Quantum search in a four-complex-dimensional subspace

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Abstract

For there to be $M > 1$ target items to be searched in an unsorted database of size N, with $M/N \ll 1$ for a sufficiently large N, we explore the performance of Grover's search algorithm when considering some possible situations that may arise in a four-complex-dimensional subspace, for which in the case of identical rotation angles $\phi = \theta$, we give the maximum success probabilities of finding a desired state and their corresponding numbers of Grover iterations in an approximate fashion. Our analysis reveals that the case of identical rotation angles $\phi = \theta$ is energetically favorable compared to the case $|\theta - \phi| \gg 0$ for boosting the probability to detect a desired state.

Keywords: *Grover's search algorithm; Identical rotation angles; Four-complex-dimensional subspace; Minimal polynomial.*

1 Introduction

The original Grover's search algorithm [1] exhibits a quadratic speedup over classical counterparts for searching an unsorted database, i.e., the problem of finding a single target item in an unsorted database containing N items takes $O(\sqrt{N})$ operations on a quantum computer, and it was shown to be optimal [2–4]. Farhi *et al.* [5] gave an analog analogue of the Grover's search algorithm result by studying a Hamiltonian version of the Grover problem with a time dependent Hamiltonian.

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Pati [6] recast Grover's search algorithm in a geometric language and discussed the bounds on quantum search. As explained in Refs. [7, 8], the working of Grover's search algorithm can be understood geometrically. So far, several generalizations of the original Grover's search algorithm [1] have been developed from different aspects by some modifications mainly resulted from

(a) dealing with the case of more than one desired state [9, 10];

(b) substituting *almost* any unitary transform for the Walsh-Hadamard transform [10, 11];

(c) introduction of the concept of amplitude amplification [10, 12];

(d) further speedup for repeated quantum search by means of quantum computations in parallel [13, 14];

(e) replacing the phase inversion by arbitrary phase rotations [15];

(f) allowing for an arbitrary complex initial amplitude distribution, instead of the uniform initial amplitude distribution [16, 17];

(g) investigating the case of an arbitrarily entangled initial state [18].

Long *et al.* [15] first presented the phase matching condition, i.e., identical rotation angles $\phi = \theta$, and subsequently Høyer [19] gave the exact phase condition $\tan(\phi/2) = \tan(\theta/2)(1-2/N)$. By virtue of a recursion equation, it was concluded that in order for the quantum search algorithm to apply, the two rotation angles must be equal [17]. The results of identical rotation angles $\phi = \theta$ [20] and the same general phase formula (see Eq. (15) of Ref. [21]) as given in Eq. (10) of Ref. [22] were derived again, respectively. Furthermore, Li *et al.* [23] derived the iterated formulas and the simpler approximate formulas and the precise formula for the amplitude in the desired state. The experimental implementation of phase matching has been achieved by using classical Fourier optics [24] and NMR system [25].

In Ref. [26], an algebraic approach to the analysis of Grover's search algorithm with an arbitrary initial quantum state was introduced by Shapira *et al.*, revealing a geometrical structure of the quantum search process that turns out to be confined to a four-dimensional subspace of the Hilbert space. In the present paper, by taking some possible situations that may arise in a four-complex-dimensional subspace into consideration, we deduce approximately both the maximum success probabilities of finding one of M desired states in a large database that consists of N unsorted objects and their corresponding numbers of Grover iterations for the case when $M/N \ll 1$ and $\phi = \theta$ with a different technique. We also prove that if we let the absolute difference $|\theta - \phi| \gg 0$, then this situation will lead to a deterioration of the performance of Grover's search algorithm.

2 The four-dimensional unitary matrix representation

Let M be the number of the desired states in an unsorted database of size N. Consider an initial superposition of basis states $|\gamma_0\rangle = \sum_{x=0}^{N-1} c_x |x\rangle$ for a set of complex numbers $T = \{c_0, c_1, \ldots, c_{N-1}\}$ with not all elements zero. We may write $|\gamma_0\rangle$ in the most general form

$$|\gamma_0\rangle = \cos\beta_0 |\alpha_1\rangle + e^{i\varsigma} \sin\beta_0 |\beta_1\rangle, \tag{1}$$

where ς is an arbitrary real number, $\beta_0 = \arcsin\left(\sqrt{\sum_{x\in Y} |c_x|^2}\right)$, the normalized basis vectors $|\alpha_1\rangle$ and $|\beta_1\rangle$ are given by

$$|\alpha_1\rangle = \frac{1}{\sqrt{\sum_{x\in\widetilde{Y}} |c_x|^2}} \sum_{x\in\widetilde{Y}} c_x |x\rangle$$

and

$$|\beta_1\rangle = \frac{e^{-i\varsigma}}{\sqrt{\sum_{x\in Y} |c_x|^2}} \sum_{x\in Y} c_x |x\rangle,$$

respectively. Here $\arcsin(\bullet)$ is defined as $-\pi/2 \leq \arcsin(\bullet) \leq \pi/2$, $\widetilde{Y}$ and Y are the sets of the undesired and desired states respectively, all $c_x \in T$ fulfill the normalization condition $\sum_{x\in\widetilde{Y}} |c_x|^2 + \sum_{x\in Y} |c_x|^2 = 1$ with the constraints $\sum_{x\in\widetilde{Y}} |c_x|^2 \neq 0$ and $\sum_{x\in Y} |c_x|^2 \neq 0$, and the notation $|.|$ denotes the modulus of any complex number. As to $|\gamma_0\rangle$, we shall suppose for the sake of argument that the ratio $\kappa = M/N$ and β_0 are both *small* throughout this paper.

For any two rotation angles $\phi, \theta \in (0, \pi]$, we define

$$U_\phi = I + \left(e^{i\phi} - 1\right) \sum_{x\in\, Y} |x\rangle\langle x|$$

and

$$U_\theta = I + \left(e^{i\theta} - 1\right)|\eta\rangle\langle\eta|$$

respectively, where I denotes the identity operator, i denotes the principal square root of -1, and $|\eta\rangle$ is a unit vector in the N–dimensional complex Hilbert space spanned by all the desired and undesired states and it is defined through Eq. (2). Let A be any unitary operator, and let

$$|\mu\rangle = A\,|\eta\rangle \tag{2}$$

be the result of applying A to $|\eta\rangle$. The Grover iteration $G = -AU_\theta A^{-1}U_\phi$, where the superscript $^{-1}$ refers to the inverse of an operator, then reads

$$G = -\left(I + \left(e^{i\theta} - 1\right)|\mu\rangle\langle\mu|\right)\left(I + \left(e^{i\phi} - 1\right)\sum_{x\in\ Y}|x\rangle\langle x|\right) \tag{3}$$

in view of Eq. (2).

In case that the state vector $|\mu\rangle$ lies outside the two-dimensional complex subspace L spanned by $|\alpha_1\rangle$ and $|\beta_1\rangle$, we can get the unit vector

$$|S\rangle = \frac{1}{\sqrt{1 - |\langle\alpha_1\mid\mu\rangle|^2 - |\langle\beta_1\mid\mu\rangle|^2}}\Big(|\mu\rangle - \langle\alpha_1\mid\mu\rangle\,|\alpha_1\rangle - \langle\beta_1\mid\mu\rangle\,|\beta_1\rangle\Big)$$

that is perpendicular to L by means of the Gram-Schmidt orthogonalization process. We now set

$$|\mu\rangle = \sin\omega\cos\varphi_1 e^{it_1}\,|\alpha_1\rangle + \sin\omega\sin\varphi_1 e^{it_2}\,|\beta_1\rangle + \cos\omega e^{it_3}\,|S\rangle\,, \tag{4}$$

where $\varphi_1 \in (0,\beta_0]$, $\omega \in (0,\pi/2]$, and t_1, t_2, t_3 are arbitrary real numbers. In general, for the case when $\omega \neq \pi/2$ and there are more than one desired state to be searched, the third component vector on the right-hand side of Eq. (4), corresponding to some sequence of complex numbers $d_0, d_1, \cdots, d_{N-1}$ with $\sum_{x=0}^{N-1}|d_x|^2 = 1$, $\sum_{x\in\widetilde{Y}} d_x\overline{c_x} = 0$ and $\sum_{x\in Y} d_x\overline{c_x} = 0$, has a further decomposition

$$|S\rangle = \cos\varphi_2\,|\alpha_2\rangle + e^{(-it_3+it_4)}\sin\varphi_2\,|\beta_2\rangle\,, \tag{5}$$

where $\overline{\bullet}$ denotes the complex conjugate of a complex number, t_4 is an arbitrary real number,

$$|\beta_2\rangle = \frac{e^{(it_3-it_4)}}{\sqrt{\sum_{x\in Y}|d_x|^2}}\sum_{x\in Y} d_x\,|x\rangle \tag{6}$$

and

$$|\alpha_2\rangle = \frac{1}{\sqrt{\sum_{x\in\widetilde{Y}} |d_x|^2}} \sum_{x\in\widetilde{Y}} d_x |x\rangle$$

are the normalized superposition of the desired states and that of the remaining ones, and here, to allow for the possibility of the change of $|S\rangle$, for all $\varphi_1 \in (0,\beta_0]$ and all $\omega \in (0,\pi/2)$ we let the angle

$$\varphi_2 = \arcsin\left(\sqrt{\sum_{x\in Y} |d_x|^2}\right) \neq 0 \tag{7}$$

satisfy the relation

$$\sin^2\omega \sin^2\varphi_1 + \cos^2\omega \sin^2\varphi_2 \leq \sin^2\beta_0 \tag{8}$$

which holds also for $\omega = \pi/2$, meaning that the probability to measure a desired state in $|\mu\rangle$ as defined by Eq. (4) is not greater than the probability to measure a desired state in $|\gamma_0\rangle$.

Parenthetically, we should note that for any given $\varphi_1 \in (0,\beta_0]$, the possible values of φ_2 depend upon the magnitude of the angle ω. In accordance with inequality (8), it is straightforward to show that φ_2 is also small when ω is small, but it is very likely to be considerably large as ω approaches $\pi/2$. In addition, when $\omega = \pi/2$, $|S\rangle$ does not exist; however, $|\mu\rangle$ can always be expressed in terms of $|\alpha_1\rangle$ and $|\beta_1\rangle$ as $|\mu\rangle = \cos\varphi_1 e^{it_1} |\alpha_1\rangle + \sin\varphi_1 e^{it_2} |\beta_1\rangle$, independent of the existence of $|S\rangle$ and the choice of φ_2. For this reason we will be free from Eq. (7) and take φ_2 to be an arbitrary real number for such an exceptional case.

Thus, Eq. (4) by substitution now takes the form

$$|\mu\rangle = \sin\omega\cos\varphi_1 e^{it_1} |\alpha_1\rangle + \sin\omega\sin\varphi_1 e^{it_2} |\beta_1\rangle + \cos\omega\cos\varphi_2 e^{it_3} |\alpha_2\rangle + \cos\omega\sin\varphi_2 e^{it_4} |\beta_2\rangle . \tag{9}$$

By exploiting Eq. (9), the matrix representation of G in Eq. (3) relative to the ordered orthonormal basis $\{|\alpha_1\rangle, |\beta_1\rangle, |\alpha_2\rangle, |\beta_2\rangle\}$ is computed to be

$$Qt = \begin{pmatrix} Qt_{11} & Qt_{12} & Qt_{13} & Qt_{14} \\ Qt_{21} & Qt_{22} & Qt_{23} & Qt_{24} \\ Qt_{31} & Qt_{32} & Qt_{33} & Qt_{34} \\ Qt_{41} & Qt_{42} & Qt_{43} & Qt_{44} \end{pmatrix}, \tag{10}$$

whose entries are

$$
\begin{aligned}
Qt_{11} &= -1 + \lambda \sin^2 \omega \cos^2 \varphi_1, \\
Qt_{12} &= \lambda e^{i\phi} e^{(it_1 - it_2)} \sin^2 \omega \sin \varphi_1 \cos \varphi_1, \\
Qt_{13} &= \lambda e^{(it_1 - it_3)} \sin \omega \cos \omega \cos \varphi_1 \cos \varphi_2, \\
Qt_{14} &= \lambda e^{i\phi} e^{(it_1 - it_4)} \sin \omega \cos \omega \cos \varphi_1 \sin \varphi_2, \\
Qt_{21} &= \lambda e^{(-it_1 + it_2)} \sin^2 \omega \sin \varphi_1 \cos \varphi_1, \\
Qt_{22} &= \lambda e^{i\phi} \sin^2 \omega \sin^2 \varphi_1 - e^{i\phi}, \\
Qt_{23} &= \lambda e^{(it_2 - it_3)} \sin \omega \cos \omega \sin \varphi_1 \cos \varphi_2, \\
Qt_{24} &= \lambda e^{i\phi} e^{(it_2 - it_4)} \sin \omega \cos \omega \sin \varphi_1 \sin \varphi_2, \\
Qt_{31} &= \lambda e^{(-it_1 + it_3)} \sin \omega \cos \omega \cos \varphi_1 \cos \varphi_2, \\
Qt_{32} &= \lambda e^{i\phi} e^{(-it_2 + it_3)} \sin \omega \cos \omega \sin \varphi_1 \cos \varphi_2, \\
Qt_{33} &= -1 + \lambda \cos^2 \omega \cos^2 \varphi_2, \\
Qt_{34} &= \lambda e^{i\phi} e^{(it_3 - it_4)} \cos^2 \omega \sin \varphi_2 \cos \varphi_2, \\
Qt_{41} &= \lambda e^{(-it_1 + it_4)} \sin \omega \cos \omega \cos \varphi_1 \sin \varphi_2, \\
Qt_{42} &= \lambda e^{i\phi} e^{(-it_2 + it_4)} \sin \omega \cos \omega \sin \varphi_1 \sin \varphi_2, \\
Qt_{43} &= \lambda e^{(-it_3 + it_4)} \cos^2 \omega \sin \varphi_2 \cos \varphi_2, \\
Qt_{44} &= \lambda e^{i\phi} \cos^2 \omega \sin^2 \varphi_2 - e^{i\phi},
\end{aligned}
$$

where $\lambda = 1 - e^{i\theta}$.

3 The performance of Grover's search algorithm in the four-dimensional complex subspace

Theorem 1 *Let $|\gamma_0\rangle$ and $|\mu\rangle$ be defined as in Eqs. (1) and (9), respectively, and suppose that N is sufficiently large and $M \ll N$. Then, in the case of $\phi = \theta \in (0, \pi]$, the maximum success probability of Grover's search algorithm is approximately equal to $1/\left(1 + \cot^2 \omega \cos^2 \varphi_2\right)$ for any fixed $\omega \in (0, \pi/2)$, any $\varphi_1 \in (0, \beta_0]$, and all possible values of φ_2 satisfying the condition of inequality (8).*

Proof From inequality (8), it follows that

$$\sin \omega \cos \omega \sin \varphi_1 \sin \varphi_2 \leq \sin^2 \beta_0 / 2 \tag{11}$$

and

$$\cos^2\omega\sin^2\varphi_2 < \sin^2\beta_0 \tag{12}$$

for all $\varphi_1 \in (0,\beta_0]$ and all $\omega \in (0,\pi/2)$. Neglecting the higher-order terms of φ_1 and the product terms associated respectively with the two quantities on the left-hand sides of the above inequalities in the elements of the unitary matrix Qt given in Eq. (10), we may approximately write this matrix as

$$Qt \doteq Qt' = \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned}
q_{11} &= -1+\lambda\sin^2\omega, & q_{12} &= \varphi_1\lambda e^{i\phi}e^{(it_1-it_2)}\sin^2\omega,\\
q_{13} &= \lambda e^{(it_1-it_3)}\sin\omega\cos\omega\cos\varphi_2, & q_{14} &= \lambda e^{i\phi}e^{(it_1-it_4)}\sin\omega\cos\omega\sin\varphi_2,\\
q_{21} &= \varphi_1\lambda e^{(-it_1+it_2)}\sin^2\omega, & q_{22} &= -e^{i\phi},\\
q_{23} &= \varphi_1\lambda e^{(it_2-it_3)}\sin\omega\cos\omega\cos\varphi_2, & q_{24} &= 0,\\
q_{31} &= \lambda e^{(-it_1+it_3)}\sin\omega\cos\omega\cos\varphi_2, & q_{32} &= \varphi_1\lambda e^{i\phi}e^{(-it_2+it_3)}\sin\omega\cos\omega\cos\varphi_2,\\
q_{33} &= -1+\lambda\cos^2\omega, & q_{34} &= \lambda e^{i\phi}e^{(it_3-it_4)}\cos^2\omega\sin\varphi_2\cos\varphi_2,\\
q_{41} &= \lambda e^{(-it_1+it_4)}\sin\omega\cos\omega\sin\varphi_2, & q_{42} &= 0,\\
q_{43} &= \lambda e^{(-it_3+it_4)}\cos^2\omega\sin\varphi_2\cos\varphi_2, & q_{44} &= -e^{i\phi}.
\end{aligned}$$

It is convenient to multiply the matrix Qt' by a global phase factor $-e^{-i(\theta+\phi)/2}$ so that Eq. (13) can be rewritten in a tractable form

$$Q \doteq -e^{-i(\theta+\phi)/2}Qt' = Q_1 - \Omega Q_2 - RQ_3\,, \tag{14}$$

where

$$Q_1 = \begin{pmatrix} e^{i(\theta-\phi)/2} & 0 & 0 & 0 \\ 0 & e^{-i(\theta-\phi)/2} & 0 & 0 \\ 0 & 0 & e^{i(\theta-\phi)/2} & 0 \\ 0 & 0 & 0 & e^{-i(\theta-\phi)/2} \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & e^{i\xi_1} & 0 & \dfrac{e^{i\xi_2}\cot\omega\sin\varphi_2}{\varphi_1} \\ -e^{-i\xi_1} & 0 & -e^{-i\xi_3}\cot\omega\cos\varphi_2 & 0 \\ 0 & e^{i\xi_3}\cot\omega\cos\varphi_2 & 0 & \dfrac{e^{i\xi_4}\cot^2\omega\sin(2\varphi_2)}{2\varphi_1} \\ -\dfrac{e^{-i\xi_2}\cot\omega\sin\varphi_2}{\varphi_1} & 0 & -\dfrac{e^{-i\xi_4}\cot^2\omega\sin(2\varphi_2)}{2\varphi_1} & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} -\cos^2\omega & 0 & e^{(it_1-it_3)}\sin\omega\cos\omega\cos\varphi_2 & 0 \\ 0 & 0 & 0 & 0 \\ e^{(-it_1+it_3)}\sin\omega\cos\omega\cos\varphi_2 & 0 & -\sin^2\omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R = e^{-i(\theta+\phi)/2} - e^{i(\theta-\phi)/2},$$

and Ω is given by Eq. (16), where ξ_1, ξ_2, ξ_3, ξ_4, and Ω are defined through

$$\begin{cases} -\varphi_1\left(e^{i(\theta+\phi)/2} - e^{-i(\theta-\phi)/2}\right)e^{(it_1-it_2)}\sin^2\omega = \Omega e^{i\xi_1} \\ -\varphi_1\left(e^{i(\theta+\phi)/2} - e^{-i(\theta-\phi)/2}\right)e^{(it_1-it_4)}\sin^2\omega = \Omega e^{i\xi_2} \\ -\varphi_1\left(e^{i(\theta+\phi)/2} - e^{-i(\theta-\phi)/2}\right)e^{(it_3-it_2)}\sin^2\omega = \Omega e^{i\xi_3} \\ -\varphi_1\left(e^{i(\theta+\phi)/2} - e^{-i(\theta-\phi)/2}\right)e^{(it_3-it_4)}\sin^2\omega = \Omega e^{i\xi_4} \end{cases}, \tag{15}$$

whence

$$\Omega = \varphi_1\sin^2\omega\sqrt{2\left(1-\cos\theta\right)} \tag{16}$$

and

$$\xi_1 - \xi_3 = \xi_2 - \xi_4 = t_1 - t_3. \tag{17}$$

By applying inequality (12) we find that

$$(Q_3)^2 \doteq -Q_3, \tag{18}$$

and furthermore, that

$$\Omega R Q_2 Q_3 \doteq O \qquad \text{and} \qquad \Omega R Q_3 Q_2 \doteq O \tag{19}$$

with the aid of Eq. (17), where O denotes a 4×4 zero matrix.

In the special case when $\phi = \theta$, Eq. (14) becomes

$$Q_{\phi=\theta} \doteq I - \Omega Q_2' - R' Q_3 \approx e^{-\Omega Q_2'} - R' Q_3, \tag{20}$$

in which $R' = -1 + e^{-i\theta}$, and

$$Q_2' = \begin{pmatrix} 0 & e^{i\xi_1'} & 0 & \dfrac{e^{i\xi_2'}\cot\omega\sin\varphi_2}{\varphi_1} \\ -e^{-i\xi_1'} & 0 & -e^{-i\xi_3'}\cot\omega\cos\varphi_2 & 0 \\ 0 & e^{i\xi_3'}\cot\omega\cos\varphi_2 & 0 & \dfrac{e^{i\xi_4'}\cot^2\omega\sin(2\varphi_2)}{2\varphi_1} \\ -\dfrac{e^{-i\xi_2'}\cot\omega\sin\varphi_2}{\varphi_1} & 0 & -\dfrac{e^{-i\xi_4'}\cot^2\omega\sin(2\varphi_2)}{2\varphi_1} & 0 \end{pmatrix}, \tag{21}$$

where ξ_1', ξ_2', ξ_3', and ξ_4' are defined via

$$\begin{cases} -\varphi_1\left(e^{i\theta}-1\right)e^{(it_1-it_2)}\sin^2\omega = \Omega e^{i\xi_1'} \\ -\varphi_1\left(e^{i\theta}-1\right)e^{(it_1-it_4)}\sin^2\omega = \Omega e^{i\xi_2'} \\ -\varphi_1\left(e^{i\theta}-1\right)e^{(it_3-it_2)}\sin^2\omega = \Omega e^{i\xi_3'} \\ -\varphi_1\left(e^{i\theta}-1\right)e^{(it_3-it_4)}\sin^2\omega = \Omega e^{i\xi_4'} \end{cases}$$

which follows from Eq. (15) putting $\phi = \theta$. Likewise for this case

$$\xi_1' - \xi_3' = \xi_2' - \xi_4' = t_1 - t_3, \tag{22}$$

$$\Omega R' Q_2' Q_3 \doteq O \qquad \text{and} \qquad \Omega R' Q_3 Q_2' \doteq O \tag{23}$$

can also be obtained. As a result of the foregoing calculations, for any positive integer $j \geq 1$, $Q^j_{\phi=\theta}$ can now be compactly expressed as

$$Q^j_{\phi=\theta} \doteq e^{-j\Omega Q_2'} - \left(\left(R'+1\right)^j - 1\right) Q_3. \tag{24}$$

To calculate $Q^j_{\phi=\theta}$ we need to determine the matrix exponential $e^{-j\Omega Q_2'}$. To do this, taking Eq. (22) into account and letting the characteristic polynomial of the complex skew-symmetric matrix Q_2' equal to zero, i.e., the determinant $\det(Q_2' - \rho I) = 0$, yields the following eigenvalues of Q_2': $\rho_1 = \rho_2 = 0$, $\rho_3 = i\sqrt{\tau}$, and $\rho_4 = -i\sqrt{\tau}$, where

$$\tau = \left(1 + \cot^2\omega\cos^2\varphi_2\right)\left(1 + \frac{\cot^2\omega\sin^2\varphi_2}{\varphi_1^2}\right). \tag{25}$$

Since

$$Q_2'\left(Q_2'-\rho_3 I\right)\left(Q_2'-\rho_4 I\right)=\left(Q_2'\right)^3+\rho_3\rho_4 Q_2'=O$$

ensures that the minimal polynomial of Q_2' takes on the form $\rho\left(\rho-\rho_3\right)\left(\rho-\rho_4\right)$, we may set

$$e^{-j\Omega\rho}=a+b\rho+c\rho^2 \tag{26}$$

with undetermined coefficients a, b, and c. Upon substitution of the values of ρ_1, ρ_3 and ρ_4 into Eq. (26), respectively, we arrive at a nonhomogeneous system of linear equations

$$\begin{cases} e^{-j\Omega\rho_1}=a+b\rho_1+c\left(\rho_1\right)^2 \\ e^{-j\Omega\rho_3}=a+b\rho_3+c\left(\rho_3\right)^2 \\ e^{-j\Omega\rho_4}=a+b\rho_4+c\left(\rho_4\right)^2 \end{cases},$$

from which, we obtain $a=1$, $b=-\sin\left(j\Omega\sqrt{\tau}\right)/\sqrt{\tau}$, and $c=\left(1-\cos\left(j\Omega\sqrt{\tau}\right)\right)/\tau$. As a consequence, substituting Q_2' for ρ in Eq. (26) directly gives

$$e^{-j\Omega Q_2'}=aI+bQ_2'+c\left(Q_2'\right)^2.$$

Accordingly, we appeal to this result and Eq. (22), and some rearrangement, to deduce that

$$Q^j_{\phi=\theta}\doteq\begin{pmatrix} qq_{11} & qq_{12} & qq_{13} & qq_{14} \\ qq_{21} & qq_{22} & qq_{23} & qq_{24} \\ qq_{31} & qq_{32} & qq_{33} & qq_{34} \\ qq_{41} & qq_{42} & qq_{43} & qq_{44} \end{pmatrix}, \tag{27}$$

where

$$qq_{11} = \left(e^{-i\theta j} - 1\right)\cos^2\omega + \frac{\cot^2\omega\cos^2\varphi_2 + \cos\left(j\Omega\sqrt{\tau}\right)}{1+\cot^2\omega\cos^2\varphi_2},$$
$$qq_{12} = -e^{i\xi_1'}\sin\left(j\Omega\sqrt{\tau}\right)/\sqrt{\tau},$$
$$qq_{13} = -e^{\left(i\xi_1' - i\xi_3'\right)}\left(\left(e^{-i\theta j}-1\right)\sin\omega\cos\omega\cos\varphi_2 + \frac{\cot\omega\cos\varphi_2\left(1-\cos\left(j\Omega\sqrt{\tau}\right)\right)}{1+\cot^2\omega\cos^2\varphi_2}\right),$$
$$qq_{14} = -e^{i\xi_2'}\sin\left(j\Omega\sqrt{\tau}\right)\cot\omega\sin\varphi_2/\left(\varphi_1\sqrt{\tau}\right),$$
$$qq_{21} = e^{-i\xi_1'}\sin\left(j\Omega\sqrt{\tau}\right)/\sqrt{\tau},$$
$$qq_{22} = 1 - \left(1-\cos\left(j\Omega\sqrt{\tau}\right)\right)/\left(1+\cot^2\omega\sin^2\varphi_2/\varphi_1^2\right),$$
$$qq_{23} = e^{-i\xi_3'}\cot\omega\cos\varphi_2\sin\left(j\Omega\sqrt{\tau}\right)/\sqrt{\tau},$$
$$qq_{24} = -e^{\left(-i\xi_1'+i\xi_2'\right)}\frac{\left(1-\cos\left(j\Omega\sqrt{\tau}\right)\right)\cot\omega\sin\varphi_2}{\left(1+\cot^2\omega\sin^2\varphi_2/\varphi_1^2\right)\varphi_1},$$
$$qq_{31} = -e^{\left(-i\xi_1'+i\xi_3'\right)}\left(\left(e^{-i\theta j}-1\right)\sin\omega\cos\omega\cos\varphi_2 + \frac{\cot\omega\cos\varphi_2\left(1-\cos\left(j\Omega\sqrt{\tau}\right)\right)}{1+\cot^2\omega\cos^2\varphi_2}\right),$$
$$qq_{32} = -e^{i\xi_3'}\cot\omega\cos\varphi_2\sin\left(j\Omega\sqrt{\tau}\right)/\sqrt{\tau},$$
$$qq_{33} = \left(e^{-i\theta j}-1\right)\sin^2\omega + \frac{1+\cos\left(j\Omega\sqrt{\tau}\right)\cot^2\omega\cos^2\varphi_2}{1+\cot^2\omega\cos^2\varphi_2},$$
$$qq_{34} = -e^{i\xi_4'}\sin\left(j\Omega\sqrt{\tau}\right)\cot^2\omega\sin\varphi_2\cos\varphi_2/\left(\varphi_1\sqrt{\tau}\right),$$
$$qq_{41} = e^{-i\xi_2'}\sin\left(j\Omega\sqrt{\tau}\right)\cot\omega\sin\varphi_2/\left(\varphi_1\sqrt{\tau}\right),$$
$$qq_{42} = -e^{\left(i\xi_1'-i\xi_2'\right)}\frac{\left(1-\cos\left(j\Omega\sqrt{\tau}\right)\right)\cot\omega\sin\varphi_2}{\left(1+\cot^2\omega\sin^2\varphi_2/\varphi_1^2\right)\varphi_1},$$
$$qq_{43} = e^{-i\xi_4'}\sin\left(j\Omega\sqrt{\tau}\right)\cot^2\omega\sin\varphi_2\cos\varphi_2/\left(\varphi_1\sqrt{\tau}\right),$$
$$qq_{44} = \frac{1+\cos\left(j\Omega\sqrt{\tau}\right)\cot^2\omega\sin^2\varphi_2/\varphi_1^2}{1+\cot^2\omega\sin^2\varphi_2/\varphi_1^2}.$$

It then follows that

$$\langle\beta_1|\,G^j_{\phi=\theta}\,|\gamma_0\rangle \doteq \frac{e^{-i\xi_1'}\sin\left(j\Omega\sqrt{\tau}\right)\cos\beta_0}{\sqrt{\tau}} + e^{i\varsigma}\sin\beta_0\left(1-\frac{1-\cos\left(j\Omega\sqrt{\tau}\right)}{1+\cot^2\omega\sin^2\varphi_2/\varphi_1^2}\right) \tag{28}$$

and

$$\langle\beta_2|\,G^j_{\phi=\theta}\,|\gamma_0\rangle \doteq \frac{e^{-i\xi_2'}\cot\omega\sin\varphi_2}{\varphi_1}\left(\frac{\sin\left(j\Omega\sqrt{\tau}\right)\cos\beta_0}{\sqrt{\tau}} - \frac{e^{\left(i\varsigma+i\xi_1'\right)}\sin\beta_0\left(1-\cos\left(j\Omega\sqrt{\tau}\right)\right)}{1+\cot^2\omega\sin^2\varphi_2/\varphi_1^2}\right). \tag{29}$$

Obviously, when $j\Omega\sqrt{\tau}\to\pi/2$, then in this case the second terms on the right-hand sides of Eqs. (28) and (29) are completely negligible for sufficiently small β_0, so after

$$J_{(\omega,\phi=\theta)} \doteq \left\lfloor \pi/\left(2\Omega\sqrt{\tau}\right)\right\rfloor \tag{30}$$

iterations of $G_{\phi=\theta}$, $\lfloor z\rfloor$ representing the largest integer which is smaller than z, the maximum success probability of Grover's search algorithm corresponding to the case of identical rotation angles $\phi=\theta$ is given approximately by

$$\mathbf{P}_{\max}\left(j=J_{(\omega,\phi=\theta)}\right) \doteq \left|\langle\beta_1|\,G^j_{\phi=\theta}\,|\gamma_0\rangle\right|^2 + \left|\langle\beta_2|\,G^j_{\phi=\theta}\,|\gamma_0\rangle\right|^2 \doteq \frac{1}{1+\cot^2\omega\cos^2\varphi_2}, \tag{31}$$

where we have used Eq. (25).

Theorem 1 follows. □

It remains to see that Eqs. (30) and (31) also hold true for $\omega=\pi/2$. We now list a few immediate consequences of Theorem 1.

Corollary 1 *Let N be sufficiently large and let $M\ll N$.*

(i) *We have*

$$J_{(\omega=\pi/2,\phi=\theta)} \doteq \left\lfloor \pi/\left(4\varphi_1\sin(\theta/2)\right)\right\rfloor \tag{32}$$

and

$$J_{(\omega=\omega',\phi=\theta)} \doteq \left\lfloor J_{(\omega=\pi/2,\phi=\theta)}/\left(\sqrt{\tau_{\omega=\omega'}}\sin^2\omega'\right)\right\rfloor \tag{33}$$

for any given $\omega'\in(0,\pi/2)$ and $\varphi_1\in(0,\beta_0]$, where $\tau_{\omega=\omega'}$ is defined by Eq. (25).

(ii) *Given any $\varphi_1\in(0,\beta_0]$, if φ_2 tends to zero for any fixed $\omega'\in(0,\pi/2)$, then*

$$J_{(\omega=\omega',\phi=\theta,\varphi_2\to 0)} \doteq \left\lfloor J_{(\omega=\pi/2,\phi=\theta)}/\sin\omega'\right\rfloor \tag{34}$$

and

$$\boldsymbol{P}_{\max}\left(j = J_{(\omega=\omega',\phi=\theta,\varphi_2\to 0)}\right) \doteq \sin^2\omega'. \tag{35}$$

(iii) *Given any φ_1 and ω' as above, if there are two distinct values $\varphi_{2(1)}$, $\varphi_{2(2)}$ such that $\cos\omega'\sin\varphi_{2(1)} = \varphi_1\sin\omega'$, and $\varphi_{2(2)}\to 0$, then*

$$J_{\left(\omega=\omega',\phi=\theta,\varphi_2=\varphi_{2(1)}\right)} \doteq \left\lfloor J_{\left(\omega=\omega',\phi=\theta,\varphi_2=\varphi_{2(2)}\right)}/\sqrt{2}\right\rfloor \tag{36}$$

and

$$\boldsymbol{P}_{\max}\left(j = J_{\left(\omega=\omega',\phi=\theta,\varphi_2=\varphi_{2(1)}\right)}\right) \doteq \boldsymbol{P}_{\max}\left(j = J_{\left(\omega=\omega',\phi=\theta,\varphi_2=\varphi_{2(2)}\right)}\right) \doteq \sin^2\omega'. \tag{37}$$

(iv) *If ω' is in the range $[\pi/2-\beta_0,\pi/2)$, then for any φ_1 and all possible values of φ_2 satisfying inequality (8) we have that*

$$J_{(\omega=\omega',\phi=\theta)} \doteq \left\lfloor J_{(\omega=\pi/2,\phi=\theta)}/\left(1+\kappa_1^2\sin^2\varphi_2\right)^{1/2}\right\rfloor \tag{38}$$

with $\kappa_1 = (\pi/2-\omega')/\varphi_1$, and moreover that from the order-of-magnitude standpoint, $\boldsymbol{P}_{\max}\left(j = J_{(\omega=\omega',\phi=\theta)}\right)$ and $\boldsymbol{P}_{\max}\left(j = J_{(\omega=\pi/2,\phi=\theta)}\right)$ are equivalent.

Proof **(i)** It follows from Eq. (30) that when $\omega = \pi/2$,

$$J_{(\omega=\pi/2,\phi=\theta)} \doteq \left\lfloor \pi/\left(4\varphi_1\sin(\theta/2)\right)\right\rfloor \doteq \pi/\left(4\varphi_1\sin(\theta/2)\right),$$

and hence that

$$J_{(\omega=\omega',\phi=\theta)} \doteq \left\lfloor \pi/\left(4\varphi_1\sin(\theta/2)\sqrt{\tau_{\omega=\omega'}}\sin^2\omega'\right)\right\rfloor \doteq \left\lfloor J_{(\omega=\pi/2,\phi=\theta)}/\left(\sqrt{\tau_{\omega=\omega'}}\sin^2\omega'\right)\right\rfloor,$$

where we have used Eqs. (16) and (25) and the trigonometric identity

$$\sin(\theta/2) = \sqrt{(1-\cos\theta)/2}.$$

(ii) Since, for all $\omega'\in(0,\pi/2)$, $\sqrt{\tau_{\omega=\omega'}}\to 1/\sin\omega'$ as $\varphi_2\to 0$, formulae (34) and (35) follow from Eqs. (33) and (31), respectively. These are, of course, identical to the results given previously [27, 28].

(iii) Eqs. (33) and (31) combined with Eq. (25) and the condition $\cos\omega' \sin\varphi_{2(1)} = \varphi_1 \sin\omega'$ can yield

$$J_{\left(\omega=\omega',\phi=\theta,\varphi_2=\varphi_{2(1)}\right)} \doteq \left\lfloor J_{(\omega=\pi/2,\phi=\theta)} / \left(\sqrt{2}\sin\omega'\right) \right\rfloor$$

and $\mathbf{P}_{\max}\left(j = J_{\left(\omega=\omega',\phi=\theta,\varphi_2=\varphi_{2(1)}\right)}\right) \doteq \sin^2\omega'$. Comparing with the relations (34) and (35) for $\varphi_2 = \varphi_{2(2)} \to 0$, we arrive at the desired equations (36) and (37).

(iv) This follows immediately from Eqs. (33) and (31) respectively with the neglect of second and higher order terms of $\pi/2 - \omega'$. □

Theorem 2 *Suppose that N is sufficiently large and $M \ll N$. For any $\omega \in (0, \pi/2]$ and any $\theta, \phi \in (0, \pi]$, if $2\Delta = |\theta - \phi| \gg 0$, then the Grover's search algorithm deteriorates.*

Proof Taking advantage of the formula (14) together with Eqs. (18) and (19) we can, through the neglect of the respective higher-order terms of φ_1 and $\cos\omega\sin\varphi_2$, and the terms involving the mixed product of these two quantities, i.e. $\varphi_1 \cos\omega\sin\varphi_2$, approximately obtain

$$Q^j \approx Qq - \left(\left(R + e^{i\Delta}\right)^j - \left(e^{i\Delta}\right)^j\right) Q_3, \tag{39}$$

where

$$Qq = \begin{pmatrix} Qq_{11} & Qq_{12} & Qq_{13} & Qq_{14} \\ Qq_{21} & Qq_{22} & Qq_{23} & Qq_{24} \\ Qq_{31} & Qq_{32} & Qq_{33} & Qq_{34} \\ Qq_{41} & Qq_{42} & Qq_{43} & Qq_{44} \end{pmatrix},$$

where

$$
\begin{aligned}
&Qq_{11} = e^{ij\Delta}, && Qq_{12} = -\Omega e^{i\xi_1} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right),\\
&Qq_{13} = 0, && Qq_{14} = -\frac{\Omega\cot\omega\sin\varphi_2}{\varphi_1} e^{i\xi_2} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right),\\
&Qq_{21} = \Omega e^{-i\xi_1} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right), && Qq_{22} = e^{-ij\Delta},\\
&Qq_{23} = \Omega\cot\omega\cos\varphi_2 e^{-i\xi_3} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right), && Qq_{24} = 0,\\
&Qq_{31} = 0, && Qq_{32} = -\Omega\cot\omega\cos\varphi_2 e^{i\xi_3} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right),\\
&Qq_{33} = e^{ij\Delta}, && Qq_{34} = -\frac{\Omega\cot^2\omega\sin(2\varphi_2)}{2\varphi_1} e^{i\xi_4} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right),\\
&Qq_{41} = \frac{\Omega\cot\omega\sin\varphi_2}{\varphi_1} e^{-i\xi_2} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right), && Qq_{42} = 0,\\
&Qq_{43} = \frac{\Omega\cot^2\omega\sin(2\varphi_2)}{2\varphi_1} e^{-i\xi_4} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right), && Qq_{44} = e^{-ij\Delta}.
\end{aligned}
$$

Hence, when G^j is applied to the initial superposition $|\gamma_0\rangle$ given by Eq. (1), the success probability of Grover's search algorithm $\mathbf{P}(j)$ may be approximated by

$$
\begin{aligned}
\mathbf{P}(j) \approx & \left|\Omega e^{-i\xi_1} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right)\cos\beta_0 + e^{-ij\Delta} e^{i\varsigma}\sin\beta_0\right|^2\\
&+\left|\frac{\Omega\cot\omega\sin\varphi_2}{\varphi_1} e^{-i\xi_2} e^{-i(j-1)\Delta}\left(\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right)\cos\beta_0\right|^2\\
&\doteq \Omega^2\left|\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right|^2\left(1+\frac{\cot^2\omega\sin^2\varphi_2}{\varphi_1^2}\right)\\
&= \Omega^2\left|\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right|^2\tau/\left(1+\cot^2\omega\cos^2\varphi_2\right),
\end{aligned}
$$

owing to the fact that the value of $\mathbf{P}(j)$ is governed by the factor $\left(1-e^{i2j\Delta}\right)/\left(1-e^{i2\Delta}\right)$. We see that, in the limit $2\Delta \to 0$,

$$\lim_{2\Delta\to 0}\left|\frac{1-e^{i2j\Delta}}{1-e^{i2\Delta}}\right| = \lim_{2\Delta\to 0}\left|\frac{\sin j\Delta}{\sin\Delta}\right| = j.$$

This shows that in the case $\phi = \theta$, as j approaches $J_{(\omega,\phi=\theta)}$, $\mathbf{P}(j)$ tends towards the previously given expression (31) for any fixed $\omega \in (0,\pi/2)$, any $\varphi_1 \in (0,\beta_0]$, and all possible values of φ_2 obeying the requirement (8), provided that N is sufficiently large and $M \ll N$. However, if we let the absolute difference $2\Delta = |\theta-\phi| \gg 0$, then whatever value we choose for j, it follows after a few algebraic maneuvers that $|\sin j\Delta/\sin\Delta| \ll J_{(\omega,\phi=\theta)}$ and thus this tends to destroy the Grover's search algorithm, irrespective of whether ω is large or not.

Theorem 2 follows. □

4 Conclusions

In the case of $\phi = \theta$ and $M/N \ll 1$ for sufficiently large N, we have derived the concise formulae $\mathbf{P}_{\max}\left(j = J_{(\omega,\phi=\theta)}\right) \doteq 1/\left(1+\cot^2\omega\cos^2\varphi_2\right)$ and $J_{(\omega,\phi=\theta)} \doteq \left\lfloor \pi/\left(2\Omega\sqrt{\tau}\right)\right\rfloor$, which are used to approximately evaluate the maximum success probabilities of finding a desired state and the required numbers of iterations to attain them under the assumption that the choice of φ_2 depends on the inequality (8) for all $\varphi_1 \in (0,\beta_0]$ and all $\omega \in (0,\pi/2)$. The advantage of the use of the approach proposed in this paper is that, given any $\varphi_1 \in (0,\beta_0]$, $J_{(\omega=\omega',\phi=\theta)}$ is readily calculable via $J_{(\omega=\pi/2,\phi=\theta)}$ for any $\omega' \in (0,\pi/2)$ and all possible values of φ_2 with the restriction posed by the above inequality. Finally, we have shown that the Grover's search algorithm fails to enhance the probability of measuring a desired state provided $|\theta-\phi| \gg 0$ in the four-complex-dimensional subspace.

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