Quantum measurement theory for an implementation of Deutsch's algorithm

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We discuss projective measurement theory does not meet easy detector model for Pauli observable. We propose a solution of the problem by changing the value of the result of quantum measurements and by considering macroscopic system. We discuss how our solution is used in an implementation of Deutsch's algorithm. Especially, we systematically describe our assertion based on more mathematical analysis using raw data.

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I. INTRODUCTION

As a famous physical theory, the quantum theory (cf. [1–5]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

The double-slit experiment is an illustration of wave-particle duality. In it, a beam of particles (such as photons) travels through a barrier with two slits removed. If one puts a detector screen on the other side, the pattern of detected particles shows interference fringes characteristic of waves; however, the detector screen responds to particles. The system exhibits behaviour of both waves (interference patterns) and particles (dots on the screen).

If we modify this experiment so that one slit is closed, no interference pattern is observed. Thus, the state of both slits affects the final results. We can also arrange to have a minimally invasive detector at one of the slits to detect which slit the particle went through. When we do that, the interference pattern disappears [6]. An analysis of a two-atom double-slit experiment based on environment-induced measurements is reported [7].

We assume an implementation of double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is -1. This model is easy detector model for Pauli observable. See FIG. 1.

As for quantum computation, implementation of a quantum algorithm to solve Deutsch's problem [8] on a nuclear magnetic resonance quantum computer is reported firstly [9]. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [10]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [11]. Single-photon Bell states are prepared and measured [12]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [13]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [14].

Recently, it is discussed projective measurement theory does not meet Deutsch's algorithm [15]. In this reference, it is discussed that the expected values of two spin observables $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ cannot be measured by using projective measurement theory. And it is discussed that new measurement theory covers the problem. Let us follow the argumentations. Assume a pure spin 1/2 state. We have $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 = 1$ from the wave functional analysis of quantum mechanics. On the other hand, we have $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 = 2$ if projective measurement theory is true. Hence the expected values of two spin observables $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ cannot be measured by using projective measurement theory. But, we have $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 = 1$ when new quantum measurement theory is true. The different point is that the values of the result of quantum measurements are $\pm 1/\sqrt{2}$.

Here we consider whether an expected value of one spin observable $\langle\sigma_x\rangle$ can be measured by using projective measurement theory. So, we investigate the relation between easy detector model for Pauli observable and projective measurement theory.

In this paper, we consider whether projective measurement theory meets easy detector model for Pauli observable. We assume an implementation of double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. We assume that a source of spin-carrying particles emits them in a state, which can be described as an eigenvector of Pauli observable σ_z. We consider a single expected value of Pauli observable σ_x in the double-slit experiment. A wave function analysis says that the quantum expected value of it is zero. However, the quantum predictions within projective measurement theory cannot coexist with the value of the expected value of $\langle\sigma_x\rangle = 0$. Hence, such projective measurement theory does not meet the easy detector model. We propose a solution of the problem by considering macroscopic system. We discuss how our solution is used in an implementation of Deutsch's algorithm. Especially, we systematically describe our assertion based on more mathematical analysis using raw data.

At this stage we are in the following situation.

1. We cannot measure an expected value of a single spin observable by using projective measurement theory.

2. New measurement theory covers the problem mentioned above.

3. We can use new measurement theory for an implementation of Deutsch's algorithm.

This paper is organized as follows:

In Sec. II, we consider the relation between double-slit experiment and projective measurement theory. We cannot measure a single spin observable by using projective measurement theory.

In Sec. III, we consider many double-slit experiments. And we propose a solution of the problem concerning projective measurement theory.

In Sec. IV, we review Deutsch's algorithm along with Ref. [5].

In Sec. V, we discuss how our solution is used in an implementation of Deutsch's algorithm.

Section VI concludes this paper.

II. DOUBLE-SLIT EXPERIMENT AND PROJECTIVE MEASUREMENT THEORY

In this section, we consider the relation between double-slit experiment and projective measurement theory. We assume an implementation of double-slit experiment. See FIG. 1. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is -1.

A. A wave function analysis

Let (σ_z, σ_x) be Pauli vector. We assume that a source of spin-carrying particles emits them in a state $|\psi\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We consider a quantum expected value $\langle\sigma_x\rangle$ as

$$\langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle = 0. \tag{1}$$

The above quantum expected value is zero if we consider only a wave function analysis.

We derive a necessary condition for the quantum expected value for the system in the pure spin-1/2 state $|\psi\rangle$ given in (1). We derive the possible value of the product $\langle\sigma_x\rangle \times \langle\sigma_x\rangle = \langle\sigma_x\rangle^2$. $\langle\sigma_x\rangle$ is the quantum expected value given in (1). We derive the following proposition

$$\langle\sigma_x\rangle^2 = 0. \tag{2}$$

B. Projective measurement theory

On the other hand, a mean value E satisfies projective measurement theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \tag{3}$$

where l denotes a label and r is the result of projective measurement of the Pauli observable σ_x. We assume the value of r is ± 1 (in $\hbar/2$ unit).

Assume the quantum mean value with the system in an eigenvector ($|\psi\rangle$) of Pauli observable σ_z given in (1) admits projective measurement theory. One has the following proposition concerning projective measurement theory

$$\langle\sigma_x\rangle(m) = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}. \tag{4}$$

We can assume as follows by Strong Law of Large Numbers,

$$\langle\sigma_x\rangle(+\infty) = \langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle. \tag{5}$$

In what follows, we show that we cannot assign the truth value "1" for the proposition (4) concerning projective measurement theory.

Assume the proposition (4) is true. By changing the label l into l' and by changing the label m into m', we have same quantum mean value as follows

$$\langle\sigma_x\rangle(m') = \frac{\sum_{l'=1}^{m'} r_{l'}(\sigma_x)}{m'}. \tag{6}$$

An important note here is that the value of the right-hand-side of (4) is equal to the value of the right-hand-side of (6) because we only change the labels. We have

$$\begin{aligned}
&\langle\sigma_x\rangle(m) \times \langle\sigma_x\rangle(m') \\
&= \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m'} r_{l'}(\sigma_x)}{m'} \\
&= \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{\sum_{l=1}^{m}} \times \frac{\sum_{l'=1}^{m'} r_{l'}(\sigma_x)}{\sum_{l'=1}^{m'}} \times \frac{\delta_{ll'}}{\delta_{ll'}} \\
&= \frac{\sum_{l=1}^{m}}{m} \cdot (r_l(\sigma_x))^2 \\
&= \frac{\sum_{l=1}^{m}}{m} = 1.
\end{aligned} \tag{7}$$

Here $\delta_{ll'}$ is a delta function. We use the following fact

$$(r_l(\sigma_x))^2 = 1 \tag{8}$$

and

$$\frac{\delta_{ll'}}{\delta_{ll'}} = 1. \tag{9}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that projective measurement theory is true (in a spin-1/2 system), that is

$$\langle\sigma_x\rangle(m) \times \langle\sigma_x\rangle(m') = 1. \tag{10}$$

From Strong Law of Large Numbers, we have

$$\langle\sigma_x\rangle \times \langle\sigma_x\rangle = 1. \tag{11}$$

Hence we derive the following proposition concerning projective measurement theory

$$\langle\sigma_x\rangle^2 = 1. \tag{12}$$

We do not assign the truth value "1" for two propositions (2) (concerning a wave function analysis) and (12) (concerning projective measurement theory), simultaneously. We are in the contradiction. This implies that we cannot perform the following Deutsch's algorithm.

- The control of quantum states relies on the wave functional analysis.
- The observation of quantum states relies on projective measurement theory.

We cannot accept the validity of the proposition (4) (concerning projective measurement theory) if we assign the truth value "1" for the proposition (2) (concerning a wave function analysis). In other words, such projective measurement theory does not meet the detector model for spin observable σ_x. And we cannot perform Deutsch's algorithm. Consistency between controlability and observability is necessary for an implementation of Deutsch's algorithm. And desired consistency is not established.

III. SOLUTION OF THE PROBLEM OF PROJECTIVE MEASUREMENT THEORY IN MACROSCOPIC SYSTEM

In this section, we consider many double-slit experiments. In macroscopic system, we solve the contradiction presented in the previous section.

A. A wave function analysis

We consider an implementation of N double-slit experiments. See FIG. 2. We assume that N sources of spin-carrying particles emit them in a state, which can be described as an eigenvector of Pauli observable σ_z. We have the following state globally

$$\overbrace{|\psi\rangle|\psi\rangle\cdots|\psi\rangle}^{N} = |\psi\rangle^{\otimes N}. \tag{13}$$

Each of them can be described as an eigenvector of Pauli observable σ_z. We analyze experimental data globally. We consider a single expected value of

$$\overbrace{\sigma_x \otimes \sigma_x \otimes \cdots \otimes \sigma_x}^{N} = \sigma_x^{\otimes N} \tag{14}$$

then we have the following quantum expected value from a wave function analysis

$$\langle\psi|^{\otimes N}\sigma_x^{\otimes N}|\psi\rangle^{\otimes N} = (\langle\psi|\sigma_x|\psi\rangle)^N = \langle\sigma_x\rangle^N = 0. \tag{15}$$

Thus we have the following proposition concerning a wave function analysis

$$(\langle\sigma_x\rangle^2)^N = 0, (N \to +\infty). \tag{16}$$

B. New type of a quantum measurement

On the other hand, a mean value E satisfies a quantum measurement theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \tag{17}$$

where l denotes a label and r is the result of quantum measurement of the Pauli observable σ_x. We assume the value of r is $\pm\frac{1}{\sqrt{2}}$ (in $\hbar/2$ unit)[15]. If a particle passes one side slit, then the value of the result of measurement is $+\frac{1}{\sqrt{2}}$. If a particle passes another slit, then the value of the result of measurement is $-\frac{1}{\sqrt{2}}$.

Assume the quantum mean value with the system in an eigenvector ($|\psi\rangle$) of the Pauli observable σ_z given in (1) admits such a quantum measurement theory. One has the following proposition concerning the quantum measurement theory

$$\langle\sigma_x\rangle(m) = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}. \tag{18}$$

In what follows, we show that we can assign the truth value "1" for the proposition (18) concerning the quantum measurement theory in the macroscopic system ($N \to +\infty$).

Assume the proposition (18) is true. By changing the label l into l' and by changing the label m into m', we have same quantum mean value as follows

$$\langle\sigma_x\rangle(m') = \frac{\sum_{l'=1}^{m'} r_{l'}(\sigma_x)}{m'}. \tag{19}$$

An important note here is that the value of the right-hand-side of (18) is equal to the value of the right-hand-side of (19) because we only change the labels. We have

$$\begin{aligned}
&\langle\sigma_x\rangle(m)\times\langle\sigma_x\rangle(m')\\
&=\frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}\times\frac{\sum_{l'=1}^{m'} r_{l'}(\sigma_x)}{m'}\\
&=\frac{\sum_{l=1}^{m} r_l(\sigma_x)}{\sum_{l=1}^{m}}\times\frac{\sum_{l'=1}^{m'} r_{l'}(\sigma_x)}{\sum_{l'=1}^{m'}}\times\frac{\delta_{ll'}}{\delta_{ll'}}\\
&=\frac{\sum_{l=1}^{m}}{m}(r_l(\sigma_x))^2\\
&=1/2\frac{\sum_{l=1}^{m}}{m}=1/2.
\end{aligned} \tag{20}$$

We use the following fact

$$(r_l(\sigma_x))^2=1/2. \tag{21}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that such a quantum measurement is true (in a spin-1/2 system), that is,

$$\langle\sigma_x\rangle(m)\times\langle\sigma_x\rangle(m')=1/2. \tag{22}$$

From Strong Law of Large Numbers, we have

$$\langle\sigma_x\rangle\times\langle\sigma_x\rangle=1/2. \tag{23}$$

Therefore we have $(\langle\sigma_x\rangle^2)^N=1/2^N$ Hence we derive the following proposition concerning the quantum measurement

$$(\langle\sigma_x\rangle^2)^N=1/2^N. \tag{24}$$

Thus,

$$(\langle\sigma_x\rangle^2)^N=0,(N\rightarrow+\infty). \tag{25}$$

We can assign the truth value "1" for both two propositions (16) (concerning a wave function analysis) and (25) (concerning the quantum measurement theory), simultaneously. Hence, we solve the contradiction presented in the previous section by changing the value of the result of quantum measurements and by considering an implementation of double-slit experiments macroscopically. This implies that we can perform the following Deutsch's algorithm.

- The control of quantum states relies on the wave functional analysis.
- The observation of quantum states relies on the measurement theory.

In other words, such a measurement theory meets the detector model for spin observable σ_x. And we can perform Deutsch's algorithm. Consistency between controlability and observability is necessary for an implementation of Deutsch's algorithm. And desired consistency is established.

IV. QUANTUM COMPUTATION

In this section, we review the quantum-theoretical formulation of Deutsch's algorithm [8] as the earliest quantum computer along with Ref. [5].

Quantum parallelism is a fundamental feature of many quantum algorithms. It allows quantum computers to evaluate the values of a function $f(x)$ for many different values of x simultaneously. Suppose $f:\{0,1\}\rightarrow\{0,1\}$ is a function with a one-bit domain and range. A convenient way of computing this function on a quantum computer is to consider a two-qubit quantum computer which starts in the state $|x,y\rangle$. With an appropriate sequence of logic gates it is possible to transform this state into $|x,y\oplus f(x)\rangle$, where $\oplus$ indicates addition modulo 2. We give the transformation defined by the map $|x,y\rangle\rightarrow|x,y\oplus f(x)\rangle$ a name, U_f.

Deutsch's algorithm combines quantum parallelism with a property of quantum mechanics known as interference. Let us use the Hadamard gate to prepare the first qubit $|0\rangle$ as the superposition $(|0\rangle+|1\rangle)/\sqrt{2}$, but let us prepare the second qubit as the superposition $(|0\rangle - |1\rangle)/\sqrt{2}$, using the Hadamard gate applied to the state $|1\rangle$. The Hadamard gate is as $H = \frac{1}{\sqrt{2}}(|0\rangle\langle 1| + |1\rangle\langle 0| + |0\rangle\langle 0| - |1\rangle\langle 1|)$. Let us follow the states along to see what happens in this circuit. The input state

$$|\psi_0\rangle = |01\rangle \tag{26}$$

is sent through two Hadamard gates to give

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]. \tag{27}$$

A little thought shows that if we apply U_f to the state $|x\rangle(|0\rangle - |1\rangle)/\sqrt{2}$ then we obtain the state $(-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle)/\sqrt{2}$. Applying U_f to $|\psi_1\rangle$ therefore leaves us with one of two possibilities:

$$|\psi_2\rangle = \begin{cases} \pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) = f(1) \\ \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) \neq f(1). \end{cases} \tag{28}$$

The final Hadamard gate on the first qubit thus gives us

$$|\psi_3\rangle = \begin{cases} \pm|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) = f(1) \\ \pm|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) \neq f(1). \end{cases} \tag{29}$$

Realizing that $f(0) \oplus f(1)$ is 0 if $f(0) = f(1)$ and 1 otherwise, we can rewrite this result concisely as

$$|\psi_3\rangle = \pm|f(0) \oplus f(1)\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right], \tag{30}$$

so by measuring the first qubit we may determine $f(0) \oplus f(1)$. This is very interesting indeed: the quantum circuit has given us the ability to determine a global property of $f(x)$, namely $f(0) \oplus f(1)$, using only one evaluation of $f(x)$! This is faster than is possible with a classical apparatus, which would require at least two evaluations.

V. THE RELATION BETWEEN OUR RESULT AND DEUTSCH'S ALGORITHM

In this section, we discuss how our solution is used in an implementation of Deutsch's algorithm. Now, we can measure Pauli observable σ_x by solving the contradiction discussed in Section II. Consistency between controlability and observability is established. The values of the result of quantum measurements are $\pm 1/\sqrt{2}$. So the values can be used for the values of the result of the final measurement of Deutsch's algorithm. From the previous section, we have

$$|\psi_2\rangle = \begin{cases} \pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) = f(1) \\ \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) \neq f(1). \end{cases} \tag{31}$$

We can consider

$$\pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] = \pm|+_x\rangle, \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = \pm|-_x\rangle. \tag{32}$$

Therefore if we can measure an expected value of σ_x, then we can distinguish the two states mentioned above. From a wave function analysis, we have

$$\langle +_x|\sigma_x|+_x\rangle = +1, \langle -_x|\sigma_x|-_x\rangle = -1. \tag{33}$$

We see one measurement is enough to determine which state is realized. We can omit the final Hadamard gate on the first qubit.

VI. CONCLUSIONS

In conclusion, we have considered whether projective measurement theory meets easy detector model for spin observable. We have assumed an implementation of double-slit experiment. There has been a detector just after each slit. Thus interference figure has not appeared, and we do not have considered such a pattern. We have assumed that a source of spin-carrying particles emits them in a state, which can be described as an eigenvector of Pauli observable σ_z. We have considered a single expected value of Pauli observable σ_x in the double-slit experiment. A wave function analysis has said that the quantum expected value of it is zero. However, the quantum predictions within projective measurement theory cannot have coexisted with the value of the expected value of $\langle\sigma_x\rangle = 0$. Hence, projective measurement theory does not have met such easy detector model. We have proposed a solution of the problem by considering macroscopic system. We have discussed how our solution is used in an implementation of Deutsch's algorithm. Especially, we have systematically described our assertion based on more mathematical analysis using raw data.

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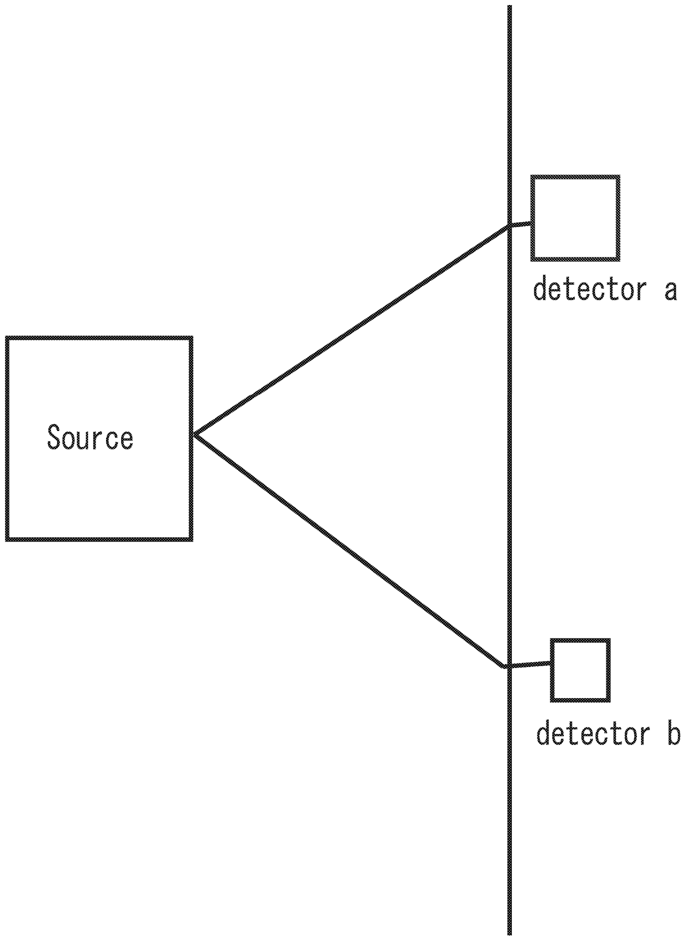

FIG. 1: A schematic diagram of an implementation of a double-slit experiment.

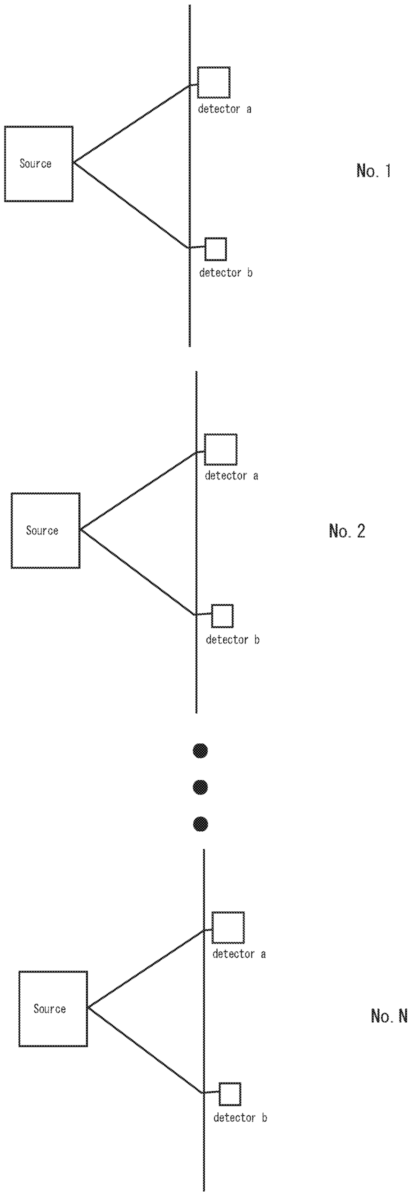

FIG. 2: A schematic diagram of an implementation of N double-slit experiments.