containing $G^\sigma = G^\sigma_{AB}$ into a denominator. Rather, we must recognize that *A* above is a *matrix inverse*, and in particular, the inverse of a Yang-Mills matrix. However, as a compact notation which will allow us to compare the form of the equations presently being developed to their Abelian counterparts such as those develop in section II for electrodynamics, we shall often write the inverse M^{-1} of a matrix *M* using a "quoted" denominator defined by $1/"M" \equiv M^{-1}$. And, when we use this compact notation, we have to keep in mind that when we de-compact, the inverse will be used to multiply from the left, as in $\left(D_\sigma D^\sigma + m^2\right)^{-1} \cdot 1$.

Proceeding, we now use (5.18) in (5.17) and reduce to:

$$\frac{\partial_\nu\partial^\mu + V_\nu{}^\mu}{"\partial_\sigma\partial^\sigma + V + m^2"} = \begin{bmatrix} \left(\partial_\sigma\partial^\sigma + V + m^2\right)\left(\partial_\nu\partial^\mu B + V_\nu{}^\mu C\right) \\ -\left(\partial^\tau\partial^\mu\partial_\nu\partial_\tau + V^{\tau\mu}\partial_\nu\partial_\tau\right) B - \left(\partial^\tau\partial^\mu V_{\nu\tau} + V^{\tau\mu} V_{\nu\tau}\right) C \end{bmatrix}. \tag{5.19}$$

We see that the numerator on the left, $\partial_\nu\partial^\mu + V_\nu{}^\mu$, can be made identical to one of the terms on the right, $\partial_\nu\partial^\mu B + V_\nu{}^\mu C$, if we set *B*=*C*. Let us do just that, and rewrite (5.19) with some further consolidation as:

$$\frac{\partial_\nu\partial^\mu + V_\nu{}^\mu}{"\partial_\sigma\partial^\sigma + V + m^2"} = \left[\left(\partial_\sigma\partial^\sigma + V + m^2\right)\left(\partial_\nu\partial^\mu + V_\nu{}^\mu\right) - \left(\partial^\tau\partial^\mu + V^{\tau\mu}\right)\left(\partial_\nu\partial_\tau + V_{\nu\tau}\right)\right] B . \tag{5.20}$$

Note, if we had reversed the order of index operation when defining $G_\nu \equiv I''_{\mu\nu} J^\mu$ in (4.20) or $G_\nu \equiv I'_{\mu\nu} J^\mu$ in (4.6) we would not have matching $\partial_\nu \partial^\mu + V_\nu{}^\mu$ terms on each side of the above. We point this out in passing for now, but later, this will be a reason for symmetrizing the inverse $I''_{\nu\tau}$ in its spacetime indexes.

Now we apply (5.1), $D^\mu D^\nu = \partial^\mu \partial^\nu + V^{\mu\nu}$, so that this compacts even further:

$$\frac{D_\nu D^\mu}{"D_\sigma D^\sigma + m^2"} = \left[\left(D_\sigma D^\sigma + m^2 \right) D_\nu D^\mu - D^\tau D^\mu D_\nu D_\tau \right] B . \tag{5.21}$$

Therefore, with yet another inverse represented by a quoted denominator which left-multiplies when represented as an inverse, and renaming of μ, ν, we obtain:

$$B = C = \frac{\dfrac{D_\beta D^\alpha}{"\left(D_\sigma D^\sigma + m^2 \right) D_\beta D^\alpha - D^\tau D^\alpha D_\beta D_\tau "}}{"D_\sigma D^\sigma + m^2"} . \tag{5.22}$$

Finally, we use (5.18) and (5.22) in $I''_{\nu\tau} = g_{\nu\tau} A + \partial_\nu \partial_\tau B + V_{\nu\tau} C = g_{\nu\tau} A + D_\nu D_\tau B$ with some further index adjustments to obtain our final result:

$$I''_{\nu\tau} = \frac{g_{\nu\tau} + \dfrac{D_\nu D_\tau D^\alpha D^\beta}{"\left(D_\sigma D^\sigma + m^2 \right) D^\alpha D^\beta - D_\sigma D^\beta D^\alpha D^\sigma "}}{"D_\sigma D^\sigma + m^2"} . \tag{5.23}$$

This inverse fully incorporates all of the non-linear features in Yang-Mills gauge theory. While certainly more complex than the Abelian inverse in (2.14), one will observe that this steroidal inverse has some important similarities in its overall form.

Given (5.1), $D^\mu D^\nu = \partial^\mu \partial^\nu + V^{\mu\nu}$, let us first expand to:

$$I''_{\nu\tau} = \frac{g_{\nu\tau} + \dfrac{\left(\partial_\nu \partial_\tau + V_{\nu\tau} \right) \left(\partial^\alpha \partial^\beta + V^{\alpha\beta} \right)}{"\left(\partial_\sigma \partial^\sigma + V + m^2 \right) \left(\partial^\alpha \partial^\beta + V^{\alpha\beta} \right) - \left(\partial_\tau \partial^\beta + V_\tau{}^\beta \right) \left(\partial^\alpha \partial^\tau + V^{\alpha\tau} \right)"}}{"\partial_\sigma \partial^\sigma + V + m^2"} . \tag{5.24}$$

We can now obtain the inverse (4.6), $I'_{\tau\nu}$, which is contained within the inverse $I''_{\tau\nu}$ of (5.23). As noted in the first paragraph of this subsection prior to (5.16), we save ourselves another inverse calculation if we simply replace $V^{\mu\nu} \to -i \partial^\mu G^\nu$ to obtain:

$$I'_{\nu\tau} = \frac{g_{\nu\tau} + \dfrac{\left(\partial_\nu \partial_\tau - i\partial_\nu G_\tau\right)\left(\partial^\alpha \partial^\beta - i\partial^\alpha G^\beta\right)}{"\left(\partial_\sigma \partial^\sigma - i\partial_\sigma G^\sigma + m^2\right)\left(\partial^\alpha \partial^\beta - i\partial^\alpha G^\beta\right) - \left(\partial_\sigma \partial^\beta - i\partial_\sigma G^\beta\right)\left(\partial^\alpha \partial^\sigma - i\partial^\alpha G^\sigma\right)"}}{"\partial_\sigma \partial^\sigma - i\partial_\sigma G^\sigma + m^2"} . \tag{5.25}$$

This re-consolidates to:

$$I'_{\nu\tau} = \frac{g_{\nu\tau} + \dfrac{\partial_\nu D_\tau \partial^\alpha D^\beta}{"\left(\partial_\sigma D^\sigma + m^2\right)\partial^\alpha D^\beta - \partial_\sigma D^\beta \partial^\alpha D^\sigma"}}{"\partial_\sigma D^\sigma + m^2"} . \tag{5.26}$$

This is based on the configuration space operator $g^{\mu\nu}\left(\partial_\sigma D^\sigma + m^2\right) - \partial^\mu D^\nu$ of the field equation $J^\nu = \partial_\mu D^{[\mu} A^{\nu]}$ obtained in (4.5) from the ordinary Euler Lagrange equation, whereas (5.23) emanates from the operator $g^{\mu\nu}\left(D_\sigma D^\sigma + m^2\right) - D^\mu D^\nu$ obtained from integrating the Yang-Mills action by parts and then employed in the field equation (4.19). Contrasting (5.23) and (5.26), it is clear that this carries straight through to the inverses.

These two inverses (5.23) and (5.26) are our two candidates to employ for the inverse field equation $G_\nu \equiv I_{\mu\nu} J^\mu$, with $I_{\mu\nu} = I'_{\mu\nu}$ or $I_{\mu\nu} = I''_{\mu\nu}$. When it comes time to taking the path integral to quantize Yang-Mills theory there is no question: the action is (4.15), period. Whatever that action (4.15) produces from the path integral will be quantum Yang-Mills theory, and that action does lead to the inverse $I''_{\mu\nu}$ of (5.23). There is a good argument to be made that the configuration space operator which *must* be used in the path integral at least *ought* to be given serious consideration for use in the classical field equation. Additionally, Yang-Mills theory is well known for its producing G^4 terms in the gauge field G^μ, see for example, the Lagrangian (3.4). The inverse $I''_{\mu\nu}$ in (5.23), with terms such as, $D_\nu D_\tau D^\alpha D^\beta$, clearly contains G^4 interactions. The $I'_{\nu\tau}$ in (5.26) clearly does note. This is another argument weighing in favor of using $I''_{\mu\nu}$ and not $I'_{\nu\tau}$ in the classical field equation. A final argument in favor of $I''_{\mu\nu}$ is that the $V^{\mu\nu}$ which is contains have the complete transformation law (5.9) under Yang-Mills gauge transformations; whereas the transformation (5.10) appears truncated.

But we do not have to make a definite choice, because the inverse $I''_{\tau\nu}$ in (5.23) contains the inverse $I'_{\tau\nu}$ of (5.26) as a special case. Thus, we can work in general from (5.23), and can always consider (5.26) if we choose. So, referring to (5.23), we can achieve a substantial simplification if we impose the gauge condition $D_\sigma D^\beta D^\alpha D^\sigma = 0$ on the operator $D_\sigma D^\beta D^\alpha D^\sigma$ which represents a fourth-gauge-covariant-derivative to obtain:

$$I''_{\nu\tau} = \frac{g_{\nu\tau} + \dfrac{D_\nu D_\tau D^\alpha D^\beta}{\text{"}\left(D_\sigma D^\sigma + m^2\right) D^\alpha D^\beta\text{"}}}{\text{"}D_\sigma D^\sigma + m^2\text{"}} = \frac{g_{\nu\tau} + \dfrac{D_\nu D_\tau}{\text{"}D_\sigma D^\sigma + m^2\text{"}}}{\text{"}D_\sigma D^\sigma + m^2\text{"}}. \tag{5.27}$$

This now starts to resemble the form of the Abelian inverse (2.14), but as we shall see in section ??, the physics of this inverse has many interesting properties not seen in Abelian gauge theory. *Most importantly, this is how it is that the nuclear force is short-ranged, even though its gauge fields are massless.* In addition to the problem of confinement, explaining the nuclear short range given massless gluons is a central challenge of the Yang-Mills mass gap problem. [7] All of this will be reviewed in section ??, but for the moment, it is worth noting that even if we set the gauge mass to zero in the above, and even without any $+i\varepsilon$ term, (5.27) still has the finite, well-behaved form:

$$I''_{\nu\tau} = \frac{g_{\nu\tau} + \dfrac{D_\nu D_\tau}{\text{"}D_\sigma D^\sigma\text{"}}}{\text{"}D_\sigma D^\sigma\text{"}} = \frac{g_{\nu\tau} + \dfrac{\partial_\nu \partial_\tau + V_{\nu\tau}}{\text{"}\partial_\sigma \partial^\sigma + V\text{"}}}{\text{"}\partial_\sigma \partial^\sigma + V\text{"}} = \frac{-g_{\nu\tau} + \dfrac{k_\nu k_\tau - V_{\nu\tau}}{\text{"}V - k_\sigma k^\sigma\text{"}}}{\text{"}k_\sigma k^\sigma - V\text{"}}. \tag{5.28}$$

Comparing with (2.37) from Abelian electrodynamics, we see that when the gauge field of a Yang-Mills theory is made massless, the perturbation scalar *V* (which is a 3x3 matrix for QCD) moves into the exact same formal position in the inverse as does the non-zero m^2, and so operates as a *pseudo mass*. More precisely, when the matrix inverses are properly calculated, V, sitting where the mass sits in QED, plays a central role in generating mass eigenvalues which one should then expect to observe in the experimental meson spectrum of QCD. In this manner, one may close the "mass gap."

But because our immediate purpose is to use this inverse to populate the magnetic monopole (3.7) with quarks as we did in section III.4 but now including all the non-linearities of Yang-Mills theory, we use (5.27) in $G_\nu \equiv I''_{\mu\nu} J^\mu$ from which this inverse originates, and also use (5.1), to form:

$$\begin{aligned} G_\mu &= \frac{g_{\mu\nu} + \dfrac{D_\nu D_\mu}{\text{"}D_\sigma D^\sigma + m^2\text{"}}}{\text{"}D_\sigma D^\sigma + m^2\text{"}} J^\nu = \frac{g_{\mu\nu} + \dfrac{\partial_\nu \partial_\mu + V_{\mu\nu}}{\text{"}\partial_\sigma \partial^\sigma + V_\sigma{}^\sigma + m^2\text{"}}}{\text{"}\partial_\sigma \partial^\sigma + V_\sigma{}^\sigma + m^2\text{"}} J^\nu \\ &= \frac{g_{\mu\nu} + \dfrac{\partial_\nu \partial_\mu - i\left(\partial_\mu \mathbf{G}_\nu + \mathbf{G}_\mu \partial_\nu\right) - \mathbf{G}_\mu \mathbf{G}_\nu}{\text{"}\partial_\sigma \partial^\sigma - i\left(\partial_\sigma \mathbf{G}^\sigma + \mathbf{G}_\mu \partial^\sigma\right) - \mathbf{G}_\sigma \mathbf{G}^\sigma + m^2\text{"}}}{\text{"}\partial_\sigma \partial^\sigma - i\left(\partial_\sigma \mathbf{G}^\sigma + \mathbf{G}_\mu \partial^\sigma\right) - \mathbf{G}_\sigma \mathbf{G}^\sigma + m^2\text{"}} J^\nu \end{aligned}. \tag{5.29}$$

It should be observed that this is not a closed expression, because G_μ is self-defined *recursively* in terms of itself, as is indicated by all of the bolded $\mathbf{G}_\mu$ in the final line. To obtain a closed expression, one would have to repeatedly insert G_μ into itself, on the right hand side in the bolded $\mathbf{G}_\mu$ positions, *ad infinitum*. It may well be possible to discern the patterns and develop a closed form of (5.29), but for the moment, we simply note this recursion as yet a fourth view of the way in which Yang-Mills gauge theory is *non-linear*, *non-commuting*, and *steroidal*, in which we now see that Yang-Mills is a *recursive* field theory.

Thus far we have use the quoted denominators formulated in (5.18) for compactness. But let us now "unpack" these. Written in terms of matrix inverses, (5.29) becomes:

$$\begin{aligned} G_\mu &= \frac{g_{\mu\nu} + \dfrac{D_\nu D_\mu}{"D_\sigma D^\sigma + m^2"}}{"D_\sigma D^\sigma + m^2"} J^\nu = \left(D_\sigma D^\sigma + m^2\right)^{-1} J_\mu + \left(D_\sigma D^\sigma + m^2\right)^{-2} D_\nu D_\mu J^\nu \\ &= \left(\partial_\sigma \partial^\sigma + m^2 + V\right)^{-1} J_\mu + \left(\partial_\sigma \partial^\sigma + m^2 + V\right)^{-2} D_\nu D_\mu J^\nu \\ &= \left(-k_\sigma k^\sigma + m^2 + V\right)^{-1} J_\mu + \left(-k_\sigma k^\sigma + m^2 + V\right)^{-2} D_\nu D_\mu J^\nu \end{aligned} \quad . \tag{5.30}$$

This form of unpacking to the left originates from $A = \left(D_\sigma D^\sigma + m^2\right)^{-1} \cdot 1$ in (5.18). We have also employed (5.1) to expand somewhat, and then have converted via $\partial^\sigma \to ik^\sigma$ to momentum space.

Now, let us return to the section III.4 where we made a first pass to populate the Yang-Mills magnetic monopoles with fermions and showed how these magnetic monopoles had many symmetries reminiscent of baryons in QCD. If we identify $G_\mu = G_{\mu AB}$ with the gluons of QCD, then these must be massless, so we need to set *m*=0 above. Additionally, let us place these gluons *on-mass shell*, so that $k_\sigma k^\sigma = 0$. Ordinarily, these two actions cause problems with inverses, and require the $+i\varepsilon$ prescription. Here they do not. Rather, (5.30) merely reduces to:

$$G_\mu = V^{-1} J_\mu + V^{-2} D_\nu D_\mu J^\nu . \tag{5.31}$$

The perturbation in (4.18) is a 3x3 matrix, and as a general rule for non-zero *V* is perfectly invertible into the finite matrix $V^{-1} = \left(-i\left(\partial_\sigma G^\sigma + G^\sigma \partial_\sigma\right) - G_\sigma G^\sigma\right)^{-1}$. Extending (5.31) with $J^\mu = \overline{\psi}\gamma^\mu\psi$ from Dirac theory, see (2.17), we now write (5.31) for on-shell, massless gluons, as:

$$G^\mu = V^{-1} J_\mu + V^{-2} D_\alpha D^\mu J^\alpha = V^{-1}\overline{\psi}\gamma^\mu\psi + V^{-2} D_\alpha D^\mu \overline{\psi}\gamma^\alpha\psi = V^{-1}\overline{\psi}\gamma^\mu\psi + V^{-2}\left(\partial_\alpha\partial^\mu + V_\alpha{}^\mu\right)\overline{\psi}\gamma^\alpha\psi . \tag{5.32}$$

This is the fully-non-linear, Yang-Mills counterpart to (2.16) from Abelian gauge theory. But here, the gauge fields may be made massless and real (on-shell). When we do so, there is no need for $+i\varepsilon$; any complex or imaginary mass values will come from complex terms in the generators of the Yang-Mills theory. The inverses are all well-behaved, and so there is an inherent mass and lifetime spectrum in the above which can be used to fill the Yang-Mills mass gap, as we shall see more fully in section ??. In section III.4, we populated the Yang-Mills magnetic monopoles with fermions by throwing "caution to the winds" and substituting (2.17) which ignored perturbations, into the magnetic monopole (3.7). We are now ready to again populate the Yang-Mills magnetic monopole with fermions that we will turn into quarks. But this, time, we will omit nothing, and will account for all the non-linear aspects of Yang-Mills theory.

VI. Magnetic Monopole Baryons for On-Shell Gluons, including all the non-linear Features of Yang-Mills Theory

Our starting point for the ensuing discussion is the final line of the magnetic monopole (4.22), into which we substitute the newly-developed, fully nonlinear G^{μ} of (5.32), which applies to massless, on-shell gluons for which we have in (5.27) imposed the gauge condition $D_{\sigma}D^{\beta}D^{\alpha}D^{\sigma}=0$ on fourth-covariant-derivatives. The result is:

$$\begin{aligned}
P^{\sigma\mu\nu} &= -i\left(D^{\sigma}\left[G^{\mu},G^{\nu}\right]+D^{\mu}\left[G^{\nu},G^{\sigma}\right]+D^{\nu}\left[G^{\sigma},G^{\mu}\right]\right)\\
&= -i\begin{pmatrix}
D^{\sigma}\left[V^{-1}\overline{\psi}\gamma^{\mu}\psi+V^{-2}D_{\alpha}D^{\mu}\overline{\psi}\gamma^{\alpha}\psi,V^{-1}\overline{\psi}\gamma^{\nu}\psi+V^{-2}D_{\alpha}D^{\nu}\overline{\psi}\gamma^{\alpha}\psi\right]\\
+D^{\mu}\left[V^{-1}\overline{\psi}\gamma^{\nu}\psi+V^{-2}D_{\alpha}D^{\nu}\overline{\psi}\gamma^{\alpha}\psi,V^{-1}\overline{\psi}\gamma^{\sigma}\psi+V^{-2}D_{\alpha}D^{\sigma}\overline{\psi}\gamma^{\alpha}\psi\right]\\
+D^{\nu}\left[V^{-1}\overline{\psi}\gamma^{\sigma}\psi+V^{-2}D_{\alpha}D^{\sigma}\overline{\psi}\gamma^{\alpha}\psi,V^{-1}\overline{\psi}\gamma^{\mu}\psi+V^{-2}D_{\alpha}D^{\mu}\overline{\psi}\gamma^{\alpha}\psi\right]
\end{pmatrix}\\
&= -i\begin{pmatrix}
D^{\sigma}V^{-1}\overline{\psi}\gamma^{[\mu}\left[\psi V^{-1}\overline{\psi}\right]\gamma^{\nu]}\psi+D^{\mu}V^{-1}\overline{\psi}\gamma^{[\nu}\left[\psi V^{-1}\overline{\psi}\right]\gamma^{\sigma]}\psi+D^{\nu}V^{-1}\overline{\psi}\gamma^{[\sigma}\left[\psi V^{-1}\overline{\psi}\right]\gamma^{\mu]}\psi\\
+D^{\sigma}V^{-2}D_{\alpha}D^{[\mu}\overline{\psi}\gamma^{\alpha}\left[\psi V^{-1}\overline{\psi}\right]\gamma^{\nu]}\psi+D^{\mu}V^{-2}D_{\alpha}D^{[\nu}\overline{\psi}\gamma^{\alpha}\left[\psi V^{-1}\overline{\psi}\right]\gamma^{\sigma]}\psi+D^{\nu}V^{-2}D_{\alpha}D^{[\sigma}\overline{\psi}\gamma^{\alpha}\left[\psi V^{-1}\overline{\psi}\right]\gamma^{\mu]}\psi\\
+D^{\sigma}V^{-1}\overline{\psi}\gamma^{[\mu}\left[\psi V^{-2}D_{\alpha}D^{\nu]}\overline{\psi}\right]\gamma^{\alpha}\psi+D^{\mu}V^{-1}\overline{\psi}\gamma^{[\nu}\left[\psi V^{-2}D_{\alpha}D^{\sigma]}\overline{\psi}\right]\gamma^{\alpha}\psi+D^{\nu}V^{-1}\overline{\psi}\gamma^{[\sigma}\left[\psi V^{-2}D_{\alpha}D^{\mu]}\overline{\psi}\right]\gamma^{\alpha}\psi\\
+D^{\sigma}V^{-2}D_{\alpha}D^{[\mu}\overline{\psi}\gamma^{\alpha}\left[\psi V^{-2}D_{\alpha}D^{\nu]}\overline{\psi}\right]\gamma^{\alpha}\psi\\
+D^{\mu}V^{-2}D_{\alpha}D^{[\nu}\overline{\psi}\gamma^{\alpha}\left[\psi V^{-2}D_{\alpha}D^{\sigma]}\overline{\psi}\right]\gamma^{\alpha}\psi\\
+D^{\nu}V^{-2}D_{\alpha}D^{[\sigma}\overline{\psi}\gamma^{\alpha}\left[\psi V^{-2}D_{\alpha}D^{\mu]}\overline{\psi}\right]\gamma^{\alpha}\psi
\end{pmatrix}.
\end{aligned}\tag{6.1}$$

Structurally, because of the $D_{\alpha}D^{\mu}$ and V^{-1} terms, this is a 3x3 matrix $P_{AB}^{\sigma\mu\nu}$ of 3x3matrices emanating from $\left(D_{\alpha}D^{\mu}\right)_{CD}$ and $\left(V^{-1}\right)_{CD}$.

Now, let us go to (5.29) in the $D_{\sigma}D^{\beta}D^{\alpha}D^{\sigma}=0$ gauge, in quoted denominator form, in the limiting case where $V_{\mu\nu}=0$. Then, the quoted denominators become ordinary denominators. In flat spacetime, and with $\partial_{\nu}J^{\nu}=0$, this becomes:

$$G^{\mu}=\frac{1}{\partial_{\sigma}\partial^{\sigma}+m^{2}}J^{\mu}=-\frac{1}{k_{\sigma}k^{\sigma}-m^{2}}J^{\mu}=-\frac{1}{k_{\sigma}k^{\sigma}-m^{2}}\overline{\psi}\gamma^{\mu}\psi\,. \tag{6.2}$$

This, of course, is the inverse (2.17) of Abelian gauge theory. If we substitute this into the top line of (6.1) while dropping back $D^\sigma \to \partial^\sigma$, then as we did in (3.13), we obtain (square-bracketed terms below are to be contrasted to the same in (6.1)):

$$\begin{aligned} P^{\sigma\mu\nu} &= -i\left(\partial^\sigma\left[G^\mu, G^\nu\right] + \partial^\mu\left[G^\nu, G^\sigma\right] + \partial^\nu\left[G^\sigma, G^\mu\right]\right) \\ &= -i\frac{1}{k_\tau k^\tau - m^2}\left(\partial^\sigma \frac{\overline{\psi}\gamma^{[\mu}\left[\psi\overline{\psi}\right]\gamma^{\nu]}\psi}{k_\tau k^\tau - m^2} + \partial^\mu \frac{\overline{\psi}\gamma^{[\nu}\left[\psi\overline{\psi}\right]\gamma^{\sigma]}\psi}{k_\tau k^\tau - m^2} + \partial^\nu \frac{\overline{\psi}\gamma^{[\sigma}\left[\psi\overline{\psi}\right]\gamma^{\mu]}\psi}{k_\tau k^\tau - m^2}\right). \end{aligned} \tag{6.3}$$

Then, following all the same steps we showed in section III.4, we can turn this into (3.19), namely:

$$P^{\sigma\mu\nu} = -i\left(\partial^\sigma\left(\overline{\psi}\gamma^{[\mu}\left(\not{p} - m\right)^{-1}\gamma^{\nu]}\psi\right) + \partial^\mu\left(\overline{\psi}\gamma^{[\nu}\left(\not{p} - m\right)^{-1}\gamma^{\sigma]}\psi\right) + \partial^\nu\left(\overline{\psi}\gamma^{[\sigma}\left(\not{p} - m\right)^{-1}\gamma^{\mu]}\psi\right)\right), \tag{6.4}$$

which in turn can be converted into (3.24), namely,

$$\mathrm{Tr}P^{\sigma\mu\nu} = -i\left(\partial^\sigma\left(\overline{\psi}_R\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\psi_R\right) + \partial^\mu\left(\overline{\psi}_G\gamma^{[\nu}\left(\not{p}_G - m_G\right)^{-1}\gamma^{\sigma]}\psi_G\right) + \partial^\nu\left(\overline{\psi}_B\gamma^{[\sigma}\left(\not{p}_B - m_B\right)^{-1}\gamma^{\mu]}\psi_B\right)\right), \tag{6.5}$$

which as discussed seems to have all of the symmetries and confinement properties that one expects to find in a baryon associated with QCD.

Because the explicit introduction of QCD color going from (6.4) to (6.5) simply entails given a R, G, B color to each of the fermions and then adding a trace, we recognize that the same path will lead to the explicit introduction of color into (6.1), that is, with color added explicitly, (6.1) will become:

$$\mathrm{Tr}P^{\sigma\mu\nu}=-i\begin{pmatrix} D^{\sigma}V^{-1}\overline{\psi}_R\gamma^{[\mu}\psi_R V^{-1}\overline{\psi}_R\gamma^{\nu]}\psi_R \\ +D^{\mu}V^{-1}\overline{\psi}_G\gamma^{[\nu}\psi_G V^{-1}\overline{\psi}_G\gamma^{\sigma]}\psi_G \\ +D^{\nu}V^{-1}\overline{\psi}_B\gamma^{[\sigma}\psi_B V^{-1}\overline{\psi}_B\gamma^{\mu]}\psi_B \\ +D^{\sigma}V^{-2}D_\alpha D^{[\mu}\overline{\psi}_R\gamma^{\alpha}\psi_R V^{-1}\overline{\psi}_R\gamma^{\nu]}\psi_R \\ +D^{\mu}V^{-2}D_\alpha D^{[\nu}\overline{\psi}_G\gamma^{\alpha}\psi_G V^{-1}\overline{\psi}_G\gamma^{\sigma]}\psi_G \\ +D^{\nu}V^{-2}D_\alpha D^{[\sigma}\overline{\psi}_B\gamma^{\alpha}\psi_B V^{-1}\overline{\psi}_B\gamma^{\mu]}\psi_B \\ +D^{\sigma}V^{-1}\overline{\psi}_R\gamma^{[\mu}\psi_R V^{-2}D_\alpha D^{\nu]}\overline{\psi}_R\gamma^{\alpha}\psi_R \\ +D^{\mu}V^{-1}\overline{\psi}_G\gamma^{[\nu}\psi_G V^{-2}D_\alpha D^{\sigma]}\overline{\psi}_G\gamma^{\alpha}\psi_G \\ +D^{\nu}V^{-1}\overline{\psi}_B\gamma^{[\sigma}\psi_B V^{-2}D_\alpha D^{\mu]}\overline{\psi}_B\gamma^{\alpha}\psi_B \\ +D^{\sigma}V^{-2}D_\alpha D^{[\mu}\overline{\psi}_R\gamma^{\alpha}\psi_R V^{-2}D_\alpha D^{\nu]}\overline{\psi}_R\gamma^{\alpha}\psi_R \\ +D^{\mu}V^{-2}D_\alpha D^{[\nu}\overline{\psi}_G\gamma^{\alpha}\psi_G V^{-2}D_\alpha D^{\sigma]}\overline{\psi}_G\gamma^{\alpha}\psi_G \\ +D^{\nu}V^{-2}D_\alpha D^{[\sigma}\overline{\psi}_B\gamma^{\alpha}\psi_B V^{-2}D_\alpha D^{\mu]}\overline{\psi}_B\gamma^{\alpha}\psi_B \end{pmatrix}. \tag{6.6}$$

Structurally, we have taken one trace, namely $\mathrm{Tr}P^{\sigma\mu\nu}=P^{\sigma\mu\nu}_{AA}$, but this is still s 3x3 matrix because of $\left(D_\alpha D^\mu\right)_{CD}$ and $\left(V^{-1}\right)_{CD}$. Certainly, there are many further manipulations and reductions that might be considered from here, but for the moment, we simply point out how this retains in all aspects the $\sigma\wedge\mu\wedge\nu\sim R\wedge G\wedge B=R[G,B]+G[B,R]+B[R,G]$ color baryon wavefunction, as discussed following (4.24).

Now, we can finally answer the question posed in the introduction to this paper. In section 2 of [1], surrounding equation (2.9), the author stated as follows:

> Now, inverse [2.7] [which is a special case variation of (5.27) above] has many interesting properties which we shall not take the time to explore here which would require an entire separate paper to do them justice [we shall do those special cases justice in section ?? to follow, here] . . . . We will also note that when working towards a quantum path integral formulation, $i\left[k^\sigma,G_\sigma\right]=\partial^\sigma G_\sigma$ in (2.7) is replaced by a gauge-invariant perturbation $-V=i\left(\partial^\sigma G_\sigma+G_\sigma\partial^\sigma\right)+G^\sigma G_\sigma$, contracted from a perturbation tensor $-V^{\mu\nu}=i\left(\partial^\mu G^\nu+G^\nu\partial^\mu\right)+G^\mu G^\nu$ [sic in [1]: the *i* was omitted from the perturbation leading to some misplaced or omitted *i*'s elsewhere]. But our interest at the moment is in the low-perturbation limit, which is specified by $...\partial_\nu G_\sigma\to 0$. Thus, using (2.7) in the inverse relation $G_\nu=I_{\sigma\nu}J^\sigma$, we "turn off" all the perturbations by setting $...\partial_\nu G_\sigma=0$. When we do so, all the inverses (quoted denominators) in (2.7) become ordinary denominators. We then reduce using the fact that in momentum space, current conservation $\partial_\mu J^\mu(x)=0$ becomes $k_\mu J^\mu(k)=0$. . . . We thus obtain: $G_\nu=-g_{\sigma\nu}J^\sigma/k^\alpha k_\alpha-m^2$ [2.9]. The above is just like the expressions we encounter for inverses with a Proca mass in

QED. It says, not unexpectedly, that in the low-perturbation limit, when we set $\partial_\nu G_\sigma \to 0$ (**and in a deeper analysis**, $-V^{\mu\nu} = i\left(\partial^\mu G^\nu + G^\nu \partial^\mu\right) + G^\mu G^\nu \to 0$) QCD looks like QED.

The "deeper analysis" referred to in [1], is now explained fully by all of the foregoing development here: When we fully consider *all* of the non-linear aspects of Yang-Mills theory, we find an inverse relationship of the form (5.29) (and even more generally (5.23) if we forego the gauge condition $D_\sigma D^\beta D^\alpha D^\sigma = 0$) which is chock full of $-V^{\mu\nu} = i\left(\partial^\mu G^\nu + G^\nu \partial^\mu\right) + G^\mu G^\nu$ perturbations which reflect the essential non-linearity of Yang-Mills theory. However, if we set $V_{\mu\nu} = 0$, and in flat spacetime, the inverse is (6.2) which is the same inverse used in [1] to populate the Yang-Mills magnetic monopoles with quarks. Certainly, by populating the magnetic monopolies in this way, we are omitting some of the non-linear features of Yang-Mills theory (those which appear in (5.23) and (5.29)) while taking advantage of other non-linearities (those that appear in (6.3)).

But the question is whether there is any physically beneficial information to be gained from populating the Yang-Mills monopoles with an inverse which ignores the perturbations by setting all of the $V^{\mu\nu} = 0$. Clearly there is benefit, because: 1) We can readily see that the essential symmetries of the Yang-Mills magnetic monopoles, once populated with fermions, and even with the perturbations turned off, are the same as the essential symmetries of baryons in QCD, including the extremely important property of confinement as reviewed in detail in section III.4 here as well as in sections 1, and 12 of [1]. 2) As good fortune would have it, the energies which are derived out of the magnetic monopoles with $V^{\mu\nu} = 0$ in the inverse $I''_{\mu\nu}$ used to populate the Yang-Mills magnetic monopoles, correlate with parts per 10^5 or 10^6 precision in AMU to at least fifteen(15) distinct light nuclide binding energies as has now been demonstrated in [2], [4], [5], [6]. So we learn that nuclear binding energies – at least to the first five or six orders of precision in AMU – are not impacted at all by the perturbations $V^{\mu\nu}$, that is, that nuclear binding energies can be obtained to high precision from the energies of Yang-Mills magnetic monopoles which have the Yang-Mills perturbations of their quarks, turned off.

At the same time, if we do want to see the complete unadulterated magnetic monopole baryon with all the perturbations included, then we need look no further than (6.1) or, with color explicit, (6.6). And even (6.1) and (6.6) do embody three limitations: 1) massless gluons, 2) on-shell gluons, and the gauge condition $D_\sigma D^\beta D^\alpha D^\sigma = 0$. If one wanted to not even make these simplifications, and wanted to consider virtual (off-shell) gluons and permit $D_\sigma D^\beta D^\alpha D^\sigma \neq 0$, and even consider massive gluons, then one would use the unreduced, complete non-linear expression (5.23) to populate the magnetic monopoles with quarks via (6.1), and would arrive at an even more formidable expression than (6.1).

Finally, as will be further develop in section, as to the question ". . . these nonlinearities are essential for generating short-range forces from a zero-mass gauge field. How you expect to get short-range forces from your approach is a mystery to me" recited in the introduction, the answer is to be found in (5.28). Here, we see that even with the gluon masses set to zero, the perturbation scalar V arising from these non-linearities naturally and "spontaneously" insinuates its way into the mass position in the vector boson inverse, which means there will be non-zero pseudo-mass eigenstates observed in relation to *V* even though the gauge fields are massless. And as is well known from weak interaction theory, once a non-zero mass (or here, perturbation energy spacetime scalar which is still a 3x3 Yang-Mills matrix) makes its way into the mass position of the vector boson propagator, the resulting interactions, in this case the nuclear interaction, will have a short range.

Draft in Progress. More to be added.

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